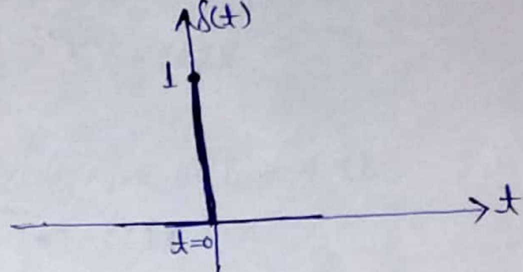
Fourier Transform of Some Common Signals:⇒/Functions:⇒

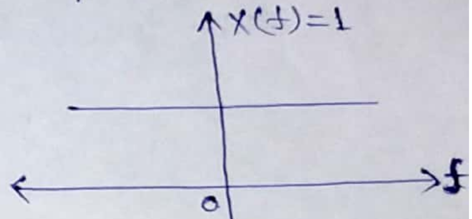
1. Delta Functions:⇒

It is defined as,



$$\delta(t) = 1 \quad \text{for } t=0$$
$$= 0 \quad \text{otherwise}$$

By the definition of fourier transform



(Amplitude Spectrum)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} . dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t} . dt$$

here, $x(t) = \delta(t)$ and value is 1 only at $t=0$ it is not necessary to take integration, Therefore

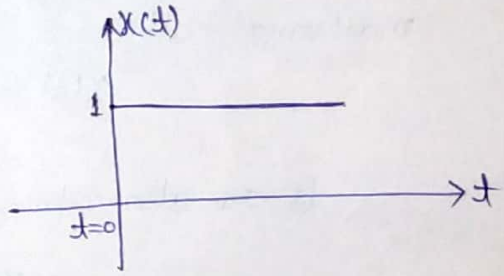
$$X(f) = 1.e^{0} = 1$$

$$\therefore \delta(t) \xleftrightarrow{F.T.} 1$$

But Delta function contains all frequencies from $-\infty$ to ∞ with equal amplitudes. The F.T. of $\delta(t)$ is a dc signal

2. Unit Step Functions:⇒

It is defined as,



$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{elsewhere}$$

Using the definition of F.T., we get

$$F[u(t)] = \int_{-\infty}^{\infty} u(t).e^{-j2\pi f t} .dt$$

here, unit step function is present only for $t \geq 0$

$$\therefore F[u(t)] = \int_{0}^{\infty} 1.\, e^{-j2\pi f t}\, dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi f}\right]_{0}^{\infty}$$

$$= \frac{1}{-j2\pi f}\left[e^{-\infty} - e^{0}\right] = \frac{1}{-j2\pi f}[0-1]$$

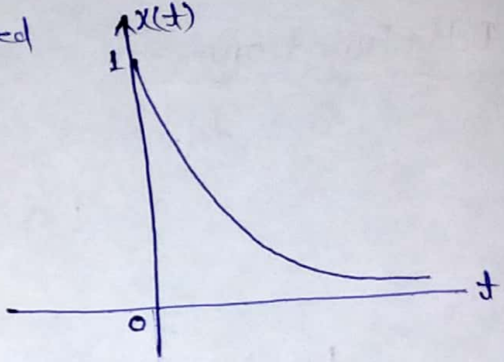
$$= \frac{1}{j2\pi f}$$

$$\therefore u(t) \xleftrightarrow{F.T.} \frac{1}{j2\pi f}$$

3. Decaying exponential Function :⇒

The exponential function can be represented mathematically,

$$x(t) = e^{-\alpha t} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



It can also represented as,

$$x(t) = e^{-\alpha t} u(t)$$

The fourier transform of x(t) is

$$F[x(t)] = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-j2\pi f t} dt = \int_0^{\infty} e^{-\alpha t} . e^{-j2\pi f t} . dt \quad \{u(t) = 1 \text{ for } t \geq 0$$

$$= \int_0^{\infty} e^{-(\alpha + j2\pi f) t} . dt = -\frac{1}{(\alpha + j2\pi f)} \left[e^{-(\alpha + 2\pi j f) t} \right]_0^{\infty}$$

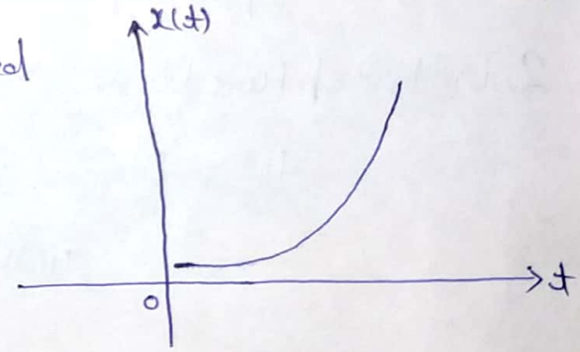
$$= \frac{-1}{(\alpha + j2\pi f)} [e^{-\infty} - e^{0}] = \frac{-1}{(\alpha + j2\pi f)} [0 - 1] = \frac{1}{\alpha + j2\pi f}$$

$$\boxed{e^{-\alpha t} u(t) \xleftrightarrow{\text{F.T.}} \frac{1}{(\alpha + j2\pi f)}} \qquad \boxed{e^{-\alpha t} u(-t) \longleftrightarrow \frac{-1}{(\alpha + j2\pi f)}}$$

