Approximations and Errors

These two are an integral part of human life. They are everywhere and unavoidable.

Error = true value – approximate value

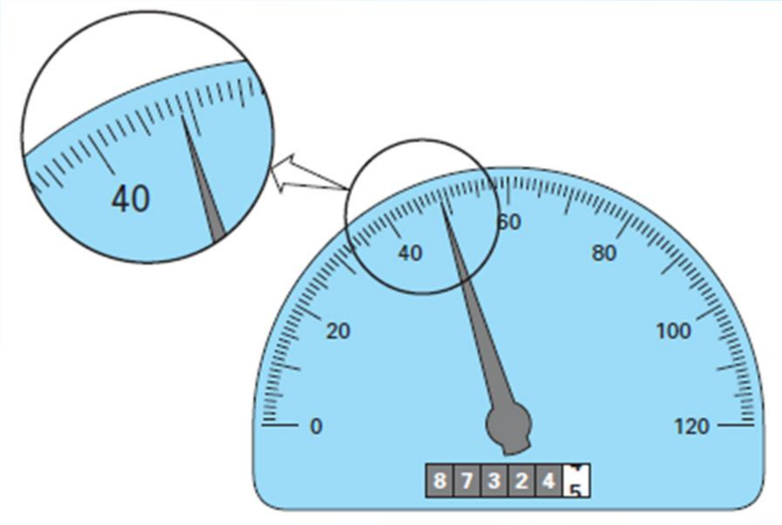
Understanding the concept of error is so important to the effective use of numerical methods because every numerical method involves an approximation.

For many applied engineering problems, we cannot obtain analytical solutions. Therefore, we cannot compute exactly the errors associated with our numerical methods. In these cases, we must settle for approximations or estimates of the errors.

For example:

A speedometer from an automobile indicates that the car is traveling between 48 and 49 km/h. One person might say 48.8, whereas another might say 48.9 km/h. Therefore, because of the limits of this instrument, only the first two digits can be used with confidence.

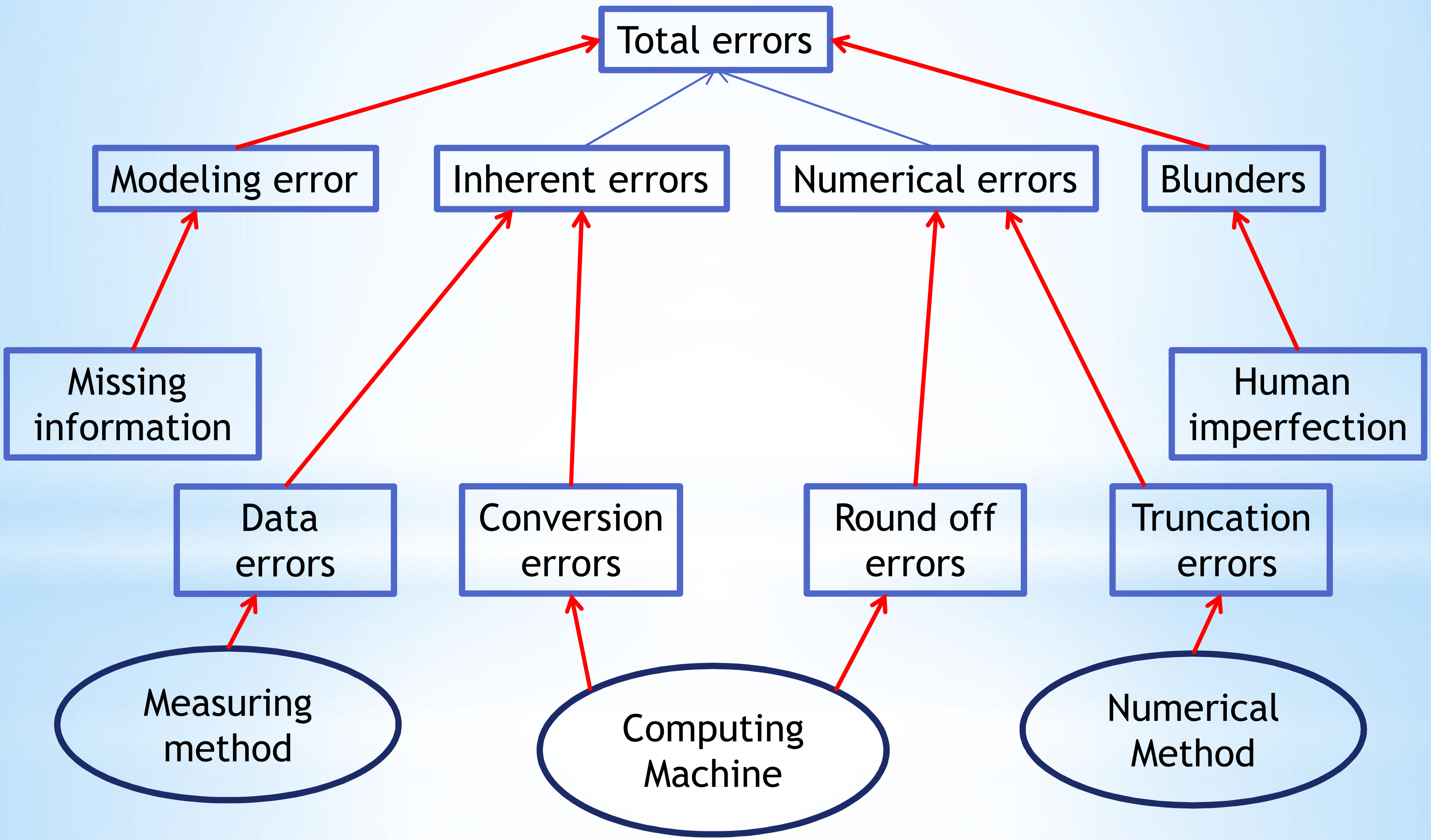
Estimates of the third digit (or higher) must be viewed as approximations.



