\# If the Centre of mass of three particle of masses 10, 20 & 30 gm be at the point (1,1,1) m then where should a fourth particle of 40 gm be placed so that the combined centre of mass be at (0,0,0)?

Sol.

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \quad \text{or} \quad 1 = \frac{0.01x_1 + 0.02x_2 + 0.03x_3}{0.01 + 0.02 + 0.03}$$

$$1 = \frac{x_1 + 2x_2 + 3x_3}{1+2+3}$$

or $x_1 + 2x_2 + 3x_3 = 6$ ---- (1)

Suppose 40 gm mass is placed at a point (x_4, y_4, z_4) so that the combined centre of mass is at (0,0,0) Now

$$0 = \frac{0.01x_1 + 0.02x_2 + 0.03x_3 + 0.04x_4}{0.01 + 0.02 + 0.03 + 0.04}$$

or $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ —— (2)

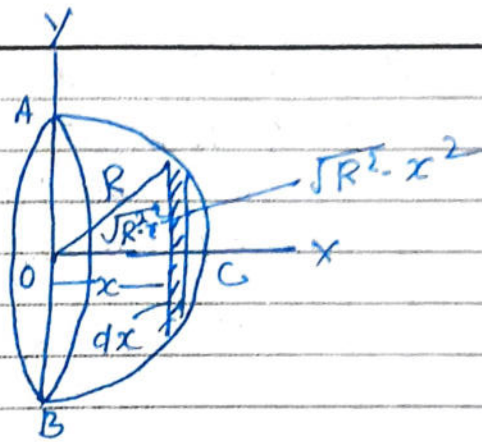
Subtracting eqn (1) from eqn (11) we have

$4x_4 = -6$ or $x_4 = -3/2$

In the same way $y_4 = -3/2$ and $z_4 = -3/2$

Therefore 40 gm mass should be placed at $(-3/2, -3/2, -3/2)$ m.

Centre of mass of a solid hemisphere →



This figure represent a hemisphere of radius R, let o be the centre of the circular section, be the origin and oc be the axis of symmetry along x-axis. Consider the portion of the hemisphere lying between two parallel planes, each perpendicular to OC, at distance x and x+dx from O, this portion will be a circular disc of radius $\sqrt{R^2-x^2}$.

face area of the disc $= \pi(R^2-x^2)$

volume of the disc $= \pi(R^2-x^2)\,dx$

If ρ be the density, then its mass

$$dm = \pi(R^2-x^2)\rho\,dx$$

the centre of mass of the hemisphere will lie on OC at a distance x from O.

$$\therefore X = \frac{\int x\,dm}{\int dm} = \frac{\int_0^R \pi(R^2-x^2)x\rho\,dx}{\int_0^R \pi(R^2-x^2)\rho\,dx} = \frac{\left[\frac{R^2x^2}{2}-\frac{x^4}{4}\right]_0^R}{\left[R^2x-\frac{x^3}{3}\right]_0^R}$$

$$= \frac{\frac{R^4}{2}-\frac{R^4}{4}}{R^3-R^3/3} = \frac{R^4}{4}\times\frac{3}{2R^3} = \frac{3R}{8}$$

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Area of disc $= \pi r^2 = \pi x^2 R^2/h^2$

volume of the disc $= \frac{\pi x^2 R^2}{h^2} \cdot dx$

mass of the disc $= \frac{\pi x^2 R^2}{h^2} \cdot dx$

Mass of the disc. $dm = \frac{\pi x^2 R^2 \rho}{h^2} \cdot dx$

where ρ is the density of the material of the cone. Hence, the centre of mass of the cone will lie on OC at a distance X from O.

$$X = \frac{\int x\,dm}{\int dm} = \frac{\int_0^h x \cdot \frac{\pi x^2 R^2 \rho}{h^2} \cdot dx}{\int_0^h \frac{\pi x^2 R^2 \rho}{h^2} \cdot dx} = \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h}$$

$$\frac{h^4}{4} \times \frac{3}{h^3} = \frac{3h}{4}$$

Hence, the centroid of a solid cone is situated at a distance $\frac{3}{4}$ of its vertical height measured from its vertex.

Put $R^2 - x^2 = t^2$ $\therefore$ $x\,dx = -t\,dt$

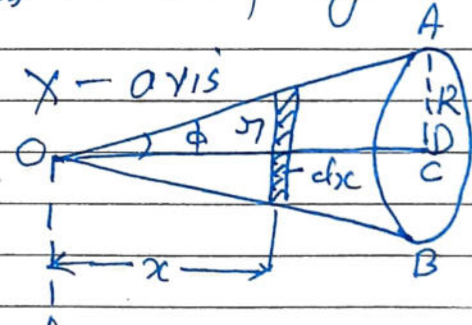
$$\therefore \int_0^R x\sqrt{R^2-x^2}\,dx = -\int_R^0 t^2\,dt = \left[\frac{r^3}{3}\right]_0^R = \frac{R^3}{3}$$

$$\text{Hence, } X = \frac{4}{\pi R^2}\cdot\frac{R^3}{3} = \frac{4R}{3\pi}$$

Hence, the centre of mass of the semi-circular lamina will lie on the line of symmetry at a distance $4R/3\pi$ from the centre O of the lamina.

