THE TIME VALUE OF MONEY

The Magic of Compounding

WHY TIME VALUE

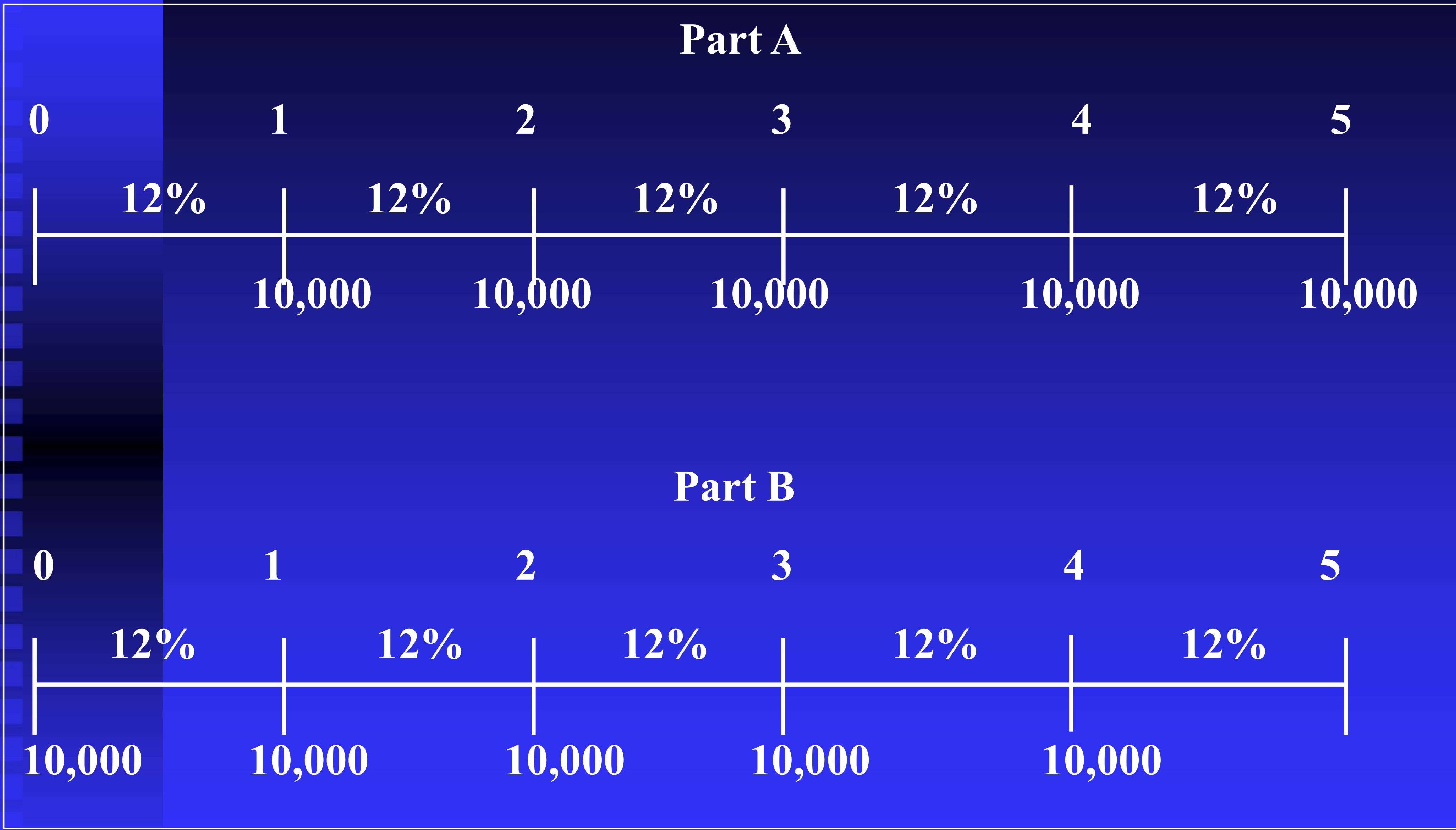
A rupee today is more valuable than a rupee a year hence.

Why ?

