Lecture - 12

Prob. 2 : Derive DIT FFT flow graph for N = 4 hence find DFT of x (n) = {1, 2, 3, 4}

Soln. :

First stage of decimation :

We have the equations for first stage of decimation.

$$X(k) = F_1(k) + W_N^k F_2(k), \qquad k = 0, 1 \ldots \frac{N}{2} - 1 \qquad \ldots(1)$$

and $$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \qquad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(2)$$

Here N = 4

$$\therefore \quad X(k) = F_1(k) + W_4^k F_2(k), \qquad k = 0, 1 \qquad \ldots(3)$$

and $$X(k+2) = F_1(k) - W_4^k F_2(k), \qquad k = 0, 1 \qquad \ldots(4)$$

Putting values of k in Equation (3) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_4^0 F_2(0) \\ \text{and} \quad X(1) &= F_1(1) + W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(5)$$

Similarly putting values of k in Equation (4) we get,

$$\left.\begin{aligned} X(2) &= F_1(0) - W_4^0 F_2(0) \\ \text{and} \quad X(3) &= F_1(1) - W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(6)$$

This signal flow graph is shown in Fig. G-12(a)

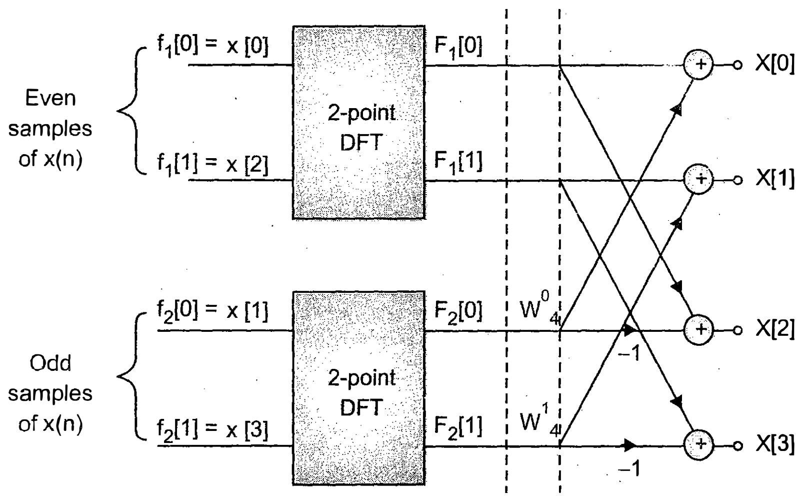


Fig. G-12(a)

Now we will replace each 2-point DFT by butterfly structure as shown in Fig. G-12(b).

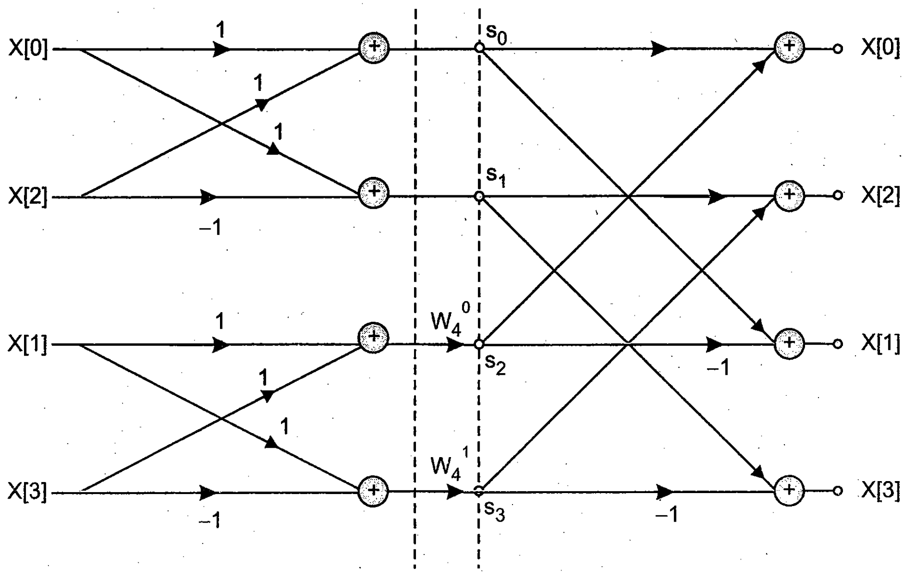


Fig. G-12(b)

