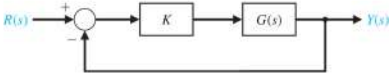
Outline

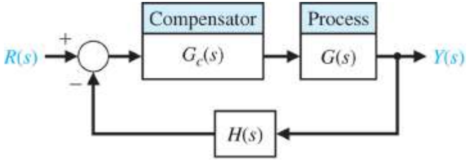
Compensators

- Early in the course we provided some useful guidelines regarding the relationships between the pole positions of a system and certain aspects of its performance
- Using root locus techniques, we have seen how the pole positions of a closed loop can be adjusted by varying a parameter



- What happens if we are unable to obtain that performance that we want by doing this?
  - Ask ourselves whether this is really the performance that we want
  - Ask whether we can change the system, say by buying different components
  - seek to compensate for the undesirable aspects of the process

Cascade compensation

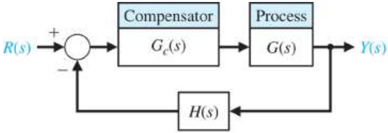


- Usually, the plant is a physical process
- If commands and measurements are made electrically, compensator is often an electric circuit
- General form of the (linear) compensators we will consider is

$$G_c(s) = \frac{K_c \prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)}$$

- Therefore, the cascade compensator adds open loop poles and open loop zeros
- These will change the shape of the root locus

Compensator design

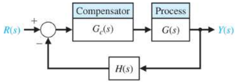


- Where should we put new poles and zeros to achieve desired performance?
- That is the art of compensator design
- We will consider first order compensators of the form

$$G_c(s) = \frac{K_c(s+z)}{(s+p)} = \frac{\tilde{K}_c(1+s/z)}{(1+s/p)}, \quad \text{where } \tilde{K}_c = K_c z/p$$

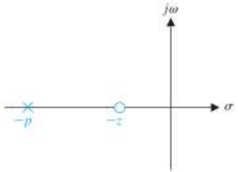
  - with the pole $-p$ in the left half plane
  - and the zero, $-z$ in the left half plane, too
- For reasons that will soon become clear
  - when $|z| < |p|$: phase lead network
  - when $|z| > |p|$: phase lag network

Lead compensation



$$G_c(s) = \frac{K_c(s+z)}{(s+p)}$$

with $|z| < |p|$. That is, zero closer to origin than pole



Let $p = 1/\tau_p$ and $z = 1/(\alpha_{\text{lead}}\tau_p)$. Since $z < p$, $\alpha_{\text{lead}} > 1$.
Define $\tilde{K}_c = K_c z/p = K_c/\alpha_{\text{lead}}$. Then

$$G_c(s) = \frac{K_c(s+z)}{(s+p)} = \frac{\tilde{K}_c(1+\alpha_{\text{lead}}\tau_p s)}{(1+\tau_p s)}$$

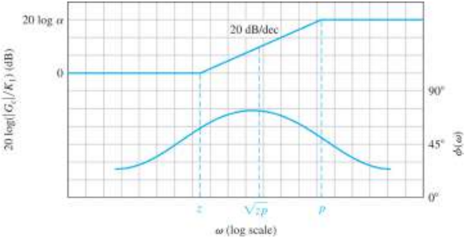
Lead compensation

With $|z| < |p|$, $\alpha_{\text{lead}} > 1$, $G_c(s) = \frac{K_c(s+z)}{(s+p)} = \frac{\tilde{K}_c(1+\alpha_{\text{lead}}\tau_p s)}{(1+\tau_p s)}$

- Frequency response:

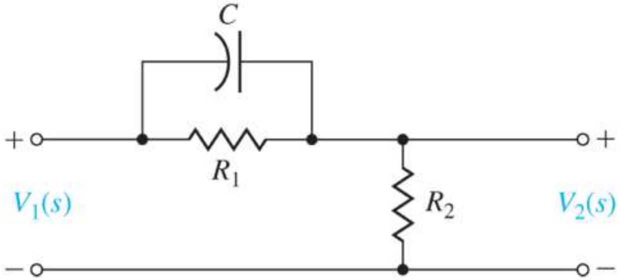
$$G_c(j\omega) = \frac{\tilde{K}_c(1 + j\omega\alpha_{\text{lead}}\tau_p)}{(1 + j\omega\tau_p)}$$

- Bode diagram (in the figure, $K_1 = \tilde{K}_c$)



- Between $\omega = z$ and $\omega = p$, $|G_c(j\omega)| \approx \tilde{K}_c\omega\alpha_{\text{lead}}\tau_p$
- What kind of operator has a frequency response with magnitude proportional to ω? Differentiator
- Note that the phase is positive. Hence "phase lead"

A passive phase lead network



Homework: Show that $\frac{V_2(s)}{V_1(s)}$ has the phase lead characteristic

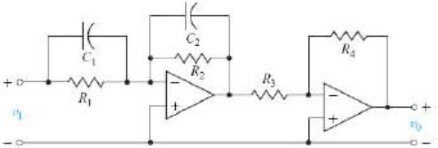
Active lead and lag networks

Here's an example of an active network architecture.

Lead or lag $\quad G_c = \dfrac{R_4R_2(R_1C_1s + 1)}{R_3R_1(R_2C_2s + 1)}$

Lead if $R_1C_1 > R_2C_2$

Lag if $R_1C_1 < R_2C_2$



Principles of Lead design via Root Locus

- The compensator adds poles and zeros to the $P(s)$ in the root locus procedure.
- Hence we can change the shape of the root locus.

- If we can capture desirable performance in terms of positions of closed loop poles
- then compensator design problem reduces to:
  - changing the shape of the root locus so that these desired closed-loop pole positions appear on the root locus
  - finding the gain that places the closed-loop pole positions at their desired positions

- What tools do we have to do this?
- Phase criterion and magnitude criterion, respectively

Root Locus Principles

- The point s_0 is on the root locus of $P(s)$ if $1 + KP(s_0) = 0$.
- In first order compensator design with $G(s) = \frac{K_G \prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$ and $G_c(s) = \frac{K_c(s+z)}{(s+p)}$, we have $P(s) = \frac{(s+z)}{(s+p)} \frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$ and $K = K_c K_G$. We will restrict attention to the case of $K > 0$
- **Phase cond.** s_0 is on root locus if $\angle P(s_0) = 180^\circ + \ell\, 360^\circ$:

$$\sum_{i=1}^{M}(\text{angle from } -z_i \text{ to } s_0) - \sum_{j=1}^{n}(\text{angle from } -p_j \text{ to } s_0)$$
$$+ (\text{angle from } -z \text{ to } s_0) - (\text{angle from } -p \text{ to } s_0)$$
$$= 180^\circ + \ell\, 360^\circ$$

- **Mag. cond.** If s_0 satisfies phase condition, the gain that puts a closed-loop pole at s_0 is $K = 1/|P(s_0)|$:

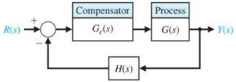
$$K = \frac{\prod_{j=1}^{n}(\text{dist from } -p_j \text{ to } s_0)}{\prod_{i=1}^{M}(\text{dist from } -z_i \text{ to } s_0)} \times \frac{(\text{dist from } -p \text{ to } s_0)}{(\text{dist from } -z \text{ to } s_0)}$$

RL design: Basic procedure

1. Translate design specifications into desired positions of dominant poles
2. Sketch root locus of uncompensated system to see if desired positions can be achieved
3. If not, choose the positions of the pole and zero of the compensator so that the desired positions lie on the root locus (phase criterion), if that is possible
4. Evaluate the gain required to put the poles there (magnitude criterion)
5. Evaluate the total system gain so that the steady-state error constants can be determined
6. If the steady state error constants are not satisfactory, repeat

This procedure enables relatively straightforward design of systems with specifications in terms of rise time, settling time, and overshoot; i.e., the transient response.
For systems with steady-state error specifications, Bode (and Nyquist) methods may be more straightforward (later)

Lead Comp. example



Consider a case with $G(s) = \frac{1}{s(s+2)}$ and $H(s) = 1$.
Design a lead compensator to achieve:

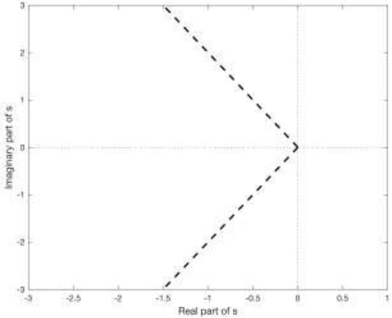
- damping coefficient $\zeta \approx 0.45$ and
- velocity error constant $K_v = \lim_{s \to 0} sG_c(s)G(s) \geq 20$
- swift transient response (small settling time)

What to do?

