Fourier Transform :⇒

Any signal is build up by addition of elementary signal which are at different frequencies, have different amplitudes and relative phases. Using fourier transform we can plot the amplitude and phase spectrums which provided us all the information about amplitudes and relative phases of such elementary signals.

Thus, Fourier Transform can be used for the analysis of a signal. It is used for transformation from time domain to frequency domain.

Definition :⇒

The fourier transform of a signal $x(t)$ is defined as

$$\text{Fourier transform: } X(j\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt$$

$$\text{OR} \qquad X(f) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi f t}.dt$$

Inverse Fourier Transform :⇒

The signal $x(t)$ can be obtained back from fourier transform $X(j\omega)$ by using the inverse fourier transform.

The inverse fourier transform is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega).e^{j\omega t}.d\omega$$

$$\text{OR} \qquad x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi f t}.df$$

or it can represented as,

$$x(t) = F^{-1}|X(j\omega)| = F^{-1}|X(f)|$$

The fourier transform is a complex function of frequency f. therefore it is possible to express it in the complex exponential form as

$$X(f) = |X(f)|.e^{j\theta(f)} \quad \text{OR} = X(j\omega) = |X(j\omega)|e^{j\theta(j\omega)}$$

where,

$|X(f)|$ = Amplitude spectrum of $x(t)$

$\theta(f)$ = The phase spectrum of $x(t)$.

Existence of Fourier Transform :⇒

A function $x(t)$ is said to be fourier transformable, if it is statified ~~similarly~~ same Dirichelet Conditions of fourier series of $x(t)$, then only it is possible to obtain the fourier transform of $x(t)$.

Properties of Fourier Transform :⇒

1. Linearity or Superposition :⇒

If $x_1(t) \xleftrightarrow{F.T.} X_1(f)$ and $x_2(t) \xleftrightarrow{F.T.} X_2(f)$, then

$$[a_1 x_1(t) + a_2 x_2(t)] \xleftrightarrow{F.T.} [a_1 X_1(f) + a_2 X_2(f)]$$

Proof.

$$F[a_1 x_1(t) + a_2 x_2(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] \cdot e^{-j2\pi f t} \cdot dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t) \cdot e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} a_2 x_2(t) \cdot e^{-j2\pi f t} dt$$

$$= a_1 X_1(f) + a_2 X_2(f)$$

Proved

2. Time Scaling :⇒

If $x(t) \xleftrightarrow{F.T.} X(f)$, then

$$x(\alpha t) \xleftrightarrow{F.T.} \frac{1}{|\alpha|} X(f/\alpha)$$

Proof.

'α' being a constant, can be +ive or -ive

* When α is +ive OR ($\alpha > 0$)

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\alpha t) \cdot e^{-j2\pi f t} \cdot dt$$

Now, substitute $\tau = \alpha t$, then

$$d\tau = \alpha dt \; ; \therefore dt = \frac{d\tau}{\alpha}$$

$$\therefore F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau/\alpha} \cdot \frac{d\tau}{\alpha}$$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi (f/\alpha)\tau} d\tau$$

$$= \frac{1}{\alpha} X(f/\alpha)$$

when α is -ive OR ($\alpha < 0$)

$$F[x(\alpha t)] = -\frac{1}{\alpha} X(f/\alpha)$$

hence,

$$F[x(\alpha t)] = \frac{1}{|\alpha|} X(f/\alpha)$$

3. Time Shifting :⇒

The time shifting property states that if $x(t) \xleftrightarrow{F.T.} X(f)$, then

$$x(t-t_0) \xleftrightarrow{F.T.} e^{-j2\pi f t_0} X(f)$$

Proof

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) . e^{-j2\pi f t} dt$$

Let, $(t-t_0) = \tau$

$t = t_0 + \tau$

$dt = d\tau$

$$\therefore F[x(t-t_0)] = \int_{-\infty}^{\infty} x(\tau) . e^{-j2\pi f (t_0+\tau)} = e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau$$

$$= e^{-j2\pi f t_0} X(f)$$

Proved

4. Frequency Shifting :⇒

The frequency shifting property states that if $x(t) \xleftrightarrow{F.T.} X(f)$, then

$$e^{j2\pi f_0 t} . x(t) \xleftrightarrow{F.T.} X(f-f_0) \quad ; \ f_0 = \text{real constant.}$$

Proof.

$$F[e^{j2\pi f_0 t} . x(t)] = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} . x(t) . e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f-f_0) t} dt$$

$$= X(f-f_0)$$

5. Differentiation in Time Domain :⇒

This property is applicable if and only if the derivative of $x(t)$ is fourier transformable.