(4) Centre of mass of right circular cone — Right circular cone having circular base AB of radius R and centre C. Let O be the apex of the cone be the origin and OC the x-axis, OC is the area of symmetry.

Consider two parallel planes perpendicular to x-axis and at distances x and $x+dx$ from O. the portion of the cone lying between two planes is a ~~cuiser~~ circular disc



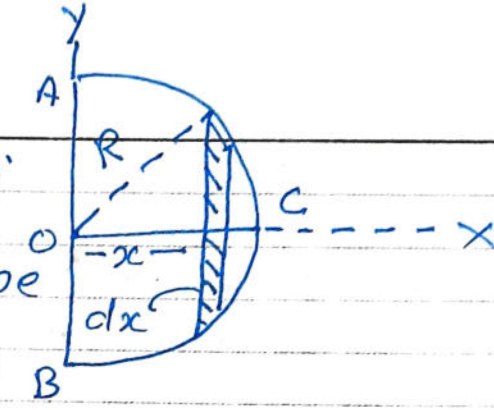
OC is the area of symmetry

Radius of the disc $= r = x\tan\phi$

where ϕ is the semi-vertical angle of the cone. If h be the height of the cone, then

$$\tan\phi = R/h$$

$$\therefore\ r/x = R/h \quad \text{or} \quad r = xR/h$$



A semicircular lamina with AOB as its diameter and OC is a radius perpendicular to it. The radius OC is evidently the axis of symmetry and, therefore, centre of mass of the lamina lies on it. Let the X-axis be along OC and Y axis along OA.

$$\text{Area of lamina} = \pi R^2/2$$

If M be the mass of the lamina, its mass per unit area $= 2M/\pi R^2$

Consider a strip of thickness dx at a distance x from the origin then,

$$\text{Length of the strip} = 2\sqrt{R^2-x^2}$$

$$\text{Area of strip} = 2\sqrt{R^2-x^2}\cdot dx$$

$$\text{Mass of the strip} = \frac{2\sqrt{R^2-x^2}\cdot dx\, 2M}{\pi R^2}$$

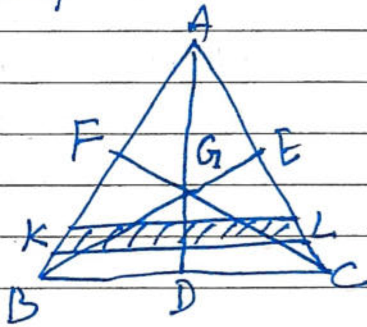
$$\therefore X = \frac{\int x\,dm}{M} = \frac{\int_0^R x\cdot 2\sqrt{R^2-x^2}\cdot dx\cdot \frac{2M}{\pi R^2}}{M}$$

$$= \frac{4}{\pi R^2}\int_0^R x\sqrt{R^2-x^2}\,dx.$$

(2) Centre of mass of a thin triangular plate.— Let the triangular plate ABC be divided into a very large number of narrow strip KL, each parallel to BC. The centre of mass of each strip lies at its middle point hence the centre of mass of the entire plate lies on the median AD. Similarly it will also lie on the median CF and BE. thus, the centre of mass of the triangular plate lies at the point of intersection G of the three median and G is called the centroid of the triangle. The point G divided each median internally in the ratio 1:2 so that

$$\boxed{\frac{AD}{AG} \quad \frac{DG}{AG} = \frac{1}{2} \text{ or } \frac{DG+AG}{AG} = \frac{1+2}{2}} \text{ or } \frac{AD}{AG} = \frac{3}{2}$$

$$\therefore AG = \frac{2}{3} AD$$



(3) ~~Centre of mass of a thin triangular plate:-~~

Centre of mass of a semi-circular lamina or plate :—

Hence, the distance of the centre of mass G of the rod from O is

$$X = \frac{\int x\,dm}{\int dm} = \frac{\int_0^l x \cdot \frac{M}{l} \cdot dx}{M} = \frac{\frac{M}{l}\left[\frac{x^2}{2}\right]_0^l}{M} = \frac{l}{2}$$

Hence, the centre of mass of the rod is situated in the middle.

from equ (5) $Mv = $ a const or $v = $ Const

equ (5) is again diff w.r.t. to time

$$\frac{dP}{dt} = M\frac{dv}{dt} = \sum_N \frac{d}{dt}(m_n v_n) = \sum_N F_n$$

F_n = is the force acting on any particle n of the system hence the total force $\sum F_n$ on all the particles of the system will be equal to the external force F_{ext}

$$F_{ext} = \frac{dP}{dt} = M\frac{dv}{dt} = M a_{cm}$$

Thus the centre of mass of a system of particles moves as if it were a particle of mass equal to the total mass of the system.