Error came in variety of forms and sizes; some are avoidable some are not.
For example:
Data conversion and round off error can not be avoided but human error can be eliminated.

Although certain errors can not be eliminated completely, we must at least know the bound of these error to make use of our final solution.

It is therefore essential to know how errors arises, how they grow during the numerical process and how they affect the accuracy of a solution.



Taxonomy of error

Data error: When data for a problem are obtained by some experimental means and have a limited accuracy and precision. This is due to some limitation of instruments and reading.

Conversion error: Arises due to limitation of the computer to store the data exactly.

Round off error: When a fixed no. of digit are used to represent exact no..

For example,

$$\pi = 3.14159265358979323846264 3\ldots$$

Truncation error: Arises from using an approximation in place of an exact mathematical procedure.

$$e^x = \sum_{i=0}^{n} \frac{x^n}{n!}$$

Inherent error: Error that are present in the data supplied to the model.

Numerical error: This error introduced during the process of implementation of numerical method.

Modeling Error

Mathematical models are the basis for numerical solution.

- Model formulated to represent physical process using certain parameter involved in the situation.
- In many situation, it is impractical or impossible to include all the real problem and therefore, certain simplifying assumption are made.

For Example:

Developing a model for calculating the force acting on a falling body.

- We may not be able to estimate the air resistance coefficient (drag coefficient) properly or determine the direction and magnitude of wind force acting on the body and so on.

To simplify the model, we may assume that resistance is linearly proportional to the velocity of the falling body or we may assume that there is no wind force acting on the body.

Blunders

Error that are caused due to human imperfection. Such error may cause a very serious disaster in the result.

It should be possible to avoid them to a large extent by acquiring a sound knowledge of all aspect of the problem as well as numerical process.

Human error can occur at any stage of the numerical processing cycle . Some common types of errors are:-

- Lack of understanding of the problem
- Wrong assumption
- Overlooking of some basic assumption required for formulating the model
- Selecting a wrong numerical method for solving the mathematical model
- Wrong guessing of initial value

Numerical Errors (also known as procedural errors)

Introduced during the process of implementation of a numerical method. They came in two form

Round off Truncation errors

Total numerical error is the summation of these two errors. The total error can be reduced by devising suitable techniques for implementing the solution

Round off Errors

When fixed no. of digit are used to represent exact no..

Since no. are stored at every stage of computation, round off error is introduced at the end of every arithmetic operation.