- Can we achieve this with proportional control?
- If not we will attempt lead control

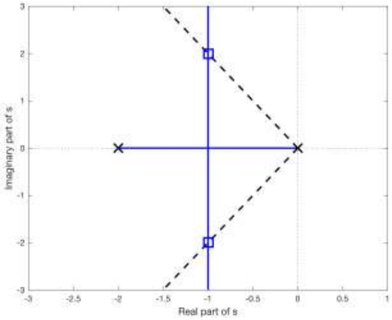
Attempt prop. control

- Closed loop poles that correspond to $\zeta = 0.45$ lie on rays of angle $\cos^{-1}(0.45) \approx 60^\circ$ to neg. real axis
- Sketch them



Attempt prop. control, II

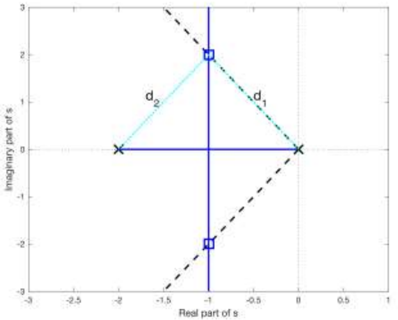
- Add sketch of root locus of $\frac{1}{s(s+2)}$



- Is there an intersection? Yes
- What is the value of $K = K_{\text{amp}} K_G$ that puts closed-loop poles at intersection point?

Attempt prop. control, III

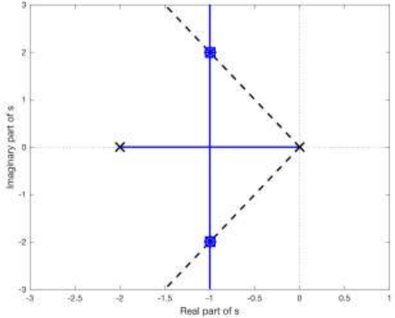
- That gain is $K = \frac{\prod \text{distances from OL poles}}{\prod \text{distances from OL zeros}}$



- $K = d_1 d_2 = 5$.
- Since $K_G = 1$, $K_{\text{amp}} = 5$.

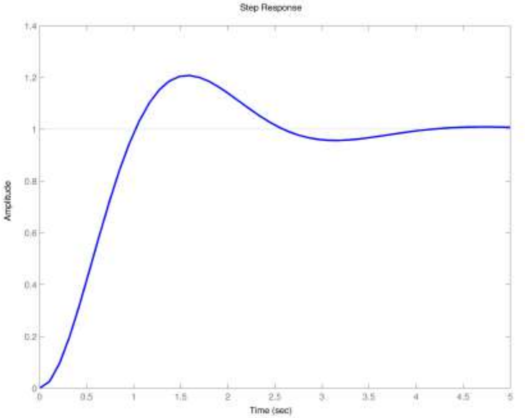
Assessing our prop. design

- $K_{\text{amp}} = 5$.
- Place actual closed loop poles on the root locus (asterisks)



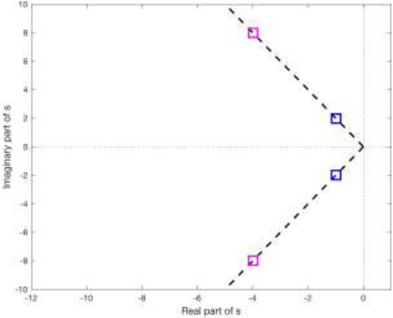
- As expected, they are at the target locations (open squares)
- What is the corresponding K_v?
- $K_v = \lim_{s \to 0} sG_c(s)G(s) = \frac{K_{\text{amp}}}{2} = 2.5$:(
- Do the closed-loop poles have responses that decay quickly? No, $T_s \approx 4$s

Prop. control, step response

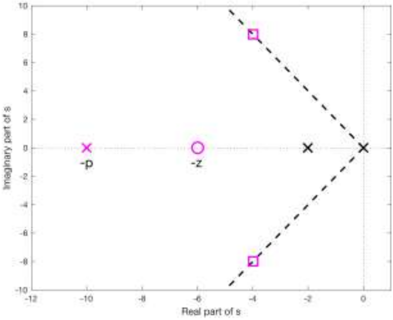


Lead compensated design

- Where should the closed-loop poles be? $\cos^{-1}(0.45) \approx 60^\circ$
- Note that the settling time is not specified; it only needs to be small. This provides design flexibility.
- However, we need a large K_v which will require large gain. Need desired positions far from open loop poles.
- Let's start with desired roots at $-4 \pm j8$ (purple squares)
- This pair has $T_s = 1$s and $\omega_n = \sqrt{4^2 + 8^2} \approx 8.9$

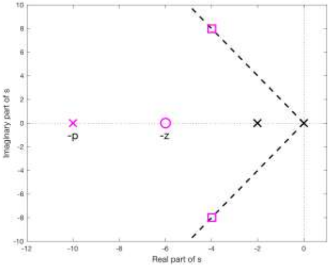


How to choose z, p and K_c



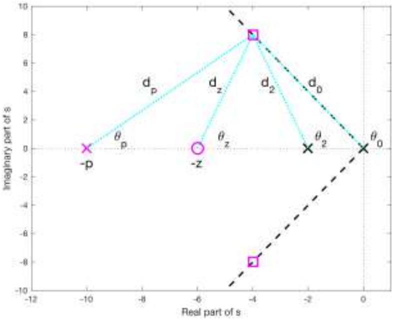
- Lead design questions:
  - How do we choose z and p to ensure that there exists a gain that will put closed loop poles at the squares?
  - Once we have done that, how do we find the gain that puts the closed-loop poles at the squares?

How to choose z, p and K_c



- We want squares to be on the root locus
- That is, if s_0 denotes the position of one of the squares, we want $1 + G_c(s_0)G(s_0) = 1 + KP(s_0) = 0$
- In other words, we want $P(s_0) = -1/K$
- Separating that complex-valued equation into magnitude and phase components, we want
  - $\angle P(s_0) = 180^\circ$; phase criterion
  - $|P(s_0)| = 1/K$; magnitude criterion

How to choose z, p and K_c



- To find z and p we use the phase criterion

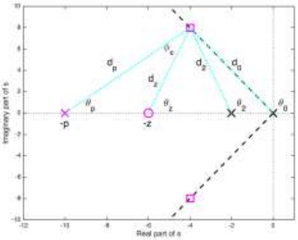
$$\sum \text{(angles from OL zeros)} - \sum \text{(angles from OL poles)} = 180^\circ$$

$$\Longrightarrow \quad \theta_z - \theta_0 - \theta_2 - \theta_p = 180^\circ$$

- Then, to find K_c we use the magnitude criterion

$$K = K_c K_G = \frac{\prod \text{distances from OL poles}}{\prod \text{distances from OL zeros}} = \frac{d_0 d_2 d_p}{d_z}$$

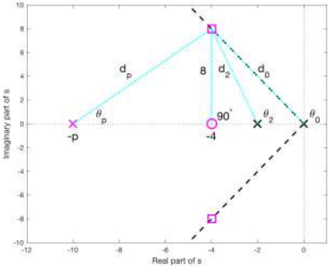
How to choose z and p



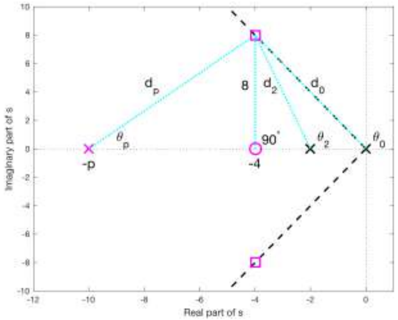
- Can we start to think of this geometrically, rather than algebraically?
- Phase condition equation at s_0: $\theta_z - \theta_p = 180^\circ + \theta_0 + \theta_2$
- One linear equation, two unknowns. Many solutions
- However, we can find out something about $\angle G_c(s_0)$
- Since $G_c(s) = K_c \frac{s+z}{s+p}$, with $K_c > 0$,
  $\angle G_c(s_0) = \angle(s_0 + z) - \angle(s_0 + p) = \theta_z - \theta_p$
  Can you see this angle in the figure? It is ϕ_c
- Since $90^\circ < \theta_0,\, \theta_2 < 180,\ \implies\ 0 < \phi_c < 180$
- That is, we need a phase lead compensator
- What does that say about z and p? $-p < -z$

Simplifying rule of thumb

- What are good choices for z and p amongst those that provide the right amount of phase lead?
- Simplifying rule of thumb: When amount of phase lead required at s_0 is less than 90°, place zero on the real axis "underneath" the desired closed-loop pole positions.
- When applicable, this reduces the complexity of the design procedure; now we only have to design the pole position; often a reasonable choice
- Can iterate on zero position as needed



