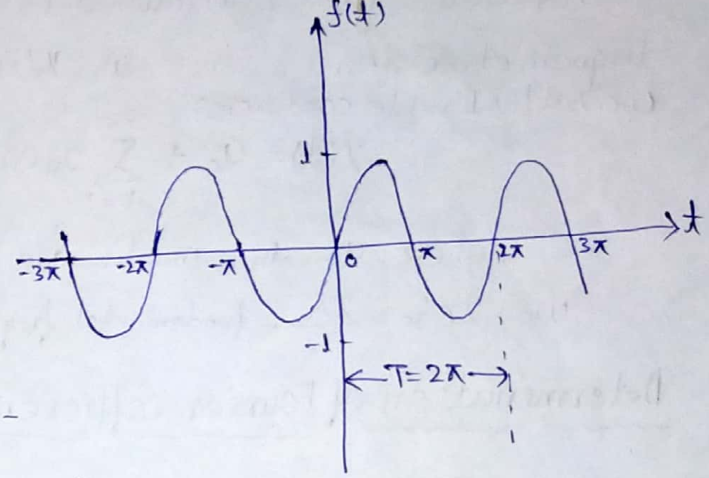
Fourier Series:⇒

If the value of each function f(t) repets itself at equal intervals in the abscissa, then f(t) is said to be a periodic function.

Fourier series represents a periodic function in form of sum of infinite number of sine and cosine terms. e.g.



$\sin x = \sin(x+2\pi) + \sin(x+4\pi) + \cdots$

Fourier series is a "mathmetical tool" used to analyze any periodic signal. after the analysis we obtain the following information about the signal:-

1. Frequency components
2. Their amplitudes
3. Phase difference b/w these frequency components

here we will express a non-sinusoidal periodic signal in to a fundamental and its harmonics —

$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + a_n\cos nx + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots + b_n\sin nx + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx.$$

where, a_0, a_n and b_n are known as fourier constants, and x is the independent variables.

Types of Fourier Series:⇒

There are three types of fourier series used for the analysis of periodic signals.

1. Trignometric OR Quadrature Fourier Series
2. Polar Fourier Series
3. Exponential Fourier Series

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Trignometric OR Quadrature Fourier Series:⇒

A periodic signal $x(t)$ with a period of "T_0" is represented by the trignometric fourier series as, $x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \cdots + a_n \cos n\omega_0 t + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \cdots + b_n \sin n\omega_0 t + \cdots$
(i.e. satifies Dirichlet conditions)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad (t \to t+T_0)$$

where, a_0, a_n, and b_n are known as fourier cofficients and

$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ = fundamental frequency, $2\omega_0, 3\omega_0, 4\omega_0 \cdots$ are called the hormonics of ω_0

Determination of Fourier cofficients:- (By EULER's Formulae)

1. The value of a_0:-

$$a_0 = \frac{1}{T_0} \int_t^{t+T_0} x(t)\,dt$$

It is called average value OR DC component of $x(t)$.

2. The value of a_n:-

$$a_n = \frac{2}{T_0} \int_t^{t+T_0} x(t).\cos(n\omega_0 t)\,dt$$

3. The value of b_n:-

$$b_n = \frac{2}{T_0} \int_t^{t+T_0} x(t).\sin(n\omega_0 t)\,dt$$

Now, a periodic signal $x(t)$ canbe expressed as

$$x(t) = \underbrace{a_0}_{\text{DC Component}} + \underbrace{a_1 \cos(\omega_0 t)}_{\text{Fundamental component}} + \underbrace{a_2 \cos(2\omega_0 t)}_{\text{Second harmonic}} + \cdots + \underbrace{b_1 \sin(\omega_0 t)}_{\text{Fundamental component}} + \underbrace{b_2 \sin(2\omega_0 t)}_{\text{Second hormonic}} + \cdots$$

* It is suitable to plot the line spectrum.
* a_0 has a zero frequency, hence it is called DC components of $x(t)$

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Polar Fourier Series :⇒

The polar fourier series is derived from the trignometric fourier series by combining the sine and cosine terms of same frequency.

2⇒ The polar fourier series represented as

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n)$$

where,

C_0 = Average value of $x(t) = a_0$

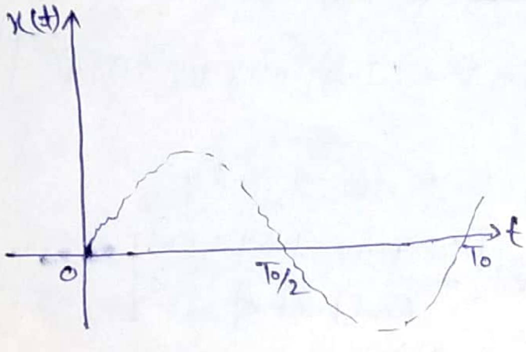
$C_n = \sqrt{a_n^2 + b_n^2}$ = spectral amplitudes of $x(t)$ i.e. amplitude of spectral component $C_n \cos(n\omega_0 t - \phi_n)$ having frequency $n\omega_0$.