Even though an individual round off error could be very small the cumulative effect of a series of computation can be very significant.

Rounding a number can be done in two ways

Chopping Symmetric rounding

Chopping:-

Extra digits are chopped. This is called truncating the number

Example:

42.7893 will be stored as 42.78 and the digit 93 will be dropped.

Express the no. 42.7893 in floating point form as

$x = 0.427893 \times 10^2$

$= (0.4278 + 0.000093) \times 10^2$

$= (0.4278 + 0.93 \times 10^{-4}) \times 10^2$

This can be expressed in general form as

$\text{True } x = (f_x + g_x \times 10^{-d}) \times 10^E$

$= f_x \times 10^E \quad + \quad g_x \times 10^{E-d}$

$=$ approximate x + error

f_x is the mantissa
d is the length of mantissa
E is the exponent
$0 \leq g_x \leq 1$

Absolute error introduced depend on the following

- Size of the digit dropped
- Number of digits in mantissa
- Size of the number

Symmetric Rounding

Last retained significant digit is rounded up by 1 if the first discarded digit is larger or equal to 5, otherwise the last retained digit is unchanged.

Example :

42.7893 become 42.79

46.5432 become 46.54

Value of unrounded no. can be expressed as

True $x = f_x \times 10^E + g_x \times 10^{E-d}$

when $g_x \leq 0.5$ entire g_x is truncated

Approximate $x = f_x \times 10^E$

Error = $g_x \times 10^{E-d}$ $g_x < 0.5$

When $g_x \geq 0.5$ last digit in the mantissa is increased by 1 and therefore

Approximate $x = f_x \times 10^E + 10^{E-d}$

Error $= (f_x \times 10^E + g_x \times 10^{E-d}) - f_x \times 10^E + 10^{E-d}$

$= (g_x - 1) \times 10^{E-d}$

So absolute error $\leq 0.5 \times 10^{E-d}$

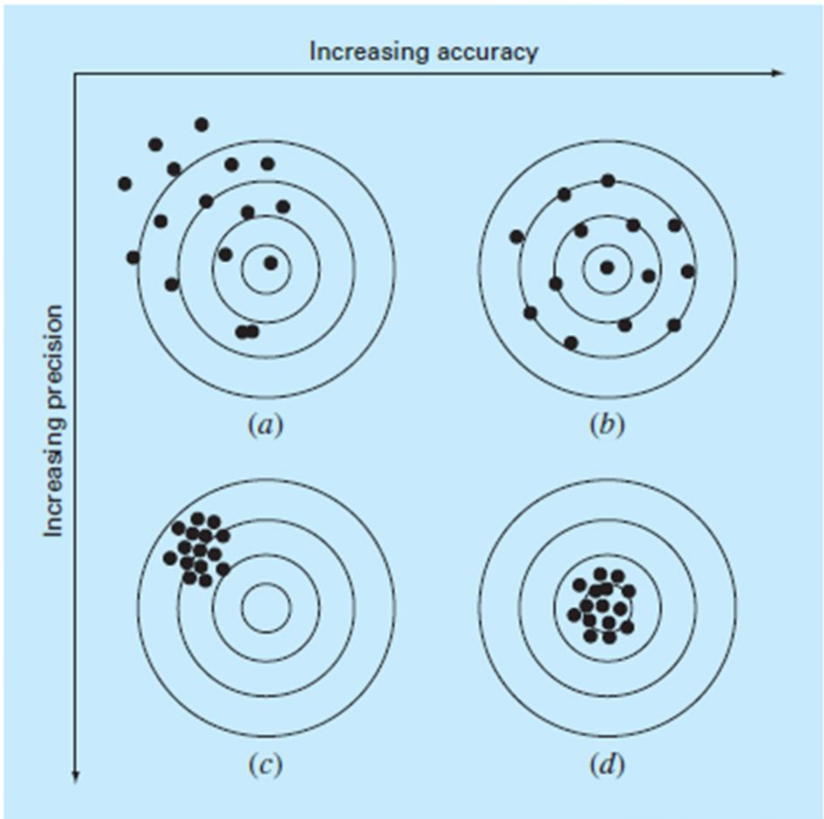
Symmetric rounding error is, at worst, one half the chopping error.