- Preference for current consumption over future consumption
- Productivity of capital
- Inflation

Many financial problems involve cash flows occurring at different points of time. For evaluating such cash flows, an explicit consideration of time value of money is required

TIME LINE



Part A
0 1 2 3 4 5
12% 12% 12% 12% 12%
10,000 10,000 10,000 10,000 10,000
Part B
0 1 2 3 4 5
12% 12% 12% 12% 12%
10,000 10,000 10,000 10,000 10,000

NOTATION

PV : Present value

FV_n : Future value *n* years hence

C_t : Cash flow occurring at the end of year *t*

A : A stream of constant periodic cash flow over a given time

r : Interest rate or discount rate

g : Expected growth rate in cash flows

n : Number of periods over which the cash flows occur.

FUTURE VALUE OF A SINGLE AMOUNT

		Rs
First year:	Principal at the beginning	1,000
	Interest for the year	
	(Rs.1,000 x 0.10)	100
	Principal at the end	1,100
Second year:	Principal at the beginning	1,100
	Interest for the year	
	(Rs.1,100 x 0.10)	110
	Principal at the end	1,210
Third year:	Principal at the beginning	1,210
	Interest for the year	
	(Rs.1,210 x 0.10)	121
	Principal at the end	1,331

FORMULA

FUTURE VALUE = PRESENT VALUE $(1+r)^n$

VALUE OF $FV_{r,n}$ FOR VARIOUS COMBINATIONS OF *r* AND *n*

n/*r*	6 %	8 %	10 %	12 %	14 %
2	1.124	1.166	1.210	1.254	1.300
4	1.262	1.361	1.464	1.574	1.689
6	1.419	1.587	1.772	1.974	2.195
8	1.594	1.851	2.144	2.476	2.853
10	1.791	2.518	2.594	3.106	3.707

DOUBLING PERIOD

Thumb Rule : Rule of 72

$$\text{Doubling period} = \frac{72}{\text{Interest rate}}$$

Interest rate : 15 percent

$$\text{Doubling period} = \frac{72}{15} = 4.8 \text{ years}$$

A more accurate thumb rule : Rule of 69

$$\text{Doubling period} = 0.35 + \frac{69}{\text{Interest rate}}$$

Interest rate : 15 percent

$$\text{Doubling period} = 0.35 + \frac{69}{15} = 4.95 \text{ years}$$

PRESENT VALUE OF A SINGLE AMOUNT

$$PV = FV_n \, [1/\,(1 + r)^n]$$

n/r	*6%*	*8%*	*10%*	*12%*	*14%*
2	0.890	0.857	0.826	0.797	0.770
4	0.792	0.735	0.683	0.636	0.592
6	0.705	0.630	0.565	0.507	0.456
8	0.626	0.540	0.467	0.404	0.351
10	0.558	0.463	0.386	0.322	0.270
12	0.497	0.397	0.319	0.257	0.208

PRESENT VALUE OF AN UNEVEN SERIES

$$PV_n = \frac{A_1}{(1+r)} + \frac{A_2}{(1+r)^2} + \ldots\ldots + \frac{A_n}{(1+r)^n}$$

$$= \sum_{t=1}^{n} \frac{A_t}{(1+r)^t}$$

Year	*Cash Flow Rs.*	*PVIF* $_{12\%,n}$	*Present Value of Individual Cash Flow*
1	1,000	0.893	893
2	2,000	0.797	1,594
3	2,000	0.712	1,424
4	3,000	0.636	1,908
5	3,000	0.567	1,701
6	4,000	0.507	2,028
7	4,000	0.452	1,808
8	5,000	0.404	2,020
Present Value of the Cash Flow Stream			13,376

FUTURE VALUE OF AN ANNUITY

- An annuity is a series of periodic cash flows (payments and receipts) of equal amounts

1	2	3	4	5
1,000	1,000	1,000	1,000	1,000

+ 1,100
+ 1,210
+ 1,331
+ 1,464

Rs.6,105

- Future value of an annuity = $\frac{A\,[(1+r)^{n}-1}{r}$

WHAT LIES IN STORE FOR YOU

Suppose you have decided to deposit Rs.30,000 per year in your Public Provident Fund Account for 30 years. What will be the accumulated amount in your Public Provident Fund Account at the end of 30 years if the interest rate is 11 percent ?

