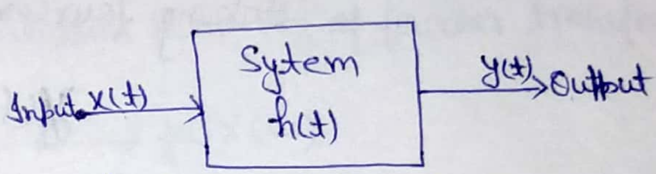
Application of Fourier Series and Fourier Transform to the System Analysis.

Frequency Response of an LTI System :⇒

Consider an LTI system having impulse response $h(t)$ driven an input $x(t)$ such that,

$$x(t) = e^{j2\pi f t}$$



The relation between $x(t)$, $h(t)$, and $y(t)$ is as fallows,

$$y(t) = x(t) * h(t)$$

i.e. $y(t)$ is obtained by taking convolution of $x(t)$ and $h(t)$

Now, the convolution of the two signals is defined as

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau).x(t-\tau)\,d\tau$$

Therefore,

$$y(t) = \int_{-\infty}^{\infty} h(\tau).e^{j2\pi f(t-\tau)}\,d\tau$$

$$\text{or,} \quad y(t) = e^{j2\pi f t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

Let us define,

$$H(f) = \int_{-\infty}^{\infty} h(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$\therefore \quad y(t) = H(f).e^{j2\pi f t} = \underline{H(f).x(t)}$$

$$\boxed{\text{OR} \quad H(f) = \frac{y(t)}{x(t)}}$$

This is called as transfer function of the system.

* that is the response of LTI system to a complex exponential function of frequency f, therefore the same complex expatential function multiplied by a constant cofficient $H(f)$, which is called as the transfer function of the system.

This is follows:- $H(f) = \int_{-\infty}^{\infty} h(t)\,e^{-j2\pi f t}\,dt$ and $h(t) = \int_{-\infty}^{\infty} H(f)\,e^{j2\pi f t}\,df$

hence,

The transfer function is defined alternatively as

$$H(f) = \left.\frac{y(t)}{x(t)}\right|_{x(t)=e^{j2\pi ft}}$$

in time domain,

$$y(t) = h(t) * x(t)$$

taking fourier transform of both side we get

$$Y(f) = H(f) \cdot X(f)$$

$$\text{or, } H(f) = \frac{Y(f)}{X(f)}$$

i.e. it is possible to describe LTI system in the frequency domain.

Amplitude and Phase Response of a system:-

The transfer function $H(f)$ is a complex quantity expressed as

$$H(f) = |H(f)| e^{j\theta(f)}$$

where,

$|H(f)|$ = Amplitude response

$\theta(f)$ = Phase response

The phase response is given by,

$$\theta(f) = \arg |H(f)|$$

Response of Differential Equations :⇒

Many physical processes can be easily modelled by n^{th} order constant coefficient differential equations, such as

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \quad \text{--- ①}$$

according to differentiation property of fourier transform

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T.} j\omega X(\omega)$$

Applying differentiation property in eq ①, we get

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

we know that $\frac{Y(\omega)}{X(\omega)}$ gives system transfer function $H(\omega)$

therefore,

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

This eq. is used to calculate the frequency response and impulse response of system. It is also a ratio of polynomials in $(j\omega)$.

Q.1) Calculate system transfer function and impulse response of the system of LTI system, which is initially at rest is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = \frac{d}{dt}x(t) + 3x(t)$$

Sol:

Taking fourier transform of both sides (applying differentiation pro.)

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = (j\omega)X(\omega) + 3X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 3(j\omega) + 2\right] = X(\omega)\left[3 + (j\omega)\right]$$

$$\therefore \frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{3 + (j\omega)}{(j\omega)^2 + 3(j\omega) + 2}$$

First we obtain roots of denominator

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$\text{roots} = \frac{-3+1}{2}, \frac{-3-1}{2} = -1, -2$$

$$\therefore H(\omega) = \frac{3 + (j\omega)}{(j\omega+1)(j\omega+2)}$$

In the partial fraction expansion we can write,

$$H(\omega) = \frac{A}{j\omega+1} + \frac{B}{j\omega+2}$$

Calculate A and B

$$A = (j\omega+1)H(\omega)\Big|_{j\omega=-1}$$

$$= \frac{(j\omega+1).(3+j\omega)}{(j\omega+1)(j\omega+2)}\Bigg|_{j\omega=-1} = \frac{3-1}{-1+2} = 2$$

and

$$B = \frac{(j\omega+2)(3+j\omega)}{(j\omega+1)(j\omega+2)}\Bigg|_{j\omega=-2} = \frac{3-2}{-2+1} = -1$$

$$\therefore H(\omega) = \frac{2}{j\omega+1} - \frac{1}{j\omega+2}$$

This is the system transfer function.

Now, the impulse response of system, take IFT of $H(\omega)$

$$\therefore h(t) = F^{-1}\left\{\frac{2}{j\omega+1}\right\} - F^{-1}\left\{\frac{1}{j\omega+2}\right\}$$

$$e^{-at}u(t) \xleftrightarrow{F.T.} \frac{1}{a+j\omega}$$

$$h(t) = 2e^{-t}u(t) - e^{-2t}u(t)$$

Q.2) Calculate output $y(t)$ if input $x(t) = e^{-3t}u(t)$ is applied to the system which is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

First we calculate $H(\omega)$. Taking F.T.

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 3j\omega + 2\right] = X(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

Now,

$$x(t) = e^{-3t}u(t)$$

$$\text{OR } X(\omega) = \frac{1}{3 + j\omega}$$

But,

$$Y(\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2} \cdot X(\omega)$$

$$= \frac{1}{(j\omega+1)(j\omega+2)} \cdot \frac{1}{(3+j\omega)}$$

$$= \frac{1}{(j\omega+1)(j\omega+2)(j\omega+3)}$$

In partial ~~expan~~ fraction expansion

$$Y(\omega) = \frac{A}{(j\omega+1)} + \frac{B}{(j\omega+2)} + \frac{C}{(j\omega+3)}$$

But, $A = (j\omega+1) \cdot \left.\frac{1}{(j\omega+1)(j\omega+2)(j\omega+3)}\right|_{j\omega=-1}$

$$= \frac{1}{2}$$

$$B = -1$$

$$C = \frac{1}{2}$$

$$\therefore Y(\omega) = \frac{1}{2}\frac{1}{(j\omega+1)} - \frac{1}{(j\omega+2)} + \frac{1}{2}\frac{1}{(j\omega+3)}$$

Taking I.F.T.

$$y(t) = \frac{1}{2}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t)$$