Properties of Laplace Transform :⇒

1. Linearity Property :⇒

Statement :- Consider two time domain signals $f_1(t)$ and $f_2(t)$.

If $f_1(t) \xleftrightarrow{L.T.} F_1(S)$ and $f_2(t) \xleftrightarrow{L.T.} F_2(S)$, then

$$[a_1 f_1(t) \pm a_2 f_2(t)] \xleftrightarrow{L.T.} a_1 F_1(S) \pm a_2 F_2(S)$$

Proof

According to laplace transformation

$$F(S) = L\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

here,

$$f(t) = a_1 f_1(t) \pm a_2 f_2(t)$$

$$\therefore L\{a_1 f_1(t) \pm a_2 f_2(t)\} = \int_{-\infty}^{\infty} [a_1 f_1(t) \pm a_2 f_2(t)]\, e^{-st}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1(t)\, e^{-st}\, dt \pm a_2 \int_{-\infty}^{\infty} f_2(t)\, e^{-st}\, dt$$

$$= a_1 F_1(S) \pm a_2 F_2(S)$$

2. ~~Time~~ Shifting Theorem :⇒

(i) Time shifting (Shifting in Time Domain) :⇒

If $f(t)$ is input signal whose ROC is R then

$$L\{f(t-t_0)\} = e^{-st_0} \cdot F(S) \qquad ROC = R$$

Proof :-

$$L\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$\therefore L\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-st}\, dt$$

(Let, $k = t - t_0$, $\therefore dk = dt$ and $t = k + t_0$)

$$= \int_{-\infty}^{\infty} f(k) \cdot e^{-s(k+t_0)}\, dk$$

$$= e^{-st_0} \int_{-\infty}^{\infty} f(k) \cdot e^{-sk}\, dk$$

$$= e^{-st_0} \cdot F(S) \qquad ROC = R$$

(ii) Shifting in s-Domain :⇒

$$\text{If } f(t) \xleftrightarrow{LT} F(s), \text{ ROC} = R$$

$$e^{s_0 t} f(t) \xleftrightarrow{LT} F(s-s_0) \qquad \text{ROC}: R + Re(s_0)$$

Proof

From laplace transform

$$L\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$\therefore L\{e^{\pm s_0 t} f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st} \cdot e^{\pm s_0 t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-(s \mp s_0)t}\, dt$$

$$= F(s \mp s_0)$$

Time Scalling Property :⇒

If,

$$f(t) \xleftrightarrow{LT} F(s) \; ; \quad \text{ROC} = R$$

then,

$$\{f(at)\} \xleftrightarrow{LT} \frac{1}{|a|} F\left(\frac{s}{a}\right); \qquad \text{ROC} = \frac{R}{a}$$

Proof

From laplace difinition

$$L\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-st}\, dt$$

$$\begin{cases} \text{Let, } P = at & \text{and } dt = \frac{1}{a} dP \\ \therefore \; t = \frac{P}{a} & \end{cases}$$

$$= \int_{-\infty}^{\infty} f(P) \cdot e^{-\left(s \cdot \frac{P}{a}\right)} \cdot \frac{dP}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(P)\, e^{-\left(\frac{s}{a}\right)P} \cdot dP$$

$$= \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Similarly,

$$L\{f(-at)\} = \frac{1}{|a|} F\left(\frac{s}{-a}\right); \quad \text{ROC} = \frac{R}{-a}$$

Convolution Property:⇒

If $f(t) \xleftrightarrow{LT} F(S)$, ROC; R_1

and $h(t) \xleftrightarrow{L.T} H(S)$, ROC: R_2

then $f(t) * h(t) \xleftrightarrow{LT} F(S).H(S)$, ROC; Intersection of R_1 & R_2

Proof

From the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

Taking L.T. on both sides

$$Y(s) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] . e^{-st} dt$$

changing the order of integration and noting the fact that $f(\tau)$ does not depend on 't'.

$$Y(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) [h(t-\tau) e^{-st} dt] d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) L\{h(t-\tau)\} d\tau$$

If $h(t) \xleftrightarrow{L.T} H(s)$

Then from shifting property

$$h(t-\tau) \xleftrightarrow{L.T.} e^{-s\tau} H(s)$$

$$\therefore Y(s) = \int_{-\infty}^{\infty} f(\tau) H(s) e^{-s\tau} d\tau$$

$$= H(s) \int_{-\infty}^{\infty} f(\tau) e^{-s\tau} d\tau$$

$$= F(S) . H(S)$$ ROC: $R_1 \cap R_2$

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Initial Value Theorem :⇒

If $f(t) \xleftrightarrow{L.T.} F(s)$, then initial value of $f(t)$ is given as

$$\underline{f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} [sF(s)]}$$

Proof

The differentiation property of laplace transform is given by,

$$\frac{d}{dt} f(t) \xleftrightarrow{L.T.} sF(s) - f(0^+)$$

taking limit as $s \to \infty$ both sides we get

$$\lim_{s \to \infty} L\left\{\frac{d}{dt} f(t)\right\} = \lim_{s \to \infty} \{sF(s) - f(0^+)\}$$

Using laplace transform definition

$$L.H.S = \lim_{s \to \infty} L\left\{\frac{d}{dt} f(t)\right\}$$

$$= \lim_{s \to \infty} \int_0^\infty \frac{d}{dt} f(t) \cdot e^{-st} dt$$

$$\left\{\text{But } \lim_{s \to \infty} e^{-st} = 0\right\}$$

$$= 0$$

L.H.S. = R.H.S.

$$\therefore 0 = \lim_{s \to \infty} \{sF(s) - f(0^+)\}$$

$$= \lim_{s \to \infty} sF(s) - \lim_{s \to \infty} f(0^+)$$

But in the second term, 's' is not present so we can neglect $\lim_{s \to \infty}$, therefore

$$\therefore 0 = \lim_{s \to \infty} sF(s) - f(0^+)$$

$$\text{OR } f(0^+) = \lim_{s \to 0} sF(s)$$

Final Value Theorem :⇒

If $f(t) \xleftrightarrow{L.T.} F(s)$, then final value of $f(t)$ is given as

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} [sF(s)]$$

Proof :- We know that differentiation property

$$L\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^+)$$

Taking limit of both sides $s \to 0$ we get

$$\lim_{s \to 0} L\left\{\frac{d}{dt} f(t)\right\} = \lim_{s \to 0} \{sF(s) - f(0^+)\}$$

$$= \lim_{s \to 0} sF(s) - f(0^+)$$

$$= \lim_{s \to 0} \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$ (according to L.T. Defini-

$$= \int_0^\infty \frac{d}{dt} f(t) dt . \lim_{s \to 0} e^{-st}$$

But, $\lim_{s \to 0} e^{-st} = e^0 = 1$

$$\therefore \lim_{s \to 0} L\left\{\frac{d}{dt} f(t)\right\} = \int_0^\infty \frac{d}{dt} f(t) dt = [f(t)]_0^\infty = f(\infty) - f(0^+)$$

OR $\lim_{s \to 0} [s.F(s)] = \lim_{t \to \infty} f(t)$

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