The accumulated sum will be :

Rs.30,000 $(FVIFA_{11\%,30yrs})$

$$= \text{Rs.}30{,}000 \left(\frac{(1.11)^{30} - 1}{.11} \right)$$

= Rs.30,000 [199.02]

= Rs.5,970,600

HOW MUCH SHOULD YOU SAVE ANNUALLY

You want to buy a house after 5 years when it is expected to cost Rs.2 million. How much should you save annually if your savings earn a compound return of 12 percent ?

The future value interest factor for a 5 year annuity, given an interest rate of 12 percent, is :

$$FVIFA_{n=5,\ r=12\%} = \frac{(1+0.12)^5 - 1}{0.12} = 6.353$$

The annual savings should be :

$$\frac{Rs.2000,000}{6.353} = Rs.314,812$$

ANNUAL DEPOSIT IN A SINKING FUND

Futura Limited has an obligation to redeem Rs.500 million bonds 6 years hence. How much should the company deposit annually in a sinking fund account wherein it earns 14 percent interest to cumulate Rs.500 million in 6 years time ?

The future value interest factor for a 5 year annuity, given an interest rate of 14 percent is :

$$FVIFA_{n=6,\ r=14\%} = \frac{(1+0.14)^6 - 1}{0.14} = 8.536$$

The annual sinking fund deposit should be :

$$\frac{\text{Rs.500 million}}{8.536} = \text{Rs.58.575 million}$$

FINDING THE INTEREST RATE

A finance company advertises that it will pay a lump sum of Rs.8,000 at the end of 6 years to investors who deposit annually Rs.1,000 for 6 years. What interest rate is implicit in this offer?

The interest rate may be calculated in two steps :

1. Find the $FVIFA_{r,6}$ for this contract as follows :

$$Rs.8,000 = Rs.1,000 \times FVIFA_{r,6}$$

$$FVIFA_{r,6} = \frac{Rs.8,000}{Rs.1,000} = 8.000$$

2. Look at the $FVIFA_{r,n}$ table and read the row corresponding to 6 years until you find a value close to 8.000. Doing so, we find that

$FVIFA_{12\%,6}$ is 8.115So, we conclude that the interest rate is slightly below 12 percent.

HOW LONG SHOULD YOU WAIT

You want to take up a trip to the moon which costs Rs.1,000,000 the cost is expected to remain unchanged in nominal terms. You can save annually Rs.50,000 to fulfill your desire. How long will you have to wait if your savings earn an interest of 12 percent ? The future value of an annuity of Rs.50,000 that earns 12 percent is equated to Rs.1,000,000.

$$50{,}000 \times FVIFA_{n=?,12\%} = 1{,}000{,}000$$

$$50{,}000 \times \left[\frac{1.12^n - 1}{0.12}\right] = 1{,}000{,}000$$

$$1.12^n - 1 = \frac{1{,}000{,}000}{50{,}000} \times 0.12 = 2.4$$

$$1.12^n = 2.4 + 1 = 3.4$$

$$n \log 1.12 = \log 3.4$$

$$n \times 0.0492 = 0.5315$$

$$n = \frac{0.5315}{0.0492} = 10.8 \text{ years}$$

You will have to wait for about 11 years.

PRESENT VALUE OF AN ANNUITY

$$\text{Present value of an annuity} = A \left(\frac{1 - \frac{1}{(1+r)^n}}{r} \right)$$

Value of $PVIFA_{r,n}$ for Various Combinations of *r* and *n*

n/*r*	6 %	8 %	10 %	12 %	14 %
2	1.833	1.783	1.737	1.690	1.647
4	3.465	2.312	3.170	3.037	2.914
6	4.917	4.623	4.355	4.111	3.889
8	6.210	5.747	5.335	4.968	4.639
10	7.360	6.710	6.145	5.650	5.216
12	8.384	7.536	6.814	6.194	5.660