Motion of the Centre of mass :– Diffing equ^n (1) with respect to time t, we get

$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + \cdots + m_N\frac{dr_N}{dt}$$

Now writeing v for the velocity dR/dt of the centre of mass and $v_1, v_2 \ldots$ for the velocities of the particles we have

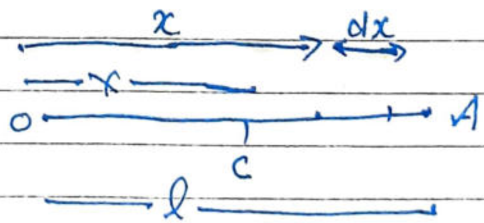
$$MV = m_1v_1 + m_2v_2 + \cdots + m_Nv_N$$

$$MV = \sum m_n v_n = P \qquad \text{—— (5)}$$

$$V = \frac{\sum m_n v_n}{M} = \frac{m_1v_1 + m_2v_2 + \cdots m_Nv_N}{m_1 + m_2 + \cdots m_N} \qquad \text{—— (6)}$$

This equ^n gives the velocity of the Centre of mass

Centre of mass of a thin uniform rod :– Let the x-axis lie along the length OA of the rod and O be the origin, If M and l represent its mass and length, then mass per unit length = M/L



Consider a small element dx at a distance x from the origin, then its mass

$$dm = \frac{M}{l}\cdot dx$$

so that

$$X = \frac{m_1x_1 + m_2x_2 + \cdots + m_Nx_N}{m_1 + m_2 + \cdots m_N}$$

$$= \frac{\sum m_n x_n}{\sum m_n}$$

$$= \frac{m_1y_1 + m_2y_2 + \cdots + m_Ny_N}{m_1 + m_2 + \cdots + m_N}$$

$$= \frac{\sum m_n y_N}{\sum m_n} \quad \text{——— (3)}$$

$$Z = \frac{m_1z_1 + m_2z_2 + \cdots + m_Nz_N}{m_1 + m_2 + \cdots m_N} = \frac{\sum m_n z_n}{\sum m_n}$$

A rigid body may be Considered as a system of closely packed particles, having continious distribution of mass. The centre of mass of such a body will be given on replacing the sign of summation (Σ) by the sign of integration in equ'l (3) i.e

$$X = \frac{\int x\,dm}{\int dm} = \frac{1}{M}\int x\,dm; \quad Y = \frac{\int y\,dm}{\int dm} = \frac{1}{M}\int y\,dm,$$

$$Z = \frac{\int z\,dm}{\int dm} = \frac{1}{M}\int z\,dm. \quad \text{——— (4)}$$

Centre of mass:—

Let us consider a system of N-particles, having masses $m_1, m_2, \ldots m_N$ and instantaneous position vectors $r_1, r_2, \ldots r_3 \ldots r_N$ respectively. The centre of mass of this system at this instant is the point whose position vector R is given

$$MR = m_1 r_1 + m_2 r_2 + \cdots + M_N r_N$$

Where M is the total mass of the system i.e

$$M = \sum m_n = m_1 + m_2 + \cdots m_N$$

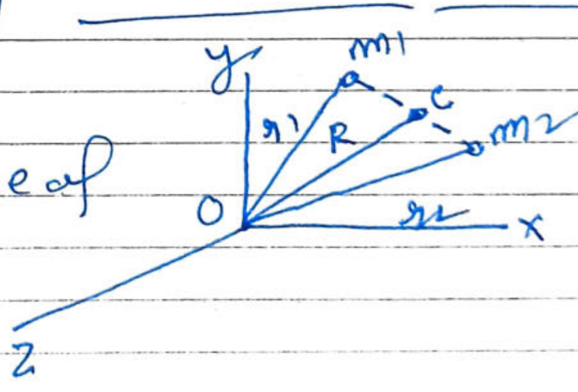
Therefore

$$\boxed{R = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_N r_N}{m_1 + m_2 + \cdots M_N} = \frac{\sum_{n=1}^{N} m_n r_n}{\sum_{n=1}^{N} m_n}} \quad (1)$$

The sum $\sum m_n r_n$ is called the first moment of mass for the system.

For a system of two particles

$$\boxed{R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}} \quad (2)$$



If (x, y, z) are the coordinates of the centre of mass, then

$$R = x\hat{i} + y\hat{j} + z\hat{k}$$