How to choose p



With rule of thumb in place

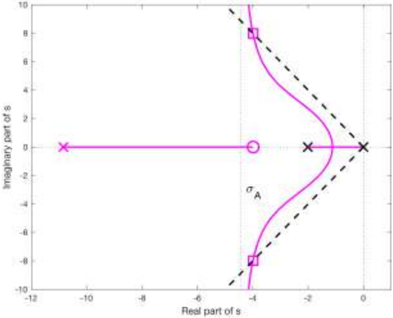
- Find θ_p using

$$\sum \text{angles from OL zeros} - \sum \text{angles from OL poles} = 180^\circ$$
$$\sim 90 - (116 + 104 + \theta_p) = 180$$
$$\implies \theta_p \approx 50$$

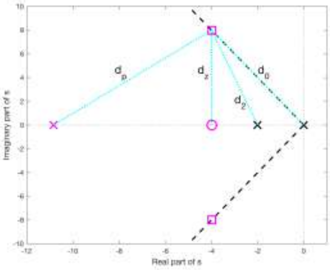
- Hence, pole at $-p = -4 - 8/\tan(\theta_p) \approx -10.86$

Checking our work

Does the root locus for the compensated system go through the desired positions?



How to choose K_c



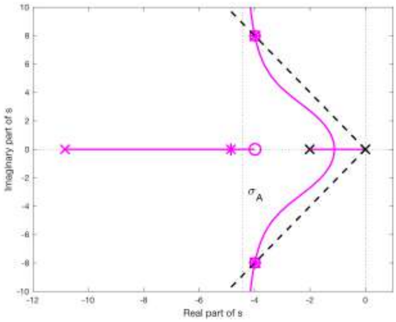
- What is the gain that puts closed-loop poles in the boxes? Recall

$$K = K_c K_G = \frac{\prod \text{distances from OL poles}}{\prod \text{distances from OL zeros}} = \frac{d_0 d_2 d_p}{d_z}$$

- In this example $K_G = 1$
- Therefore, $K_c = \frac{d_0 d_2 d_p}{d_z} \approx \frac{8.94(8.25)(10.54)}{8} \approx 97.1$

Summarizing initial design

- Our compensator is $G_c(s) = \frac{97.1(s+4)}{(s+10.86)}$
- The compensated open loop is $G_c(s)G(s) = \frac{97.1(s+4)}{s(s+2)(s+10.86)}$
- Mark all closed-loop poles on the root locus (asterisks)
  Note that conjugate pair hit the target (as designed),
  and that the real pole is not far from the (open/closed loop) zero

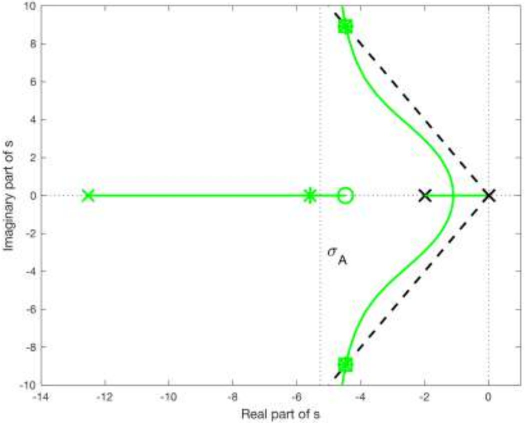


- Velocity constant: $K_v = \lim_{s \to 0} sG_c(s)G(s) \approx 17.9$:(

What to do now?

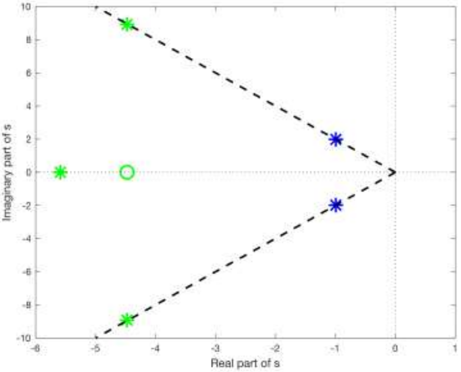
- We tried hard, but did not achieve the design specs
- Let's go back and re-examine our choices
- Zero position of compensator was chosen via rule of thumb
- Can we do better?
  Yes, but two parameter design becomes trickier.
- What were other choices that we made?
- We chose desired poles to be of magnitude $\omega_n \approx 8.9$
- We could choose them to be further away;
  larger gain to get there (and faster transient response)
- By how much?
- Show that when desired poles have $\omega_n = 10$ as well as the required $\zeta \approx 0.45$, then the choice of $z \approx 4.47$, $p \approx 12.5$ and $K_C \approx 125$ results in $K_v \approx 22.3$

Root Locus, new lead comp.



Comparisons to prop. design

Closed-loop pole and zero positions

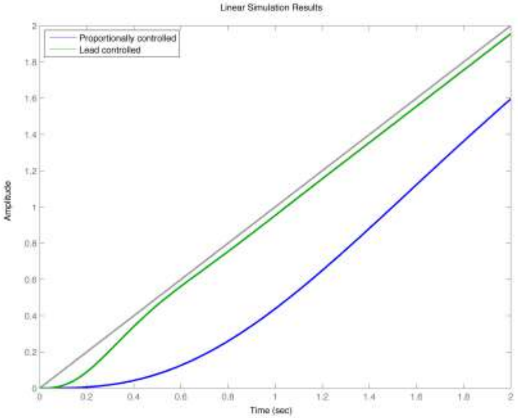


Comparisons

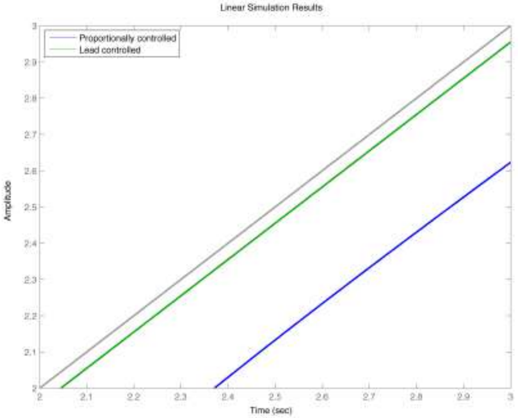
	Prop.-contr.	Lead contr.
Controller, $G_C(s)$	5	$\frac{125(s+4.47)}{(s+12.5)}$
OL TF, $G_C(s)G(s)$	$\frac{5}{s(s+2)}$	$\frac{125(s+4.47)}{(s+12.5)}\ \frac{1}{s(s+2)}$
CL TF, $\frac{Y(s)}{R(s)}$	$\frac{5}{s(s+2)+5}$	$\frac{125(s+4.47)}{s(s+2)(s+12.5)+125(s+4.47)}$
CL poles	$-1 \pm j2$	$-4.47 \pm j8.94,\ -5.59$
CL zeros	∞, ∞	$-4.47, \infty, \infty$
CL TF, again	$\frac{5}{s^2+2s+5}$	$\frac{131(1+0.013s)}{s^2+8.94s+100} - \frac{1.71}{s+5.59}$

- Complex conjugate poles still dominate
- Closed-loop zero at -4.47 (which is also an open-loop zero) reduces impact of closed-loop pole at -5.59; residue of that pole in partial fraction expansion is small

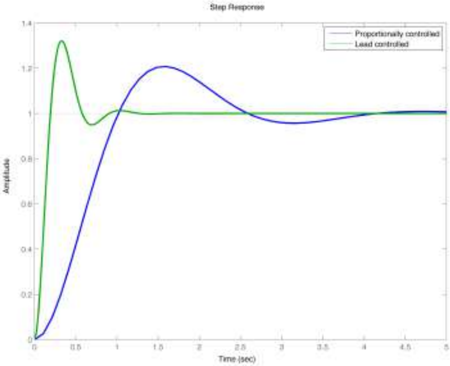
New lead comp., ramp response



New lead comp., ramp response, detail



New lead comp., step response



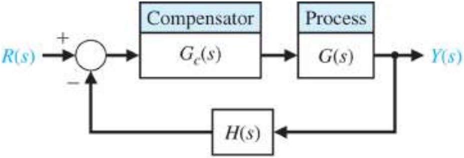
Note faster settling time than prop. controlled loop,
However, the CL zero has increased the overshoot a little

Perhaps we should go back and re-design for $\zeta \approx 0.40$
in order to better control the overshoot