LOAN AMORTISATION SCHEDULE

Loan : 1,000,000 r = 15%, n = 5 years

$$1{,}000{,}000 = A \times PVIFA_{n=5},\ r = 15\%$$

$$= A \times 3.3522$$

$$A = 298{,}312$$

Year	*Beginning Amount*	*Annual Instalment*	*Interest*	*Principal Repayment*	*Remaining Balance*
	(1)	*(2)*	*(3)*	*(2)-(3) = (4)*	*(1)-(4) = (5)*
1	1,000,000	298,312	150,000	148,312	851,688
2	851,688	298,312	127,753	170,559	681,129
3	681,129	298,312	102,169	196,143	484,986
4	484,986	298,312	727,482	225,564	259,422
5	259,422	298,312	38,913	259,399	23*

a Interest is calculated by multiplying the beginning loan balance by the interest rate.

b. Principal repayment is equal to annual instalment minus interest.

* Due to rounding off error a small balance is shown

EQUATED MONTHLY INSTALMENT

Loan = 1,000,000, Interest = 1% p.m, Repayment period = 180 months

$$1{,}000{,}000 = \frac{A \text{ x } 1\text{-}1/(0.01)^{180}}{0.01}$$

A = Rs.12,002

PRESENT VALUE OF A GROWING ANNUITY

A cash flow that grows at a constant rate for a specified period of time is a growing annuity. The time line of a growing annuity is shown below:

0	1	2	3	n
	$A(1+g)$	$A(1+g)^2$		$A(1+g)^n$

The present value of a growing annuity can be determined using the following formula :

$$\text{PV of a Growing Annuity} = A\,(1+g)\left[\frac{1-\dfrac{(1+g)^n}{(1+r)^n}}{r-g}\right]$$

The above formula can be used when the growth rate is less than the discount rate ($g < r$) as well as when the growth rate is more than the discount rate ($g > r$). However, it does not work when the growth rate is equal to the discount rate ($g = r$) – in this case, the present value is simply equal to $n\,A$.