Accuracy and Precision

The errors associated with both calculations and measurements can be characterized with respect to their accuracy and precision.

Accuracy refers to how closely a computed or measured value agrees with the true value.

Precision refers to how closely individual computed or measured values agree with each other.



Error

The numerical error is equal to the discrepancy between the truth and the approximation.

True value = approximation + error

Absolute error Relative error

true error e_t = true value (x_t) – approximation (x_a)

Value of error may be + or – depends on value of x_t and x_a.

Most of the time magnitude of error is important not the sign. So mode vale of error is known as *absolute error*.

$$e_t = |x_t - x_a|$$

In many cases this may not reflect its influence correctly as it does not take account the order of magnitude of the value under study.

For example: 1 gm is much more significant in the weight of 10 gm gold chain than in the weight of a bag of rice.

One way to account these magnitudes of the quantities is to normalize the error to the true value, which is known as relative or normalized error.

$$\text{Relative or normalized error } e_r = \frac{\text{absolute error}}{\text{true value}}$$

$$e_r = \frac{|x_t - x_a|}{|x_t|}$$

The relative error can also be multiplied by 100 percent to express it as

$$e_r = \frac{|x_t - x_a|}{|x_t|} 100\%$$

It is often convenient to simplify the Taylor series by defining a step size $h = x_{i+1} - x_i$ and expressing as

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{n}(x_i)}{n!}h^n + R_n$$

where the remainder term is now

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

The derivative of velocity of a falling parachutist is approximated by a finite-divided-difference equation.

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \cdots + R_n$$

Now let us truncate the series after the first derivative term:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

Equation can be solved for

$$v'(t_i) = \underbrace{\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}}_{\text{First-order approximation}} - \underbrace{\frac{R_1}{t_{i+1} - t_i}}_{\text{Truncation error}}$$

The truncation error associated with this approximation of the derivative.

$$\frac{R_1}{t_{i+1} - t_i} = \frac{v''(\xi)}{2!}(t_{i+1} - t_i)$$

or

$$\frac{R_1}{t_{i+1} - t_i} = O(t_{i+1} - t_i)$$

Thus, the estimate of the derivative has a truncation error of order (t_{i+1} $-ti$). In other words, the error of our derivative approximation should be proportional to the step size.

Error Propagation

How errors in numbers can propagate through mathematical functions.

For example: If we multiply two numbers that have errors, we would like to estimate the error in the product.

- **Functions of a Single Variable**

Suppose that we have a function f(x) that is dependent on a single independent variable x. Assume that $\bar{x}$ is an approximation of x. We, therefore, would like to assess the effect of the discrepancy between x and $\bar{x}$ on the value of the function. We would like to estimate

$$\Delta f(\bar{x}) = f(x) - f(\bar{x})$$

The problem with evaluating $\Delta f(\bar{x})$ is that $f(x)$ is unknown because x is unknown. We can overcome this difficulty if $\bar{x}$ is close to x and $f(\bar{x})$ is continuous and differentiable. If these conditions hold, a Taylor series can be employed to compute $f(x)$ near $f(\bar{x})$, as in

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\bar{x})}{2}(x - \bar{x})^2 + \cdots$$

Dropping the second- and higher-order terms and rearranging yields

$$f(x) - f(\bar{x}) \cong f'(\bar{x})(x - \bar{x})$$

Or

$$\Delta f(\bar{x}) = |f'(\bar{x})|\Delta \bar{x}$$

where $\Delta f(\bar{x}) = |f(x) - f(\bar{x})|$ represents an estimate of the error of the function and $\Delta(\bar{x}) = |(x) - (\bar{x})|$ represents an estimate of the error of *x*.

- **Functions of More than One Variable**

$$f(u_{i+1}, v_{i+1}) = f(u_i, v_i) + \frac{\partial f}{\partial u}(u_{i+1} - u_i) + \frac{\partial f}{\partial v}(v_{i+1} - v_i)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial u^2}(u_{i+1} - u_i)^2 + 2\frac{\partial^2 f}{\partial u \partial v}(u_{i+1} - u_i)(v_{i+1} - v_i) + \frac{\partial^2 f}{\partial v^2}(v_{i+1} - v_i)^2\right] + \cdots$$