Outcomes

- Root locus approach to phase lead design was reasonably successful in terms of putting dominant poles in desired positions; e.g., in terms of ζ and ω_n
- We did this by positioning the pole and zero of the lead compensator so as to change the shape of the root locus
- However, root locus approach does not provide independent control over steady-state error constants (details upcoming)
- That said, since lead compensators reduce the DC gain (they resemble differentiators), they are not normally used to control steady-state error.
- The goal of our lag compensator design will be to increase the steady-state error constants, without moving the other poles too far

Cascade compensation



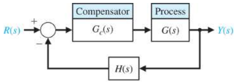
- Throughout this lecture, and all the discussion on cascade compensation, we will consider the case in which $H(s) = 1$.
- We will consider first order compensators of the form

$$G_c(s) = \frac{K_c(s + z)}{(s + p)}$$

with the pole, $-p$, and the zero, $-z$, both in the left half plane

- when $|z| < |p|$: phase lead network
- when $|z| > |p|$: phase lag network

Steady-state errors



If closed loop stable, steady state error for input $R(s)$:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{R(s)}{1 + G_C(s)G(s)}$$

Let $G(s) = \frac{K_G \prod_i (s+z_i)}{\prod_j (s+p_j)}$ and consider $G_C(s) = \frac{K_C(s+z)}{(s+p)}$

- Consider the case in which $G(s)$ is a type-0 system.
  - Steady state error due to a step $r(t) = Au(t)$: $e_{ss} = \frac{A}{1+K_{\text{posn}}}$, where

$$K_{\text{posn}} = G_C(0)G(0) = \frac{K_C z}{p} \frac{K_G \prod_i z_i}{\prod_j p_j}$$

- Note that for a lead compensator, $z/p < 1$,
- So lead compensation may degrade steady-state error performance

Aside: What about e_{ss} of step for Type ≥ 1 systems?

Steady-state error

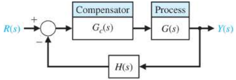
- Now, consider the case in which $G(s)$ is a type-1 system, $G(s) = \frac{K_G \prod_i (s+z_i)}{s \prod_j (s+p_j)}$
- Steady-state error due to a ramp $r(t) = At$: $e_{ss} = A/K_v$, where the velocity constant is

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = \frac{K_C z}{p} \frac{K_G \prod_i z_i}{\prod_j p_j}$$

- Once again, lead compensation may degrade steady-state error performance

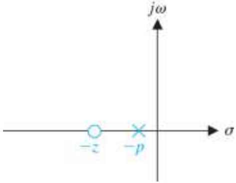
- Is there a way to increase the value of these error constants while leaving the closed loop poles in essentially the same place as they were in an uncompensated system? Perhaps $|z| > |p|$?

Lag compensation



$$G_c(s) = \frac{K_c(s+z)}{(s+p)}$$

with $|z| > |p|$. That is, pole closer to origin than zero



Let $z = 1/\tau_z$ and $p = 1/(\alpha_{\text{lag}}\tau_z)$. Since $z > p$, $\alpha_{\text{lag}} > 1$.
Define $\tilde{K}_c = K_c z/p = K_c\alpha_{\text{lag}}$. Then

$$G_c(s) = \frac{K_c(s+z)}{(s+p)} = \frac{\tilde{K}_c(1+\tau_z s)}{(1+\alpha_{\text{lag}}\tau_z s)}$$

Frequency response

$$G_c(j\omega) = \frac{\tilde{K}_C(1 + j\omega\tau_z)}{(1 + j\omega\alpha_{\text{lag}}\tau_z)}$$

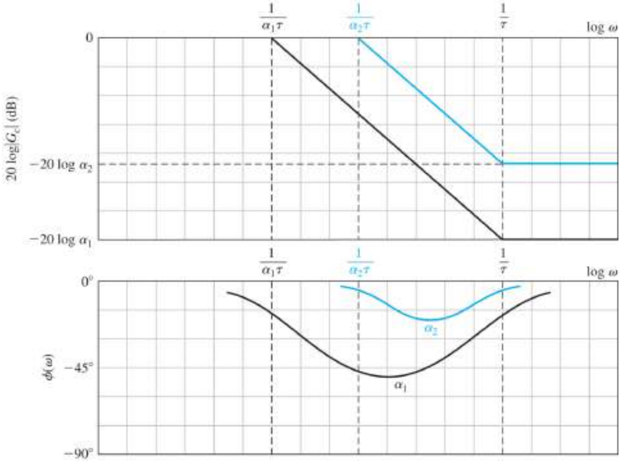
Magnitude

- Low frequency gain: $\tilde{K}_C$
- Corner freq. in denominator at $\omega_p = p = 1/(\alpha_{\text{lag}}\tau_z)$
- Corner freq. in numerator at $\omega_z = z = 1/\tau_z$
- $\omega_p < \omega_z$
- High frequency gain: $\tilde{K}_C/\alpha_{\text{lag}} = K_C$

Phase

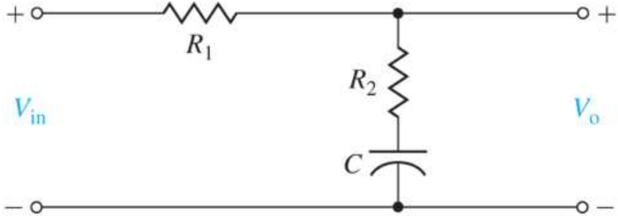
- $\phi(\omega) = \text{atan}(\omega\tau_z) - \text{atan}(\alpha_{\text{lag}}\omega\tau_z)$
- At low frequency: $\phi(\omega) = 0$
- At high frequency: $\phi(\omega) = 0$
- In between: negative, with max. lag at $\omega = \sqrt{zp}$

Bode Diagram, with $\tilde{K}_c = 1$



Note integrative characteristic

A passive phase lag network



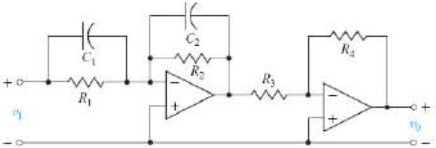
Active lead and lag networks

Here's an example of an active network architecture.

Lead or lag $\quad G_c = \dfrac{R_4 R_2 (R_1 C_1 s + 1)}{R_3 R_1 (R_2 C_2 s + 1)}$

Lead if $R_1 C_1 > R_2 C_2$

Lag if $R_1 C_1 < R_2 C_2$



Lag compensator design

- Lag compensator: $G_c(s) = K_c \frac{s+z}{s+p}$. with $|z| > |p|$.
- Recall position error constant for compensated type-0 system and velocity error constant for compensated type-1 system:

$$K_{\text{posn}} = \frac{K_C z}{p} \frac{K_G \prod_i z_i}{\prod_j p_j}, \qquad K_v = \frac{K_C z}{p} \frac{K_G \prod_i z_i}{\prod_j p_j}$$

  where in the latter case the product in the denominator is over the non-zero poles.

Design Principles

- We don't try to reshape the uncompensated root locus.
- We just try to increase the value of the desired error constant by a factor $\alpha_{\text{lag}} = z/p$ without moving the existing closed-loop poles (well not much)
- Reshaping was the goal of lead compensator design

Lag compensator design

Design principles:

- Don't reshape the root locus
  - Adding the open loop pole and zero from the compensator should only result in a small change to the angle criterion for any (important) point on the uncompensated root locus
  - Angles from compensator pole and zero to any (important) point on the locus must be similar
  - Pole and zero must be close together

- Increase value of error constant:
  - Want to have a large value for $\alpha_{\text{lag}} = z/p$.
  - How can that happen if z and p are close together?
  - Only if z and p are both small, i.e., close to the origin

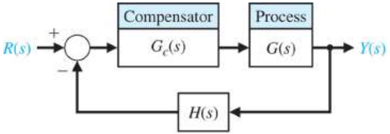
Lag comp. design via Root Locus

1. Obtain the root locus of uncompensated system
2. From transient performance specs, locate suitable dominant pole positions *on that locus*
3. Obtain the loop gain for these points, $K_{\text{unc}} = K_{\text{amp}} K_G$; hence the (closed-loop) steady-state error constant
4. Calculate the necessary increase. Hence $\alpha_{\text{lag}} = z/p$
5. Place pole and zero close to the origin (with respect to desired pole positions), with $z = \alpha_{\text{lag}} p$.
   Typically, choose z and p so that their angles to desired poles differ by less than 1°.
6. Set $K_C = K_{\text{amp}}$

What if there is nothing suitable at step 2?