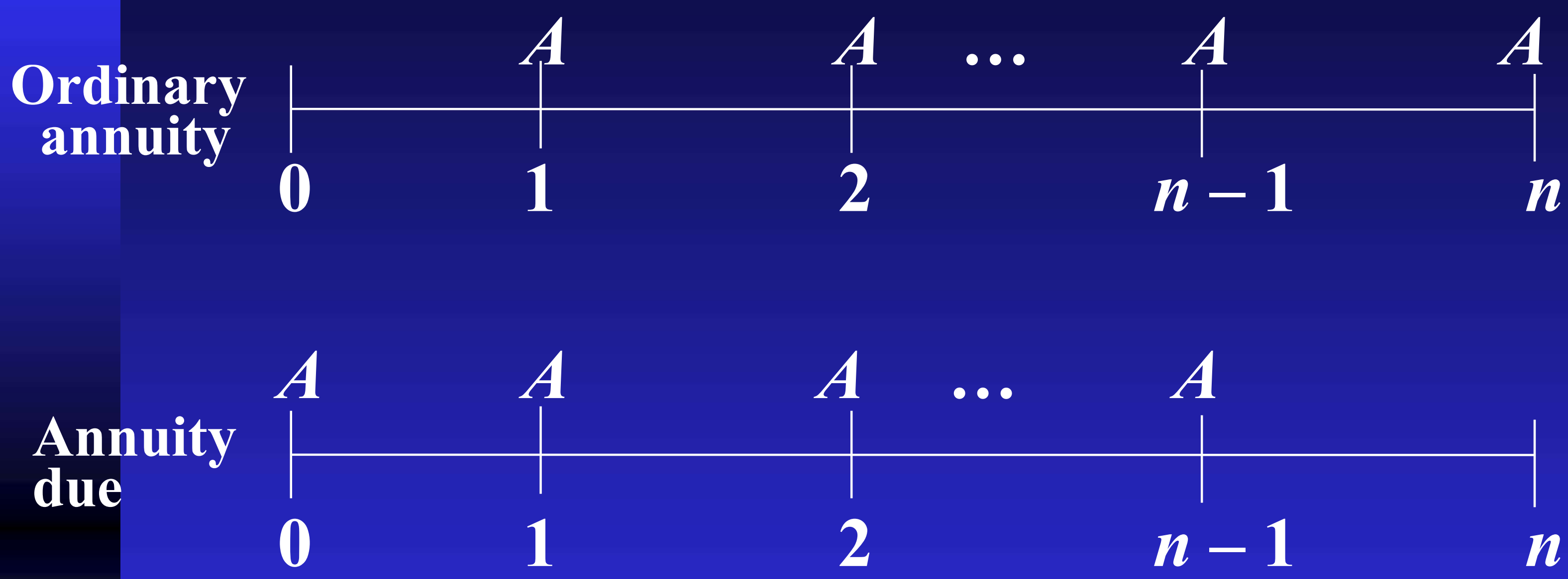
PRESENT VALUE OF A GROWING ANNUITY

For example, suppose you have the right to harvest a teak plantation for the next 20 years over which you expect to get 100,000 cubic feet of teak per year. The current price per cubic foot of teak is Rs 500, but it is expected to increase at a rate of 8 percent per year. The discount rate is 15 percent. The present value of the teak that you can harvest from the teak forest can be determined as follows:

$$\text{PV of teak} = \text{Rs } 500 \times 100{,}000\,(1.08)\left[\frac{1 - \dfrac{1.08^{20}}{1.15^{20}}}{0.15 - 0.08}\right]$$

$$= \text{Rs.}551{,}736{,}683$$

ANNUITY DUE



Thus,

Annuity due value = Ordinary annuity value $(1 + r)$

This applies to both present and future values

PRESENT VALUE OF PERPETUITY

$$\text{Present value of perpetuity} = \frac{A}{r}$$

SHORTER COMPOUNDING PERIOD

$$\text{Future value} = \text{Present value}\left(1+\frac{r}{m}\right)^{m\text{x}n}$$

Where r = nominal annual interest rate

m = number of times compounding is done in a year

n = number of years over which compounding is done

Example : Rs.5000, 12 percent, 4 times a year, 6 years

$$5000(1+0.12/4)^{4\text{x}6} = 5000\,(1.03)^{24}$$

$$= \text{Rs.}10{,}164$$

EFFECTIVE VERSUS NOMINAL RATE

$r = (1+k/m)^m - 1$

r = effective rate of interest

k = nominal rate of interest

m = frequency of compounding per year

Example : k = 8 percent, m=4

$r = (1+.08/4)^4 - 1 = 0.0824$

= 8.24 percent

Nominal and Effective Rates of Interest

		Effective Rate %		
Nominal Rate %	*Annual Compounding*	*Semi-annual Compounding*	*Quarterly Compounding*	*Monthly Compounding*
8	8.00	8.16	8.24	8.30
12	12.00	12.36	12.55	12.68

SUMMING UP

- Money has time value. A rupee today is more valuable than a rupee a year hence.
- The general formula for the future value of a single amount is :

$$\text{Future value} = \text{Present value } (1+r)^n$$

- The value of the compounding factor, $(1+r)^n$, depends on the interest rate (r) and the life of the investment (n).
- According to the rule of 72, the doubling period is obtained by dividing 72 by the interest rate.
- The general formula for the future value of a single cash amount when compounding is done more frequently than annually is:

$$\text{Future value} = \text{Present value } [1+r/m]^{m*n}$$

- An annuity is a series of periodic cash flows (payments and receipts) of equal amounts. The future value of an annuity is:

  Future value of an annuity

  $$= \text{Constant periodic flow } [(1+r)^n - 1)/r]$$

- The process of discounting, used for calculating the present value, is simply the inverse of compounding. The present value of a single amount is:

  $$\text{Present value} = \text{Future value x } 1/(1+r)^n$$

- The present value of an annuity is:

  Present value of an annuity

  $$= \text{Constant periodic flow } [1 - 1/ (1+r)^n] /r$$

- A perpetuity is an annuity of infinite duration. In general terms:

  $$\text{Present value of a perpetuity} = \text{Constant periodic flow } [1/r]$$