If $x(t) \xleftrightarrow{F.T.} X(f)$, then

$$\frac{d}{dt}x(t) \xleftrightarrow{F.T.} j2\pi f X(f)$$

Proof

By the definition of Inverse fourier transform,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$\frac{d}{dt}x(t) = \int_{-\infty}^{\infty} X(f)\left(\frac{d}{dt} e^{j2\pi f t}\right) df$$

$$= \int_{-\infty}^{\infty} [X(f).\, j2\pi f]\, e^{j2\pi f t}\, df$$

the term inside the bracket must be the F.T. of $\frac{d}{dt}x(t)$

$$\therefore F\left[\frac{d}{dt}x(t)\right] = j2\pi f\, X(f)$$

Proved

6. Integration in Time Domain :⇒

if $x(t) \xleftrightarrow{F.T.} X(f)$ and provided that $X(0) = 0$, then

$$\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.T.} \frac{1}{j2\pi f} X(f) \quad ; \quad X(0) = 0 \text{ (provided)}$$

and,

$$\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.T.} \frac{1}{j2\pi f} X(f) + \pi X(0)\,\delta(\omega) \quad ; \quad X(0) \neq 0 \text{ (provided)}$$

Proof. In order to avoid confusion let us use different variable than 't', therefore let us express $x(t)$ as,

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right]$$

Taking fourier transform of both the sides

$$X(f) = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\lambda)\,d\lambda\right\}\right]$$

using differentiation property

$$X(f) = j2\pi f\, F\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right]$$

OR

$$\frac{X(f)}{j2\pi f} = F\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right]$$

Let us consider

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Such that

$$\frac{dy(t)}{dt} = x(t)$$

we can write

$$F[x(t)] = F\left[\frac{d}{dt}y(t)\right] = X(j\omega)$$

ie $j\omega Y(j\omega) = X(j\omega)$

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega)$$

or,

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega} X(j\omega)$$

* the integration suppresses the high frequency components.

7. Convolution in the Time Domain:⇒

If $x_1(t) \xleftrightarrow{F.T.} X_1(f)$ and $x_2(t) \xleftrightarrow{F.T.} X_2(f)$, then

$$[x_1(t) * x_2(t)] \xleftrightarrow{F.T.} X_1(f).X_2(f)$$

Proof. The convolution of two signal in time domain is defined as.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda).x_2(t-\lambda)d\lambda$$

Taking F.T. of the convolution

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\lambda).x_2(t-\lambda)d\lambda\right]e^{-j2\pi f t}.dt$$

Multiply and divide by $e^{-j2\pi f\lambda}$ in R.H.S.

$$R.H.S. = \int_{-\infty}^{\infty} x_1(\lambda).e^{-j2\pi f\lambda}d\lambda.\int_{-\infty}^{\infty} x_2(t-\lambda)e^{-j2\pi f t}.e^{j2\pi f\lambda}dt$$

$$= \int_{-\infty}^{\infty} x_1(\lambda).e^{-j2\pi f\lambda}d\lambda.\int_{-\infty}^{\infty} x_2(t-\lambda).e^{-j2\pi f(t-\lambda)}dt$$

Let $t-\lambda = m$

$$= \int_{-\infty}^{\infty} x_1(\lambda)e^{-j2\pi f\lambda}d\lambda.\int_{-\infty}^{\infty} x_2(t-\lambda)e^{-j2\pi f m}dm$$

$$= X_1(f).X_2(f)$$

$$= L.H.S.$$

Proved

8. Conjugate Functions:⇒

if $x(t) \xleftrightarrow{F.T.} X(f)$ and if $x(t)$ is a complex valued function then,

$$x^*(t) \xleftrightarrow{F.T.} X^*(-f)$$

Proof. as per I.F.T.

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$

Now take complex conjugate of both the side we get

$$x^*(t) = \left[\int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df\right]^*$$

$$\therefore\ x^*(t) = \int_{-\infty}^{\infty} X^*(f)e^{-j2\pi f t}df$$

Now replace f by -f

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)e^{j\pi f t}df$$

OR $x^*(t) = F^{-1}[X^*(-f)]$; OR $x^*(t) \xleftrightarrow{F.T.} X^*(-f)$

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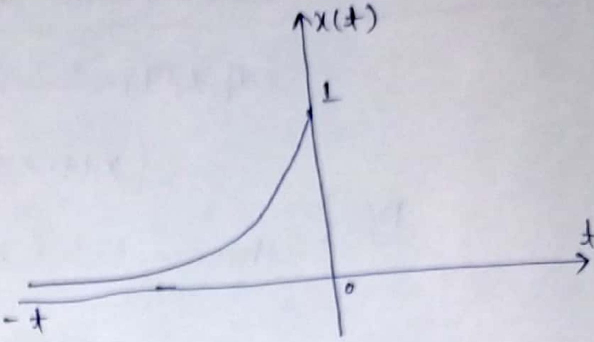
* Growing exponential Function:⇒

The exponential function can be represented as,



$$x(t) = e^{\alpha t} \quad \text{for } t \leq 0$$
$$= 0 \quad \text{for } t > 0$$

It can also expressed as,

$$x(t) = e^{\alpha t} u(-t)$$

It's fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} e^{\alpha t} u(-t) \cdot e^{-j2\pi f t}\, dt = \int_{-\infty}^{0} e^{\alpha(t)} \cdot e^{-j2\pi f t}\, dt = \int_{-\infty}^{0} e^{(\alpha - j2\pi f)t} \cdot dt$$

$$= \frac{1}{(\alpha - j2\pi f)} \left[e^{(\alpha - j2\pi f)t} \right]_{-\infty}^{0} = \frac{1}{(\alpha - j2\pi f)} \left[e^{0} - e^{-\infty} \right] = \frac{1}{(\alpha - j2\pi f)} [1 - 0]$$

$$= \frac{1}{(\alpha - j2\pi f)}$$

hence,

$$\boxed{e^{\alpha(t)} u(-t) \xleftrightarrow{F.T.} \frac{1}{(\alpha - j2\pi f)}}$$