- Perhaps do lead compensation first,
- then lag compensation on lead compensated plant.
  i.e., design a lead-lag compensator

Example



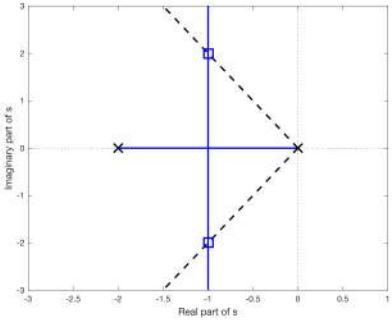
Let's consider, again, the case with $G(s) = \frac{1}{s(s+2)}$.
Design a lag compensator to achieve damping coefficient $\zeta \approx 0.45$ and velocity error constant $K_v \geq 20$

Note: we will get a different closed loop from our lead design.

First step, obtain uncompensated root locus, and locate desired dominant pole locations

Uncompensated root locus

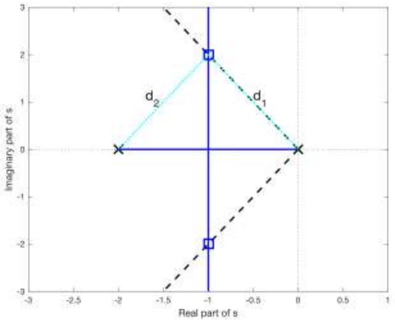
In this example, this step is the same as the first step in our lead design example



- So, yes, it is possible to achieve a damping coefficient $\zeta \approx 0.45$ using proportional control
- What is the gain that puts the closed loop poles there?

Finding the gain for prop. control

- That gain is $K = K_{\text{amp}} K_G = \dfrac{\prod \text{distances from OL poles}}{\prod \text{distances from OL zeros}}$

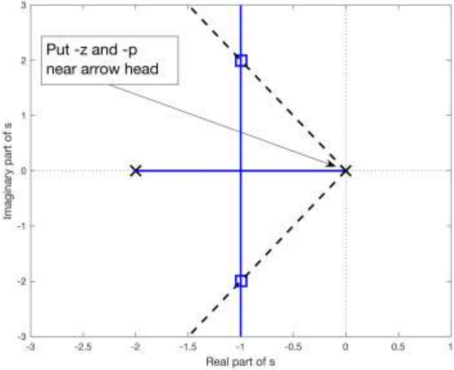


- $K = d_1 d_2 = 5$.
- Since $K_G = 1$, $K_{\text{amp}} = 5$.

Evaluate the velocity error constant, and choose z/p

- Velocity error constant of uncompensated loop: $K_{v,\text{unc}} = \lim_{s \to 0} sG_c(s)G(s) = K_{\text{amp}}K_G/2$
- Since $K_G = 1$ and $K_{\text{amp}} = 5$, $K_{v,\text{unc}} = 2.5$
- In order to obtain $K_v \geq 20$, the factor by which we need to increase $K_{v,\text{unc}}$ by at least $20/2.5 = 8$
- That implies that in the design of our lag controller, we should choose pole and zero such that $z/p \geq 8$,
- where z is chosen to be close to the origin with respect to dominant closed-loop poles, so that the root locus doesn't change too much near those dominant closed-loop poles

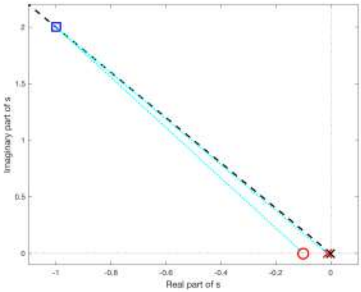
Positioning zero and pole



with $z/p \geq 8$

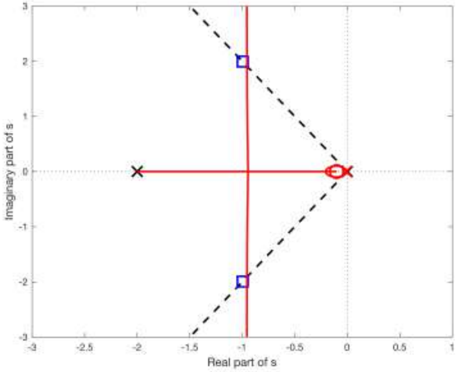
Zooming in

- Try $-z = -0.1$, along with $-p = -1/80$.



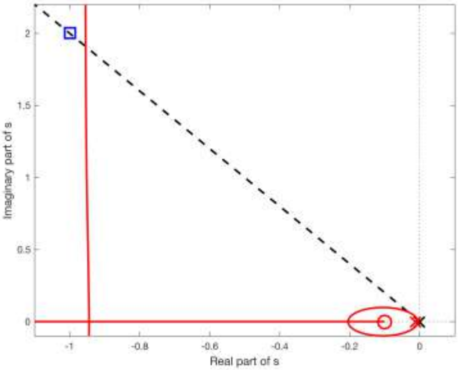
- Angles from new open-loop zero and open-loop pole to desired closed-loop pole position are pretty close.
- Therefore, their effects will nearly cancel out in phase criterion at values of *s* near box
- As a result, compensated root locus should pass close by the desired positions

Lag compensated root locus



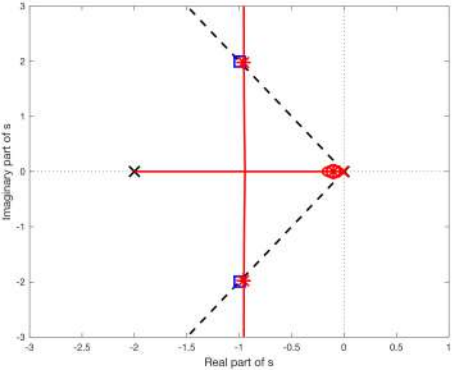
- Yes, indeed, the lag compensated root locus does pass close by the desired positions

Lag compensated root locus, zoomed in

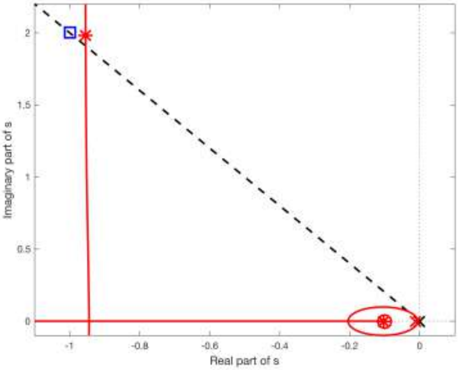


Choosing K_c

- Choose K_c to be the same as K_{amp} from the uncompensated design
- That is, $K_c = 5$
- Plot actual closed loop poles on the locus (asterisks)

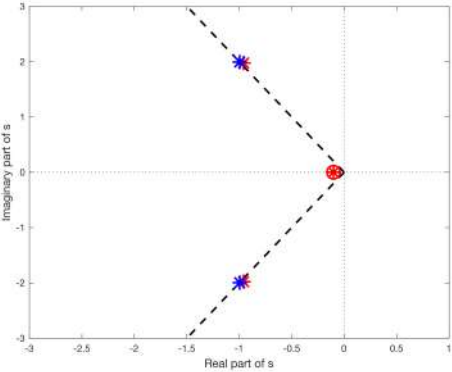


Zoomed in



Comparisons to prop. design

Closed-loop pole and zero positions

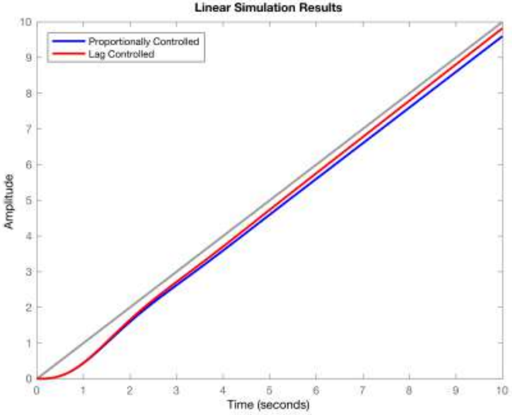


Comparisons

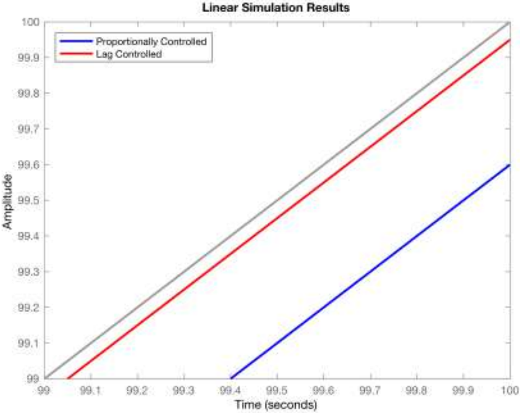
	Prop.-contr.	Lag contr.
Controller, $G_C(s)$	5	$\frac{5(s+0.1)}{(s+1/80)}$
OL TF, $G_C(s)G(s)$	$\frac{5}{s(s+2)}$	$\frac{5(s+0.1)}{(s+1/80)}\ \frac{1}{s(s+2)}$
CL TF, $\frac{Y(s)}{R(s)}$	$\frac{5}{s(s+2)+5}$	$\frac{5(s+0.1)}{s(s+2)(s+1/80)+5(s+0.1)}$
CL poles	$-1 \pm j2$	$-0.955 \pm j1.979, -0.104$
CL zeros	∞, ∞	$-0.1, \infty, \infty$
CL TF, again	$\frac{5}{s^2+2s+5}$	$\frac{4.999(1+7\times10^{-4}s)}{s^2+1.909s+4.827} + \frac{-0.004}{s+0.104}$