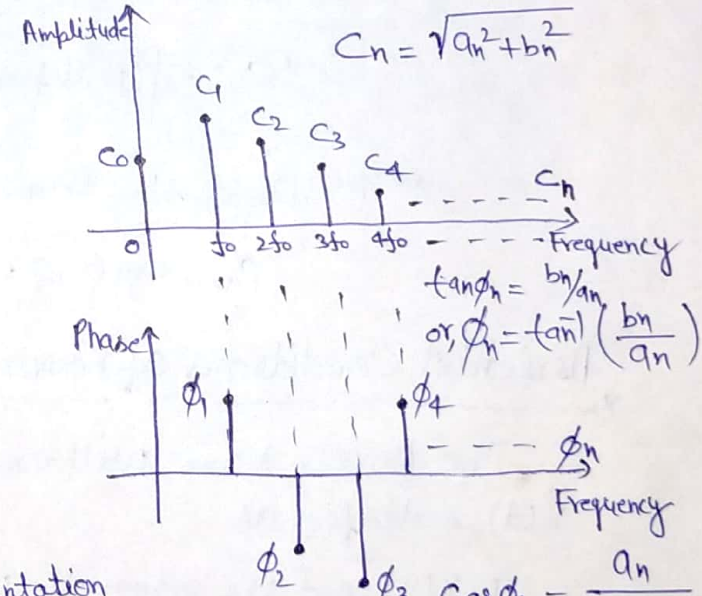
and $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ = Phase of spectral component $n\omega_0$ i.e. phase spectrum of $x(t)$

It can be expanded by opening the summation sign as

$$x(t) = C_0 + \underbrace{C_1 \cos(\omega_0 t + \phi_1)}_{\text{Fundamental component}} + \underbrace{C_2 \cos(2\omega_0 t + \phi_2)}_{\text{II}^{nd} \text{ harmonic}} + \underbrace{C_3 \cos(3\omega_0 t + \phi_3)}_{\text{III}^{rd} \text{ harmonic}} + \cdots$$

1 ⇒ Representation :⇒





$C_n = \sqrt{a_n^2 + b_n^2}$

$\tan\phi_n = b_n/a_n$

or, $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

(Line spectrum representation using polar fourier series)

$\cos\phi_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}$

and $\sin\phi_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}$

Now, Trigonometric fourier series is given as

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\text{or, } x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\cos n\omega_0 t \cos\phi_n + \sin n\omega_0 t \sin\phi_n\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(n\omega_0 t - \phi_n)$$

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Exponential Fourier Series:⇒

A periodic signal x(t) is expressed in the exponential fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0} \underset{or}{=} \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 nt}$$

where,

$$C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t).e^{-j2\pi nt/T_0}\,dt \underset{or}{=} \frac{1}{T_0}\int_{0}^{T} x(t).e^{-jn\omega_0 t}\,dt$$

Concept of negative Frequency:⇒

The above eq., observe that n is extending from $-\infty$ to $+\infty$. due to this frequencies in the frequency spectrum will extend from $-\infty$ to $+\infty$. therefore x(t) express in double sided frequency spectrum.

However the negative frequency signals do not exist physically. They are used as an important mathematical concept i.e. amplitude and phase spectrums:-

* The amplitude spectrum of the signal x(t) is denoted by,

$$|C_n| = \left[(\text{Real part of } C_n)^2 + (\text{Imaginary part of } C_n)^2\right]^{1/2}$$

* The phase spectrum of x(t) is denoted by

$$\phi_n = \arg(C_n) = \tan^{-1}\left[\frac{\text{Imaginary part of } C_n}{\text{Real part of } C_n}\right]$$

Dirichlet Conditions of Fourier Series:⇒

The fourier series will exist if and only if the periodic signal x(t) satisfies as

1. x(t) and its integrals are finite and single valued in interval $(t \to t+T_0)$ i.e. over a period of one cycle T_0
2. x(t) must have only finite number of discontinuties in the given interval of time
3. x(t) must have only finite number of maxima and minima in the given interval of time
4. x(t) is absolutely integrable, i.e.

$$\int_{-T_0/2}^{+T_0/2} |x(t)|\,dt < \infty$$

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Exponential Fourier Series :⇒

The exponential form of fourier series is simpler and more compact and hence, this is most widely used in signal analysis.

The trigonometric fourier series is given as

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

We know that, Euler's identity

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and,} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

hence,

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \quad \text{and,} \quad \sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

Therefore,

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + b_n\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{(a_n - jb_n)e^{jn\omega_0 t}}{2} + \frac{(a_n + jb_n)e^{-jn\omega_0 t}}{2}\right]$$

where

$$C_0 = a_0$$

$$C_n = \frac{1}{2}(a_n - jb_n)$$

$C_{-n} = \frac{1}{2}(a_n + jb_n)$ i.e. the conjugate of C_n

Now,

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cdot e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}$$

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Now, substituting expression of coefficients a_n and b_n in the expression C_n.

$$C_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)[\cos n\omega_0 t - j\sin n\omega_0 t]\,dt$$

or, $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{-jn\omega_0 t}\,dt$$

Similarly, $$C_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{jn\omega_0 t}\,dt$$

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