The given sequence is,

$$x(n) = \{1, 2, 3, 4\}$$

The different values of twiddle factor are as follows :

$$W_4^0 = 1$$

$$W_4^1 = e^{-\frac{j2\pi}{4} \cdot 1} = e^{-\frac{j\pi}{2}}$$

$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

The output s (n) is,

$$s_0 = x(0) + x(2) = 1 + 3 = 4$$

$$s_1 = x(0) - x(2) = 1 - 3 = -2$$

$$s_2 = [x(1) + x(3)]\, W_4^0 = 2 + 4 = 6$$

$$s_3 = [x(1) - x(3)]\, W_4^1 = (2 - 4) \cdot (-j) = 2j$$

The final output is,

$$X(0) = s_0 + s_2 = 4 + 6 = 10$$

$$X(1) = s_1 + s_3 = -2 + j2$$

$$X(2) = s_0 - s_2 = 4 - 6 = -2$$

$$X(3) = s_1 - s_3 = -2 - j2$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3)\}$$

$$\therefore X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

Prob. 3 : Let x (n) be a finite duration sequence of length 8 such that

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$

(a) Find X (k) using DITFFT flow graph.

(b) Using the result in (a) and not otherwise find DFT of sequence $x_1(n) = \{-1, 2, -4, +2\}$. Justify your answer.

(c) Using result in (b) find DFT of sequence

$x_2(n) = \{-4, +2, -1, +2\}$

Soln. :

(a) This flow graph is as shown in Fig. G-11(a)

Here $s_1(n)$ represents output of stage - 1 and $s_2(n)$ represents output of stage - 2. The different values of twiddle factor are as follows :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-\frac{j\pi}{2}} = -j$$

$$W_8^3 = e^{-\frac{j3\pi}{4}} = -0.707 - j\,0.707$$

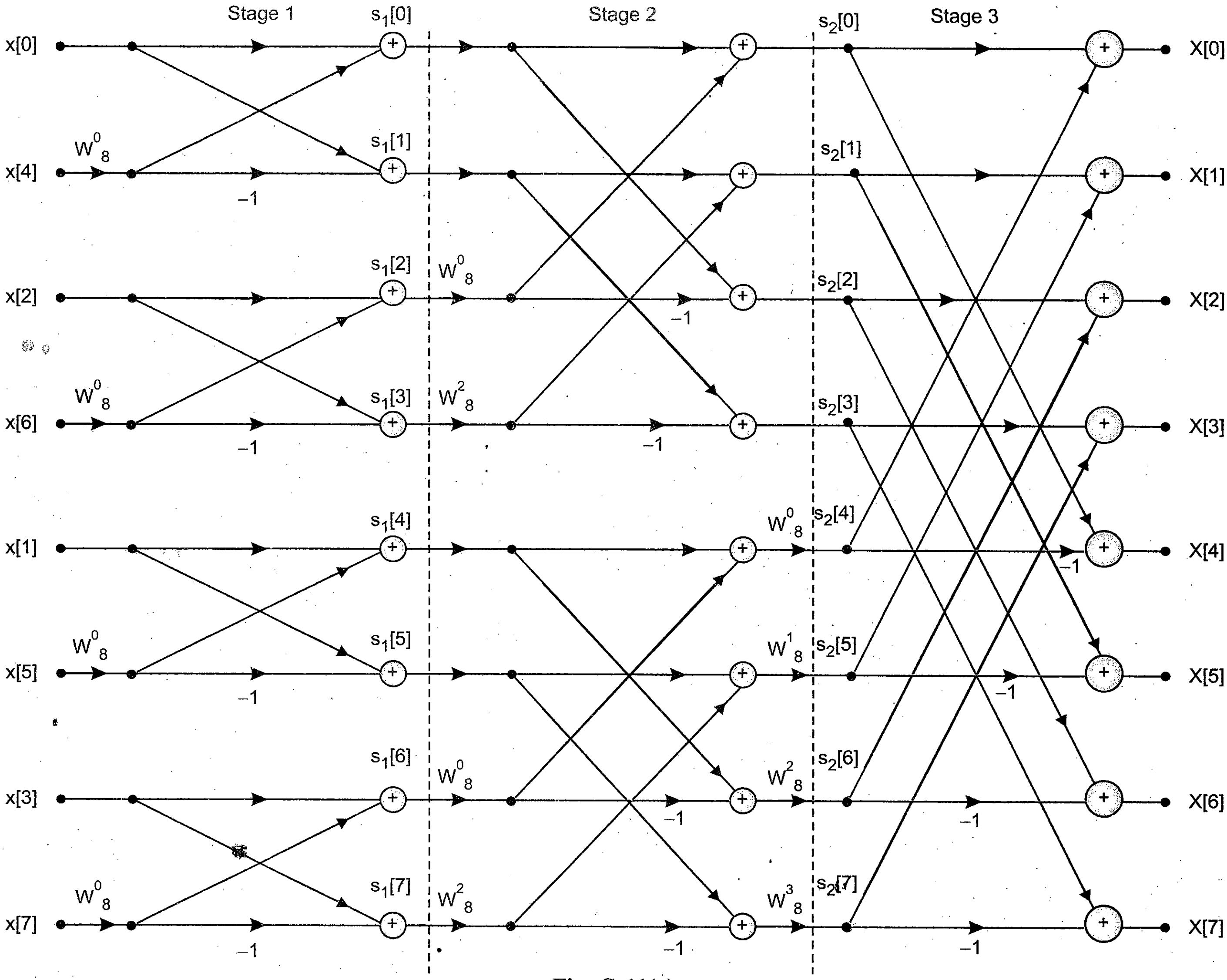
Output of stage - 1 :