- Complex conjugate poles still dominate
- Closed-loop zero at -0.1 (which is also an open-loop zero) reduces impact of closed-loop pole at -0.104; residue of that pole in partial fraction expansion is small

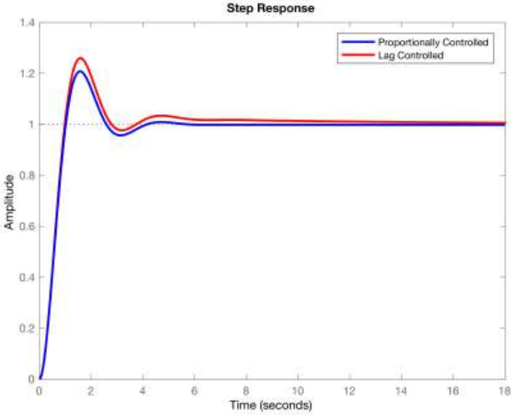
Ramp response



Ramp response, detail

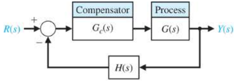


Step response



Note longer settling time of lag controlled loop, and slight increase in overshoot, due to extra closed-loop pole-zero pair that do not quite cancel each other out

Prop, Lead, Lag Design Comparisons



Recall the design example that we have considered for lead and lag designs:

For $G(s) = \frac{1}{s(s+2)}$ and with $H(s) = 1$, design a compensator to achieve:

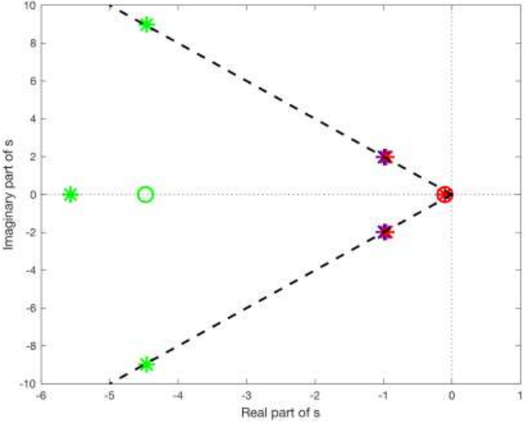
- damping coefficient $\zeta \approx 0.45$ and
- velocity error constant $K_v = \lim_{s \to 0} sG_c(s)G(s) \geq 20$
- swift transient response (small settling time)

We have done

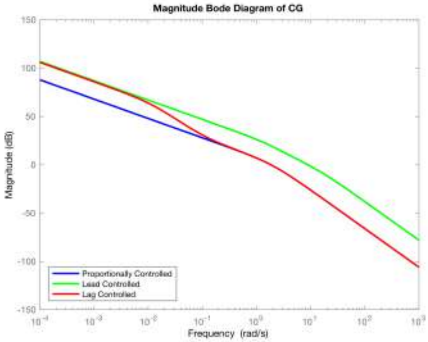
- Proportional design (blue), which failed to meet specifications
- Lead design (green)
- Lag design (red)

Prop, Lead, Lag Design Comparisons

Closed-loop pole and zero positions



Bode, open loop, $G_c(j\omega)G(j\omega)$



- Recall $K_v = \lim_{s \to 0} sG_c(s)G(s)$
- Low freq's: curves approx linear with slope -20dB/dec.
- That is $20\log_{10}(|G_c(j\omega)G(j\omega)|) \approx 20\log_{10}(A) - 20\log_{10}(\omega)$
- That means $G_c(j\omega)G(j\omega) \approx \frac{A}{j\omega}$; $G_c(s)G(s) \approx \frac{A}{s}$; $\implies K_v = A$
- Thus, when low freq. slope is -20dB/dec, “higher” curves mean larger K_v

Low freq. analysis

- Let's now do that analytically
- For each design, for small s, $G_c(s)G(s) \approx \frac{A}{s}$
- $G(s) = \frac{1}{s(s+2)}$
- **Prop:** $G_c(s) = 5$. Hence, $A = 2.5$
- **Lead:** $G_c(s) = \frac{125(s+4.47)}{(s+12.5)}$. Hence, $A = 22.3$
- **Lag:** $G_c(s) = \frac{5(s+0.1)}{(s+1/80)}$. Hence, $A = 20$

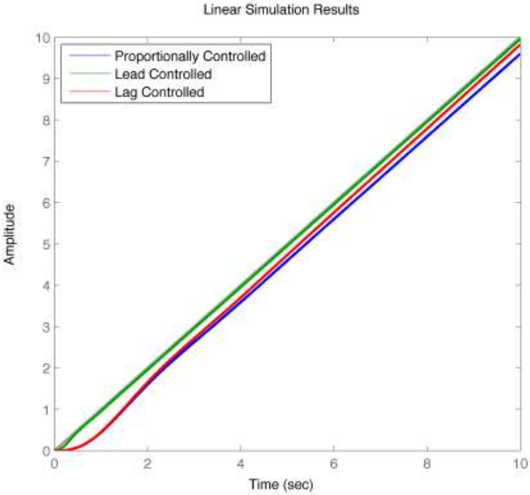
Prop, Lead, Lag Design Comparisons

For given example: $G(s) = \frac{1}{s(s+2)}$, $\zeta \approx 0.45$

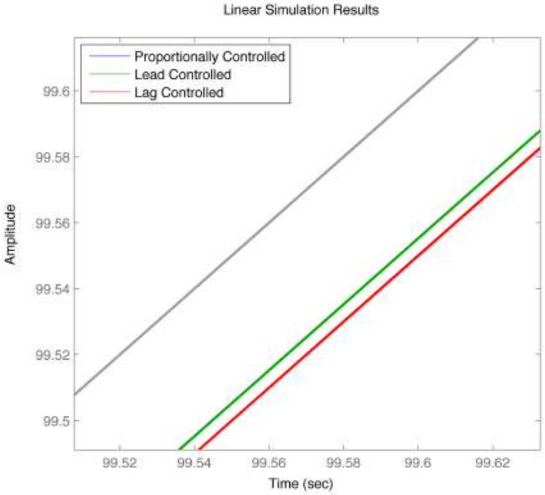
	Prop.-contr.	Lead contr.	Lag contr.
$G_C(s)$	5	$\frac{125(s+4.47)}{(s+12.5)}$	$\frac{5(s+0.1)}{(s+1/80)}$
$\frac{Y(s)}{R(s)}$	$\frac{5}{s^2+2s+5}$	$\frac{131(1+0.013s)}{s^2+8.94s+100} - \frac{1.71}{s+5.59}$	$\frac{4.999(1+7\times10^{-4}s)}{s^2+1.909s+4.827} + \frac{-0.004}{s+0.104}$
CL poles	$-1 \pm j2$	$-4.47 \pm j8.94$, -5.59	$-0.955 \pm j1.979$, -0.104
CL zeros	∞, ∞	$-4.47, \infty, \infty$	$-0.1, \infty, \infty$
$1/K_v$	0.4	0.045	0.05

- Lag design retains similar CL poles to prop. design, plus a "slow" pole with a small residue
- CL poles of lead design quite different
- Lead and lag meet K_v specification ($1/K_v = e_{\text{ss,unit ramp}}$)

