Laplace Transform of a Derivative $\left[\frac{df(t)}{dt}\right]$:⇒

Let F(s) be the laplace transform of f(t) and let $f(0^+)$ be the value of f(t) as $t \to 0$ then,

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t\to 0} f(t) = sF(s) - f(0^+)$$

Proof:-

$$F(s) = L[f(t)] = \int_0^\infty f(t).\, e^{-st}\, dt$$

From relation it is evident that, $\int u\,dv = uv - \int v\,du$

Let,

$$f(t) = u \;;\; \text{then } \left[\frac{df(t)}{dt}\right].dt = du$$

and,

$$e^{-st}\,dt = dv \;;\; \text{then } v = -\frac{1}{s} e^{-st}$$

Therefore,

$$F(s) = \int_0^\infty u\,dv = [uv]_0^\infty - \int_0^\infty v\,du$$

$$= f(t).\left[-\frac{1}{s} e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st}\left[\frac{df(t)}{dt}\right] dt$$

$$= \left[-\frac{f(t)}{s} e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= \frac{f(0^+)}{s} + \frac{1}{s}\int_0^\infty f'(t)\, e^{-st}\, dt$$

OR, $$sF(s) = f(0^+) + \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$\therefore \int_0^\infty f'(t)\, e^{-st}\, dt = sF(s) - f(0^+)$$

therefore,

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

$$L[f'(t)] = sF(s) - f(0^+)$$

Similarly, the laplace transform of second derivative of f(t) as

$$L\left[\frac{d^2f(t)}{dt^2}\right] = L\left[\frac{d}{dt}\left\{\frac{df(t)}{dt}\right\}\right] = sL\left[\frac{df(t)}{dt}\right] - \left.\frac{df(t)}{dt}\right|_{t=0}$$

$$= s[F(s) - f(0^+)] - f'(0^+) = s^2F(s) - sf(0^+) - f'(0^+)$$

where, $f'(0^+)$ is the value of the first derivative of f(t) as $t \to 0$.

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Laplace Transform of an Integral $\int f(t)\,dt$:⇒

If $L[f(t)] = F(s)$ then, the laplace transform of the first integral of $f(t)$ is given by

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Proof:-

$$L\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-st}\,dt$$

Let, $u = \int_0^t f(t)\,dt$; then $du = f(t)\,dt$

and, $dv = e^{-st}\,dt$; then $v = -\frac{1}{s}e^{-st}$

on integrating,

$$L\left[\int_0^t f(t)\,dt\right] = \int_0^\infty u\,dv$$

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= -\frac{1}{s}e^{-st}\cdot\int_0^t f(t)\,dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= 0 - 0 + \frac{1}{s}L[f(t)] = \frac{F(s)}{s}$$

Laplace transform of t^n :⇒

$$f(t) = t^n$$

from laplace definition

$$F(s) = \int_0^\infty t^n e^{-st}\,dt$$

$\int u\,dv = uv - \int v\,du$

here, $u = t^n$, $dv = e^{-st}\,dt$

$du = nt^{n-1}$, $v = -\frac{1}{s}e^{-st}$

Integration by parts yields

$$F(s) = -\frac{t^n}{s}e^{-st}\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\cdot n\cdot t^{n-1}\,dt$$

$$F(s) = 0 + \frac{n}{s}L\{t^{n-1}\}$$

$$\cancel{= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{3}{s}\cdot\frac{2}{s}\cdot\frac{1}{s}\cdot L\{t^0\}}$$

However $n = 0, 1, 2, 3, 4, \ldots$

⇒ $\boxed{L\{t^0\} = \frac{1}{s}}$

$\therefore \frac{1}{s}L\{t^0\} = \frac{1}{s^2}$, $\frac{1}{s}L\{t^1\} = \frac{2}{s^3}$, $\frac{1}{s}L\{t^2\} = \frac{6}{s^4}$ & so on

$$F(s) = \frac{n!}{s^{n+1}}$$

Integration in s- Domain Theorem :⇒ / Laplace transform of $\left[\frac{f(t)}{t}\right]$

Statement:-

If $f(t) \xleftrightarrow{L.T.} F(S)$;   ROC : R

then $\frac{f(t)}{t} \xleftrightarrow{L.T.} \int_{s}^{\infty} F(s)\,ds$,   ROC : R

Proof

we know that,

$$F(S) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

Integrating both side from s to ∞

$$\int_{s}^{\infty} F(s)\,ds = \int_{s}^{\infty}\left[\int_{0}^{\infty} f(t)\, e^{-st}\, dt\right]$$

$$= \int_{0}^{\infty} f(t)\left[\int_{s}^{\infty} e^{-st}\, ds\right] dt$$

$$= \int_{0}^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt$$

$$= \int_{0}^{\infty} f(t)\left[0 - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_{0}^{\infty} f(t) \cdot \frac{e^{-st}}{t}\, dt = L\left[\frac{f(t)}{t}\right]$$

$\therefore \frac{f(t)}{t} \xleftrightarrow{L.T.} \int_{s}^{\infty} F(s)\,ds$ ROC = R

Differentiation in S-Domain Theorem : ⇒ / Laplace transform of $[t.f(t)]$

Statement :-

If $f(t) \xleftrightarrow{L.T.} F(s)$, ROC : R

then $t.f(t) \xleftrightarrow{L.T.} -\frac{d}{ds}F(s)$, ROC : R

Proof :-

According to the definition of Laplace transform,

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad \text{— (I)}$$

differentiating both sides w.r.to s we get,

$$\frac{d}{ds}F(s) = \int_0^\infty f(t) \frac{d}{ds}(e^{-st})\, dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty [-t.f(t)]\, e^{-st}\, dt \quad \text{— (II)}$$

comparing eq. (I) and (II), we can conclude that the laplace transform of $[-t f(t)]$ is equal to $\frac{d}{ds}F(s)$.

$$\frac{d}{ds}F(s) = \int_0^\infty f(t).e^{-st} dt.(-t)$$

$$= -\int_0^\infty t.f(t).e^{-st}\, dt$$

$$= -L[t.f(t)]$$

that means,

$$t.f(t) \xleftrightarrow{L.T.} -\frac{d}{ds}F(s)$$