4. Growing exponential Function :⇒

The exponential function can be represented mathematically,

$$x(t) = e^{\alpha t} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



It can also represented as,

$$x(t) = e^{\alpha t} u(t)$$

The forier Transform of x(t) is

$$F[x(t)] = \int_{-\infty}^{\infty} e^{\alpha t} . u(t) . e^{-j2\pi f t} . dt = \int_0^{\infty} e^{\alpha t} . e^{-j2\pi f t} . dt \quad \{u(t) = 1 \text{ for } t \geq 0$$

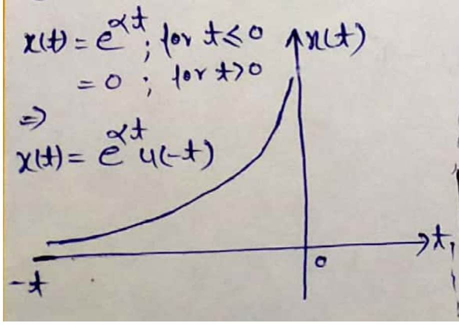
$$= \int_0^{\infty} e^{-(j2\pi f - \alpha) t} . dt = \frac{-1}{(j2\pi f - \alpha)} \left[e^{-(j2\pi f - \alpha)} \right]_0^{\infty}$$

$$= \frac{-1}{(j2\pi f - \alpha)} [e^{-\infty} - e^{0}] = \frac{-1}{(j2\pi f - \alpha)} [0 - 1] = \frac{1}{(j2\pi f - \alpha)}$$

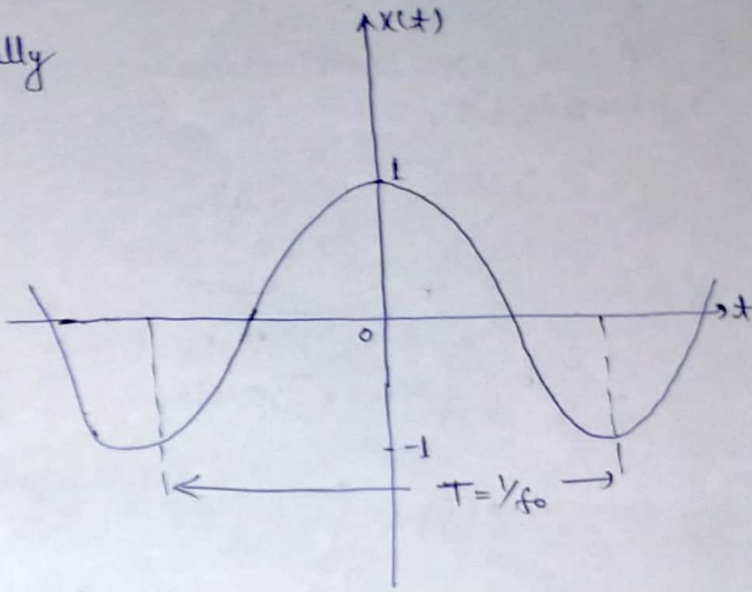
$$\boxed{e^{\alpha t} u(t) \xleftrightarrow{\text{F.T.}} \frac{1}{(j2\pi f - \alpha)}}$$

$$\boxed{e^{\alpha t} u(-t) \xleftrightarrow{\text{F.T.}} \frac{1}{\alpha - j2\pi f}}$$

$x(t) = e^{\alpha t}$; for $t \leq 0$
$= 0$; for $t > 0$
⇒ $x(t) = e^{\alpha t} u(-t)$



5. Cosine Function :⇒



A cosine wave can be mathematically represented as,

$$x(t) = A\cos(2\pi f_0 t)$$

But, A = 1

$$\therefore\ x(t) = \cos(2\pi f_0 t)$$

$$= \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

Taking fourier Transform

$$X(f) = \int_{-\infty}^{\infty}\left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right]\cdot e^{-j2\pi f t}\cdot dt$$

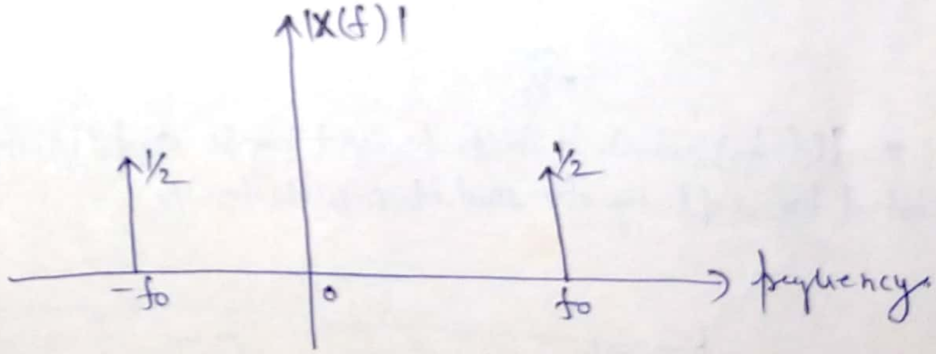
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[e^{-j2\pi(f-f_0)t} + e^{-j2\pi(f+f_0)t}\right]dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\cdot dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)}\cdot dt$$