$$s_1(0) = x(0) + W_8^0\, x(4) = -1 + 1\cdot(-4) = -5$$

$$s_1(1) = x(0) - W_8^0\, x(4) = -1 - 1(-4) = 3$$

$$s_1(2) = x(2) + W_8^0\, x(6) = 2 + 1\cdot(2) = 4$$

$$s_1(3) = x(2) - W_8^0\, x(6) = +2 - 1\cdot(2) = 0$$



Stage 1
Stage 2
Stage 3
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
W0 8
−1
s1[0]
s1[1]
s1[2]
s1[3]
s1[4]
s1[5]
s1[6]
s1[7]
W2 8
s2[0]
s2[1]
s2[2]
s2[3]
s2[4]
s2[5]
s2[6]
s2[7]
W1 8
W3 8
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

Fig. G-11(a)

$$s_1(4) = x(1) + W_8^0 x(5) = 0 + 1 \cdot (0) = 0$$

$$s_1(5) = x(1) - W_8^0 x(5) = 0 - 1(0) = 0$$

$$s_1(6) = x(3) + W_8^0 x(7) = 0$$

$$s_1(7) = x(3) - W_8^0 x(7) = 0 - 1 \cdot (0) = 0$$

Output of stage - 2 :

$$s_2(0) = s_1(0) + W_8^0 s_1(2) = -5 + 1 \cdot (4) = -1$$

$$s_2(1) = s_1(1) + W_8^2 s_1(3) = 3 - j(0) = 3$$

$$s_2(2) = s_1(0) - W_8^0 s_1(2) = -5 - 1 \cdot (4) = -9$$

$$s_2(3) = s_1(1) - W_8^2 s_1(3) = 3 + j(0) = 3$$

$$s_2(4) = s_1(4) + W_8^0 s_1(6) = 0 + 1 \cdot (0) = 0$$

$$s_2(5) = s_1(5) + W_8^2 s_1(7) = 0 - j(0) = 0$$

$$s_2(6) = s_1(4) - W_8^0 s_1(6) = 0 - 1 \cdot (0) = 0$$

$$s_2(7) = s_1(5) - W_8^2 s_1(7) = 0 + j(0) = 0$$

Final output :