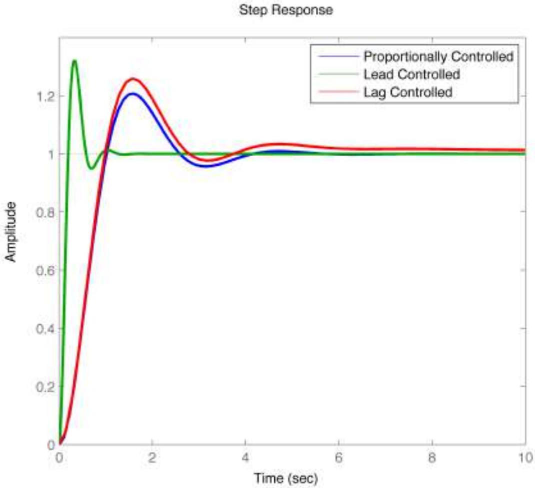
Ramp response



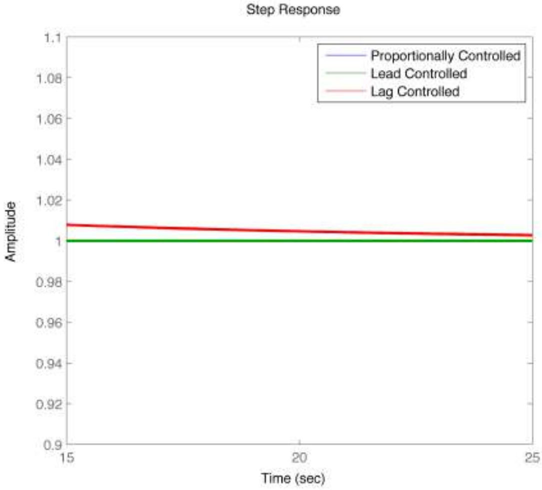
Ramp response, detail



Step response

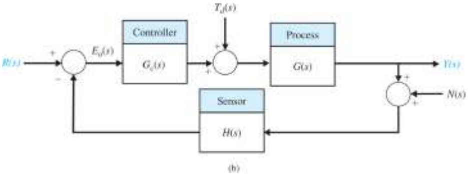


Step response, detail



Anything else to consider?

Anything else to consider?

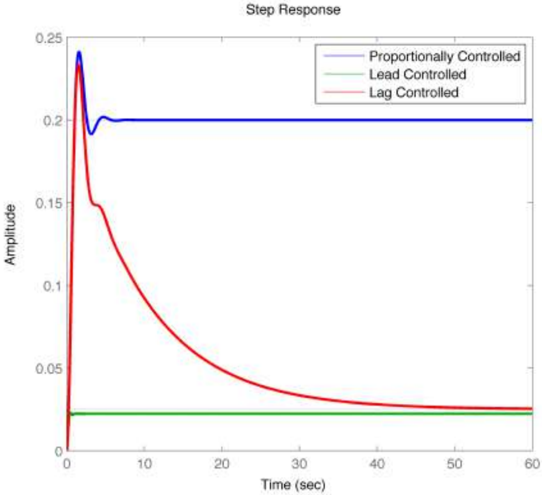


With $H(s) = 1$,

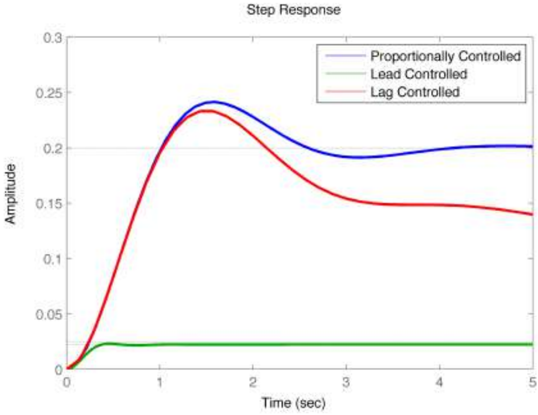
$$Y(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} R(s) + \frac{G(s)}{1 + G_c(s)G(s)} T_d(s) - \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} N(s)$$

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s) - \frac{G(s)}{1 + G_c(s)G(s)} T_d(s) + \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} N(s)$$

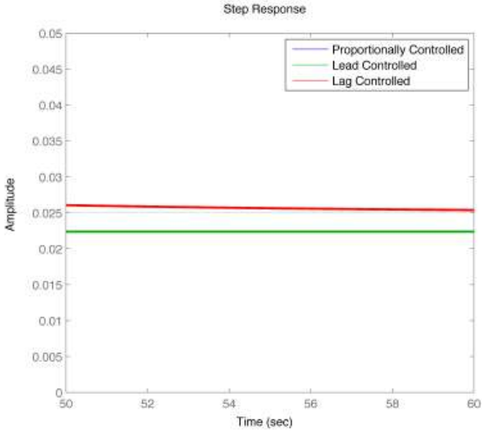
Response to step disturbance



Response to step disturbance, detail early

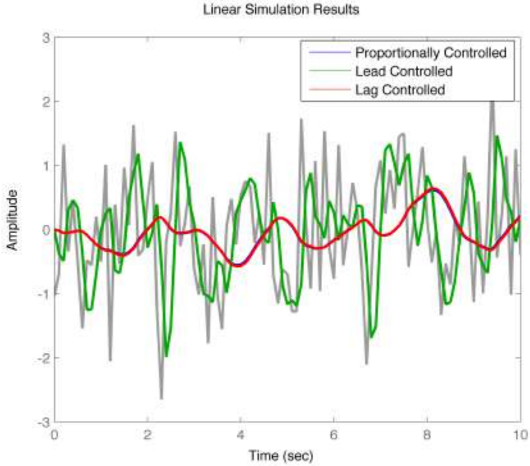


Response to step disturbance, detail late

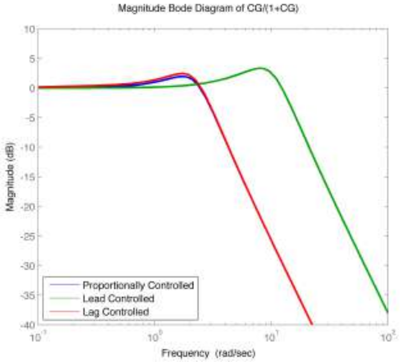


Homework: Show that e_{ss} for a step disturbance is 0.2, 0.0225 and 0.025 for prop., lead, lag, respectively

Error due to Gaussian sensor noise



Bode diagram of $G_C(s)G(s)/(1+G_C(s)G(s))$



- Prop. and lag designs do a better job at filtering out the higher frequency noise components
- You could also see this bandwidth diff. in open loop Bode plots
- Reduced bandwidth also means slower step and ramp responses

Insights

- If we would like to improve the transient performance of a closed loop
  - We can try to place the dominant closed-loop poles in desired positions
  - One approach to doing that is lead compensator design
  - However, that typically requires the use of an amplifier in the compensator, and hence requires a power supply
  - Broadening of bandwidth improves transient performance but exposes loop to noise

- If we would like to improve the steady-state error performance of a closed loop without changing the dominant transient features too much
  - We can consider designing a lag compensator to provide the required gain
  - However, that typically produces a “weak” slow pole that slows the decay to steady state

What if we want to do more?

- What happens if we want to improve transient performance **and** improve steady-state error?

- For example, what if we want to design a compensator for $G(s) = \frac{1}{s(s+2)}$ that achieves
  1. Specified maximum overshoot; minimum value for ζ
  2. Specified (2%) settling time; largest (least negative) real part of closed loop pole
  3. Specified steady-state error for ramp input; min. value for K_v, related to DC open loop "gain"

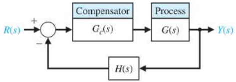
- Lead compensation gives (some) ability to address 1 and 2
- Lag compensation gives (some) ability to address 3

- What should we do?

Lead-lag compensation

- Here is one thing that we can do:
  - **Step 1:** Design a lead compensator $G_{c,\text{lead}}(s)$ for the process $G(s)$ to change the shape of the root locus and choose the gain so that the poles are in the desired position
  - **Step 2:** Design a lag compensator, $G_{c,\text{lag}}(s)$ to leave the dominant closed-loop poles of the lead-compensated process $\tilde{G}(s) = G_{c,\text{lead}}(s)G(s)$ in approximately the position but provide extra low-frequency gain
- This is called a lead-lag controller: $G_c(s) = G_{c,\text{lead-lag}}(s) = G_{c,\text{lag}}(s)G_{c,\text{lead}}(s)$

Lead-Lag Comp. example



Consider a case with $G(s) = \frac{1}{s(s+2)}$ and $H(s) = 1$.
Design a compensator to achieve:

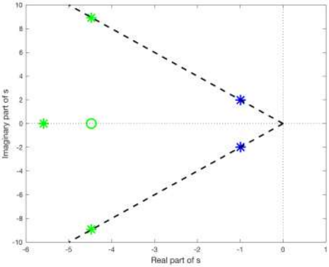
- damping coefficient $\zeta \approx 0.45$
- dominant poles with real parts ≈ -4.5, so that they correspond to a 2% setting time of $\approx \frac{4}{4.5} \sim 0.9$s
- velocity error constant $K_v = \lim_{s \to 0} sG_c(s)G(s) \geq 40$

What to do?