But, $\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} = \delta(f-f_0)$

$$X(f) = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

The frequency spectrum shows that two impluses are present one at fo and the other at -fo

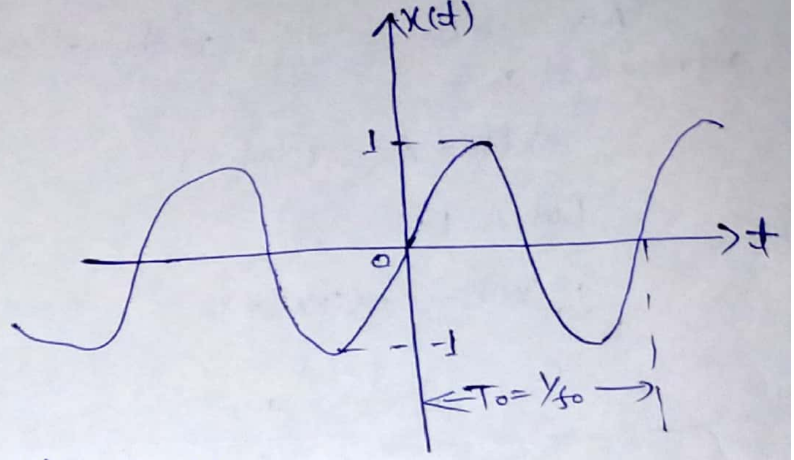


$$\boxed{\begin{aligned} &[\cos(2\pi f_0 t)] \xleftrightarrow{\text{F.T.}} \frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right] \\ &x(t)\cos(2\pi f_0 t) \xleftrightarrow{\text{F.T.}} \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right] \end{aligned}}$$

$$e^{-\alpha t}\cos(\underbrace{2\pi f_0 t}_{\omega_0 t})\,u(t) \xleftrightarrow{\text{F.T.}} \frac{\alpha + j2\pi f}{(\alpha+j2\pi f)^2 + (2\pi f_0)^2} = \frac{\alpha + j\omega}{(\alpha+j\omega)^2 + \omega_0^2}$$

6. Sinusoidal Function:⇒

A sine wave mathmetically represented as



$$x(t) = \sin(2\pi f_0 t)$$

$$= \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

taking fourier Transform

$$X(f) = \int_{-\infty}^{\infty} \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \cdot e^{-j2\pi f t} \cdot dt$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\, dt - \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} \cdot dt$$

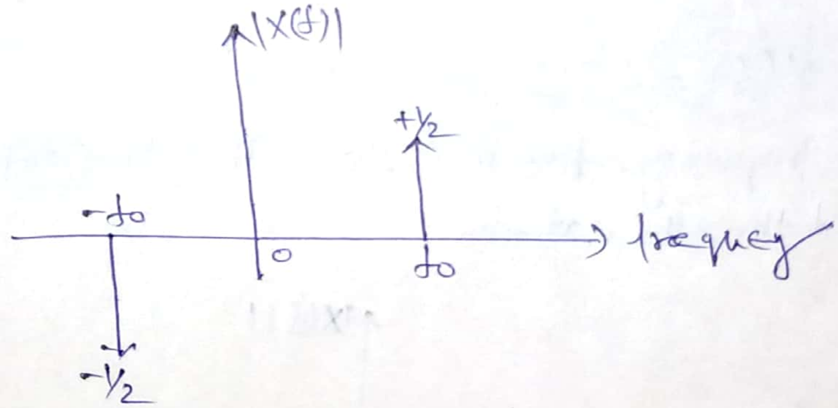
$$X(f) = \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

The amplitude spectrum of $\sin(2\pi f_0 t)$ is represented as

$$|X(f)| = \frac{1}{2}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

$\Rightarrow x(t)\sin 2\pi f_0 t \longleftrightarrow \frac{1}{2}\left[X(f-f_0) - X(f+f_0)\right]$

Therefore spectrum



* $\delta(f-f_0)$, which is shifted right by f_0 and its amplitude $+1/2$ and $\delta(f+f_0)$ shifted left by f_0 and its amplitude $-1/2$

$$\boxed{e^{-\alpha t}\sin\omega_0 t\, u(t) \xleftrightarrow{F.T.} \frac{\omega_0}{(\alpha + j\omega_0)^2 + \omega_0^2}}$$