$$X(0) = s_2(0) + W_8^0 s_2(4) = -1 + 1 \cdot (0) = -1$$

$$X(1) = s_2(1) + W_8^1 s_2(5) = 3 + (0.707 - j\,0.707) \cdot 0 = 3$$

$$X(2) = s_2(2) + W_8^2 s_2(6) = -9 - j(0) = -9$$

$$X(3) = s_2(3) + W_8^3 s_2(7) = 3 + (-0.707 - j\,0.707) \cdot 0 = 3$$

$$X(4) = s_2(0) - W_8^0 s_2(4) = -1 - 1 \cdot (0) = -1$$

$$X(5) = s_2(1) - W_8^1 s_2(5) = 3 - 1 \cdot (0) = 3$$

$$X(6) = s_2(2) - W_8^2 s_2(6) = -9 + j(0) = -9$$

$$X(7) = s_2(3) - W_8^3 s_2(7) = 3 - (-0.707 - j\,0.707) \cdot 0 = 3$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\therefore X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$$

(b) Let $x(n) = \{a, b, c, d\}$ and let its DFT be the sequence $X(k) = \{A, B, C, D\}$. If we add one zero after each sample in $x(n)$ then we will get the sequence.

$$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$$

This process is called as upsampling process. Since in this sequence one zero is added after each sample; the entire DFT repeats one time. If we will add two zeros after each sample then entire DFT will repeat two times.

$$\therefore \quad \text{DFT}\{x_1(n)\} = X_1(k) = \{A, B, C, D, A, B, C, D\}$$

In part (a) for the sequence $x(n)$.

$$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$$

We have obtained the DFT,

$$X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$$

Observe that first four DFT samples are repeated only once. This is because in $x(n)$, zero is added after each sample.

The given sequence is,

$$x_1(n) = \{-1, 2, -4, 2\}$$

So its DFT is,

$$X_1(k) = \{-1, 3, -9, 3\}$$

Justification of answer :

We will prove the property of DFT used in this example. Let $x(n) = \{a, b, c, d\}$ and $X(k) = \{A, B, C, D\}$.

Consider the sequence,

$$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$$

According to the definition of DFT we can write,

$$X_1(k) = \sum_{n=0}^{7} x_1(n) \cdot W_8^{kn} \qquad \text{...(1)}$$

We will divide the sequence $x_1(n)$ into odd part and even part. Let $x_1(2n)$ represent even part and $x_1(2n+1)$ represent odd part.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n)\, W_8^{2kn} + \sum_{n=0}^{3} x_1(2n+1)\, W_8^{(2n+1)\cdot k} \qquad \text{...(2)}$$

Observe that in the first summation 'n' is replaced by 2n and in the second summation 'n' is replaced by $(2n+1)$. But in the second summation $x_1(2n+1)$ represents odd samples of sequence $x_1(n)$ and all these samples are zero.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n)\, W_8^{2kn} \qquad \text{..(3)}$$

Now we have the property of twiddle factor.

$$W_N^{2kn} = W_{N/2}^{kn}$$

$$\therefore \quad W_8^{2kn} = W_4^{kn}$$

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x_1(2n)\cdot W_4^{kn} \qquad \text{...(4)}$$

But $x_1(2n)$ represents even samples of $x_1(n)$. That means $x_1(2n) = x(n)$.

$$\therefore \quad X_1(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} = X(k)$$

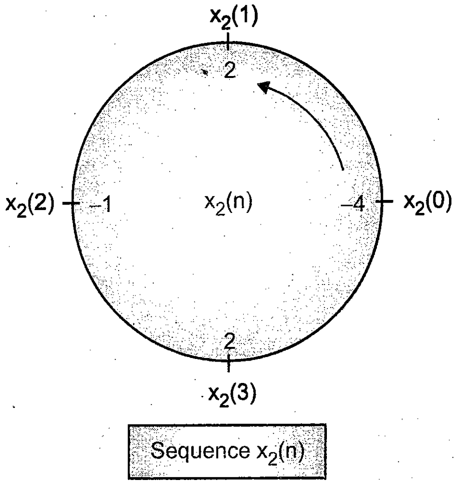
But $x_1(n)$ is eight point sequence.