- Our second lead compensator (with the green root locus), $G_{c,\text{lead}}(s) = \frac{125(s+4.47)}{(s+12.5)}$, achieves the first two requirements
- However, that design has $K_v \approx 22.3$
- Now design a lag compensator to increase K_v to 40

Lead-Lag Design

- $G_{c,\text{lead}}(s)G(s)$ has $K_v \approx 22.3$.
- Lag compensator must increase this to around 40. Therefore, we need $z_{\text{lag}}/p_{\text{lag}} \approx 1.8$.
- Looking at the closed loop poles of lead compensated plant (green, see also table on slide 33),

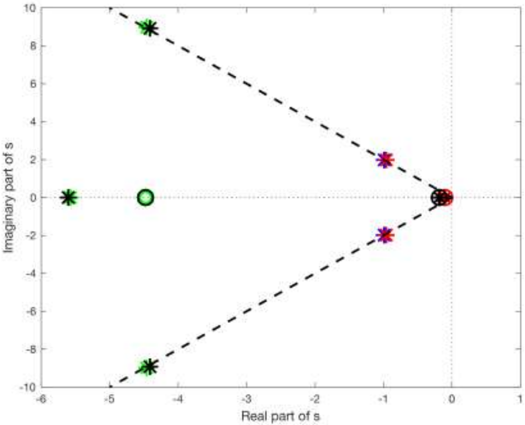


we can try $z_{\text{lag}} = 0.18$, $p_{\text{lag}} = 0.1$.

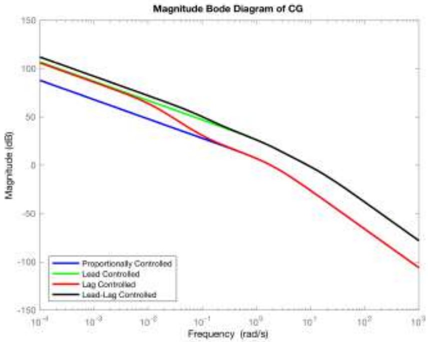
- Therefore $G_{c,\text{lead-lag}}(s) = \frac{125(s+0.18)(s+4.47)}{(s+0.1)(s+12.5)}$

Prop, Lead, Lag, Lead-Lag Design Comparisons

Closed-loop pole and zero positions



Bode, open loop, $G_c(j\omega)G(j\omega)$



- Recall $K_v = \lim_{s \to 0} sG_c(s)G(s)$
- At low freq. slope is -20dB/dec. Hence $G_c(s)G(s) \approx \frac{A}{s}$. Hence, $K_v = A$.
- Since $G_{c,\text{lead-lag}}(s) = \frac{125(s+0.18)(s+4.47)}{(s+0.1)(s+12.5)}$, $A_{\text{lead-lag}} = 40.23$
- By comparison with slide 70 (and as seen in plot), $A_{\text{lead-lag}} > A_{\text{lead}} \gtrsim A_{\text{lag}} > A_{\text{prop}}$

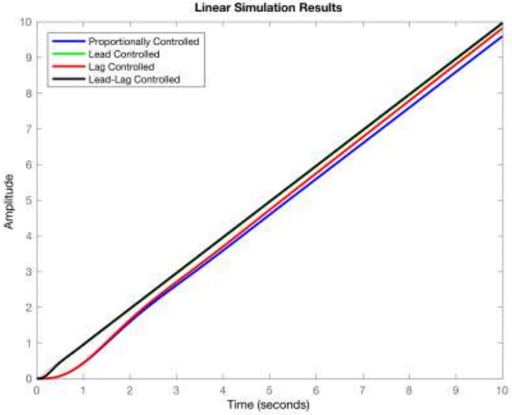
Lead, Lead-Lag Comparisons

Prop. and Lag designs are on slide 71

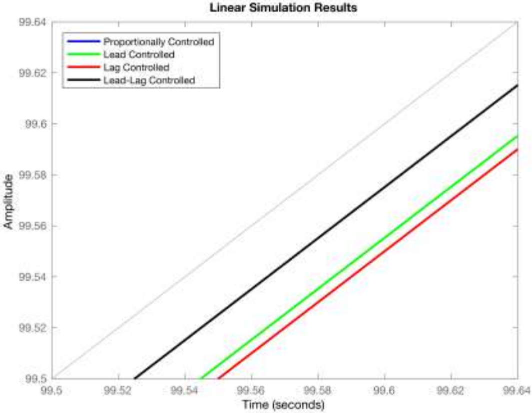
	Lead contr.	**Lead-Lag contr.**
$G_C(s)$	$\frac{125(s+4.47)}{(s+12.5)}$	$\frac{125(s+0.18)(s+4.47)}{(s+0.1)(s+12.5)}$
$\frac{Y(s)}{R(s)}$	$\frac{131(1+0.0130s)}{s^2+8.94s+100} - \frac{1.71}{s+5.59}$	$\frac{131(1+0.0132s)}{s^2+8.82s+99.46} - \frac{1.73}{s+5.60} + \frac{6.15\times10^{-4}}{s+0.1806}$
CL poles	$-4.47 \pm j8.94, -5.59$	$-4.41 \pm j8.95, -5.60, -0.1806$
CL zeros	$-4.47, \infty, \infty$	$-4.47, -0.18, \infty, \infty$
$1/K_v$	0.045	0.0249

- Lead-lag design retains similar CL poles to lead design, plus a "slow" pole with very small residue
- Lead-lag will have smaller steady-state error for a ramp input.
- Anything else? Recall larger low-frequency gain

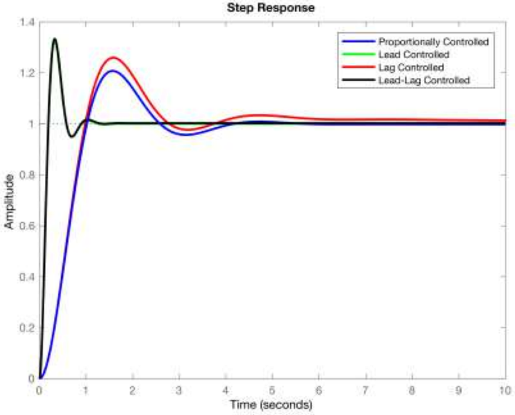
Ramp response



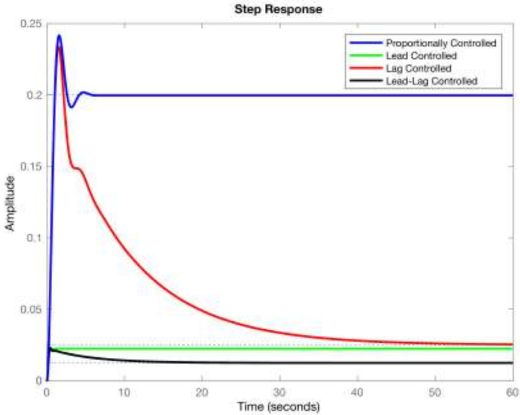
Ramp response, detail



Step response

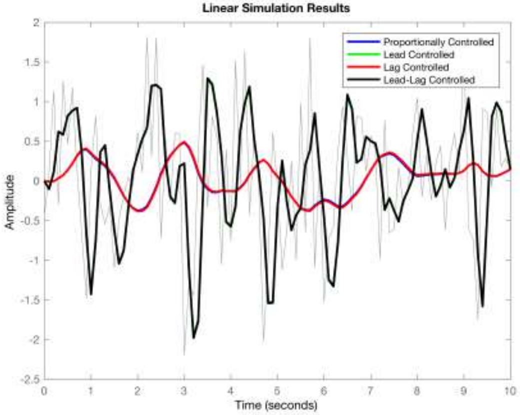


Response to step disturbance

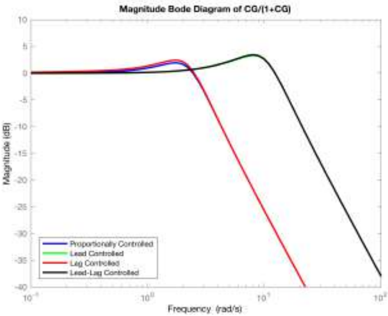


Note reduced steady-state disturbance error of lead-lag design. This is due to larger K_v, which comes from larger low-frequency “gain”

Error due to Gaussian sensor noise



Bode diagram of $G_C(s)G(s)/(1 + G_C(s)G(s))$



- Prop. and lag designs do a better job at filtering out the higher frequency noise components
- You could also see this bandwidth diff. in open loop Bode plots
- Reduced bandwidth also means slower step and ramp responses

A prelude to frequency-domain design

- In our design process there were connections between performance measures and the frequency responses of the open loop and the closed loop.
- Perhaps we might be able to build a design technique around Bode magnitude and phase diagrams of the open-loop transfer function, rather than the open-loop poles and zeros