$$\therefore \quad X_1(k) = \{A, B, C, D, A, B, C, D\}$$

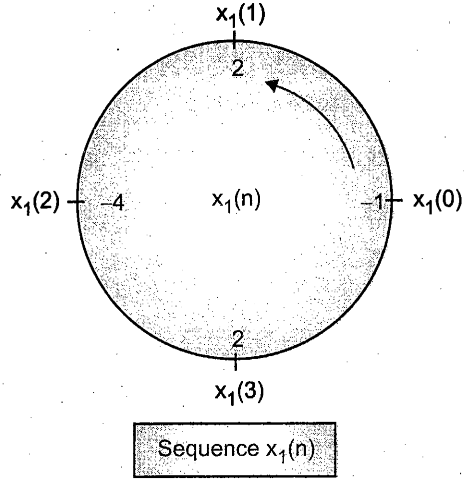
(c) Here $\quad x_2(n) = \{-4, 2, -1, 2\}$

We have $\quad x_1(n) = \{-1, 2, -4, 2\}$

Let us plot the sequences $x_1(n)$ and $x_2(n)$ as shown in Fig. G-11(b) and G-11(c).



(b)



(c)

Fig. G-11

From these diagrams we can conclude that $x_2(n)$ is obtained by circularly rotating $x_1(n)$ by 2 positions in anticlockwise direction. That means $x_2(n)$ is obtained by delaying $x_1(n)$ by 2 positions.

$$\therefore \quad x_2(n) = x_1((n-2))$$

Now according to circular time shifting property,

$$x((n-l))_N \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

Thus in this case we can write,

$$X_2(k) = X_1(k) \cdot W_4^{2k} = X_1(k) \cdot e^{-\frac{j2\pi}{4}\cdot k}$$

$$\therefore \quad X_2(k) = e^{-\frac{j\pi}{2}\cdot k} \cdot X_1(k)$$

We have, $\quad X_1(k) = \{-1, 3, -9, 3\}$

We will calculate sequence $X_2(k)$ for different values of k as follows :

For $k = 0 \Rightarrow X_2(0) = e^0 \cdot X_1(0) = -1$

For $k = 1 \Rightarrow X_2(1) = e^{-j\pi} X_1(1) = (\cos\pi - j\sin\pi) \cdot 3 = -3$

For $k = 2 \Rightarrow X_2(2) = e^{-j\pi 2} X_1(2) = (\cos 2\pi - j\sin 2\pi) \cdot (-9) = -9$

For $k = 3 \Rightarrow X_2(3) = e^{-j\pi 3} X_1(3) = (\cos 3\pi - j\sin 3\pi) \cdot 3 = -3$

$$\therefore \quad X_2(k) = \{X_2(0), X_2(1), X_2(2), X_2(3)\}$$

$$\therefore \quad X_2(k) = \{-1, -3, -9, -3\}$$

Prob. 4 : Draw flow diagram of DITFFT for N = 16.

Soln. :

(1) Here N = 16, means it is 16 point DFT.

(2) Total number of stages = 4

(3) The first stage of decimation using two 8-point DFT is shown in Fig. G-13(a).

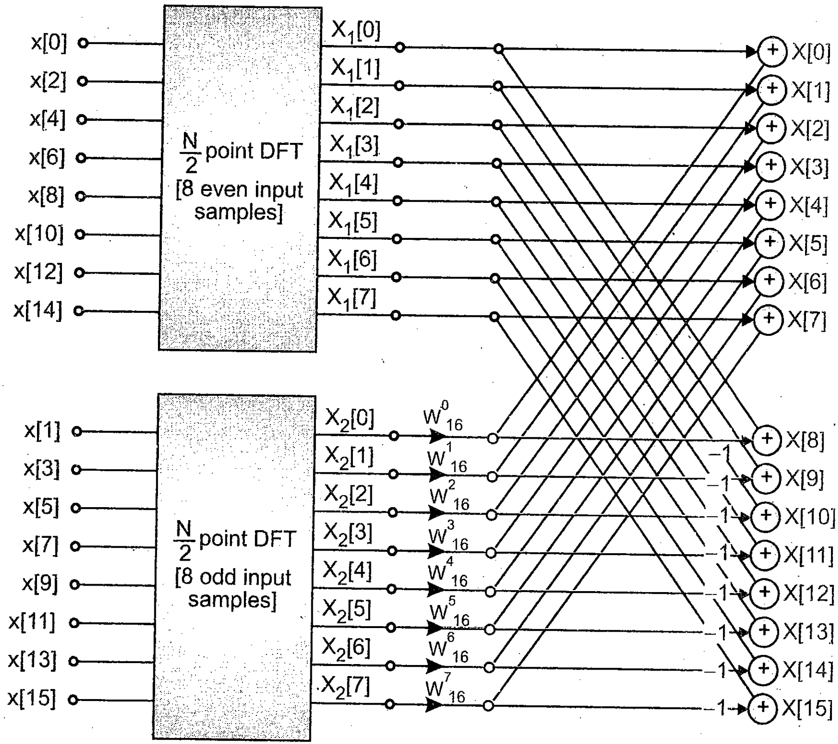


Fig. G-13(a)

(4) In the second stage each 8 point DFT is divided into 2 four point DFTs as shown in Fig. G-13(b).

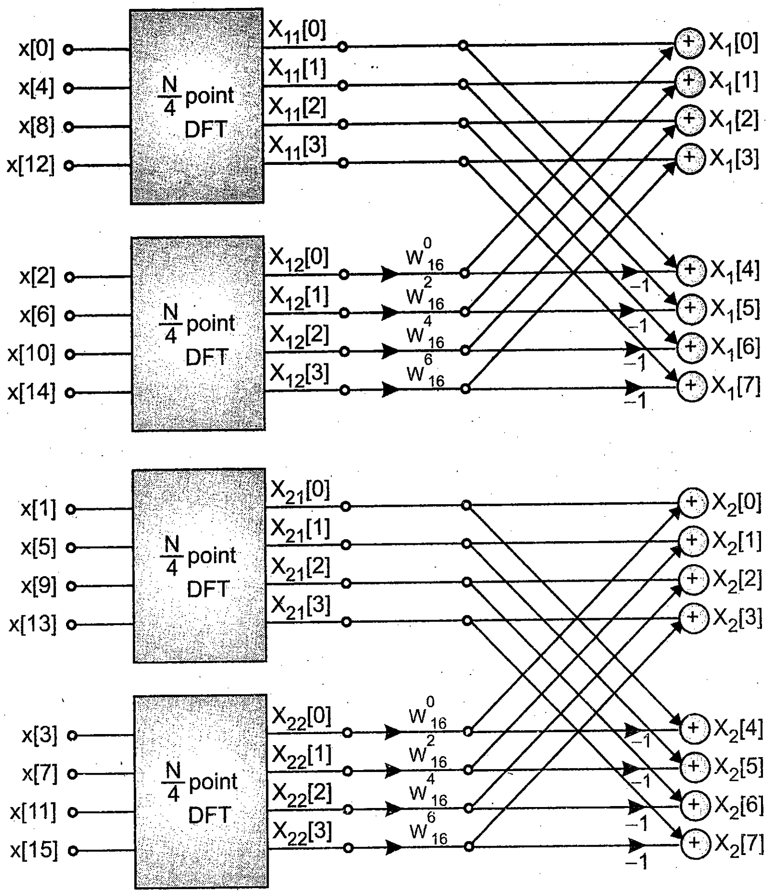
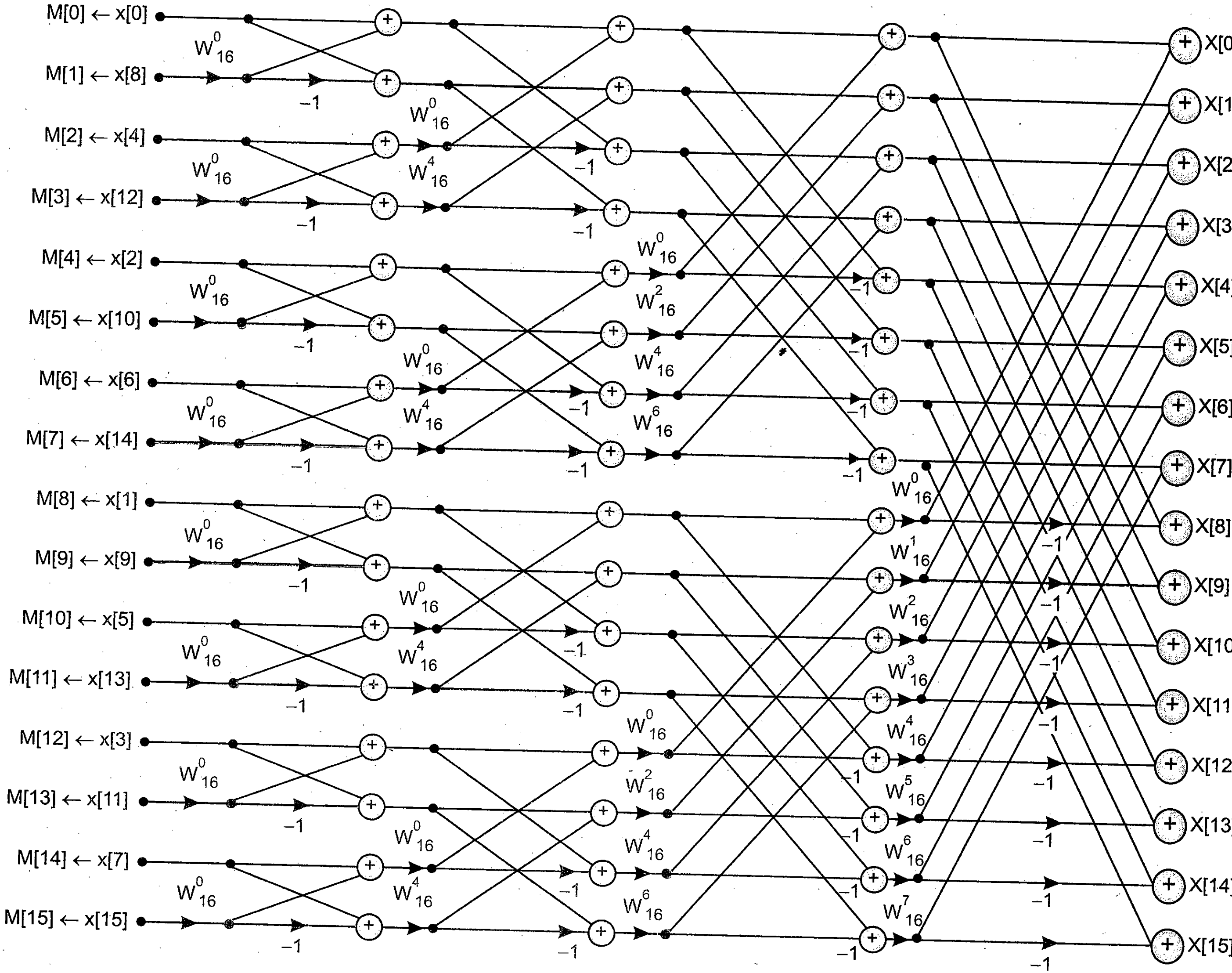


Fig. G-13(b)



M[0] ← x[0]
M[1] ← x[8]
M[2] ← x[4]
M[3] ← x[12]
M[4] ← x[2]
M[5] ← x[10]
M[6] ← x[6]
M[7] ← x[14]
M[8] ← x[1]
M[9] ← x[9]
M[10] ← x[5]
M[11] ← x[13]
M[12] ← x[3]
M[13] ← x[11]
M[14] ← x[7]
M[15] ← x[15]
W_{16}^{0}
W_{16}^{1}
W_{16}^{2}
W_{16}^{3}
W_{16}^{4}
W_{16}^{5}
W_{16}^{6}
W_{16}^{7}
−1
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
X[8]
X[9]
X[10]
X[11]
X[12]
X[13]
X[14]
X[15]

Fig. G-13(c)