ELECTROSTATIC

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INTRODUCTION

- **Electrosatic** is fundamental concepts that are applicable to static(or time-invariant) electric fields in free space(or vacuum).An electrostatic field is produced by a static charge distribution.
- Study of electrostatics by investigating the two fundamental laws governing electrostatic fields:
  - Coulomb's law, and
  - Gauss'slaw.

COULOMB'S LAW

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is:

1. Along the line joining them
2. Directly proportional to the product Q_1Q_2 of the charges
3. Inversely proportional to the square of the distance R between them.[3]

Expressed mathematically,

$$F = \frac{k\,Q_1Q_2}{R^2}$$

Where k is the proportionality constant. In SI units, charges are in coulombs (C),The distance R is in meters(m), and the force F is in newton (N) so that

$$\boxed{\begin{aligned} \varepsilon_o &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \text{F/m} \\ \text{or } k &= \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^{9}\ \text{m/F} \end{aligned}}$$

VECTOR FORM OF COULOMB'S LAW

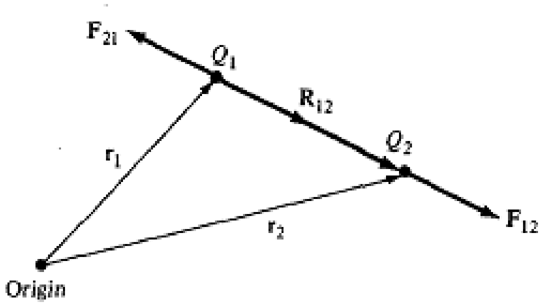
$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\,\mathbf{a}_{R_{12}}}$$

where

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$$

$$R = |\mathbf{R}_{12}|$$

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{R}$$



therefore

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\,\mathbf{R}_{12}$$

or

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o |\mathbf{r}_2 - \mathbf{r}_1|^3}$$

Coulomb's Law
(Imp. Pts to be noted)

1. $$\mathbf{F}_{21} = -\mathbf{F}_{12}$$
2. Like charges (charges of the same sign) repel each other while unlike charges attract. This is illustrated in Figure 4.2.
3. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charges.
4. Q_1 and Q_2 must be static (at rest).
5. The signs of Q_1 and Q_2 must be taken into account

Principle of Superposition

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$\mathbf{F} = \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_n)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o} \sum_{k=1}^{N} \frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}}$$

Electric Field Intensity

The **electric field intensity** (or **electric field strength**) **E** is the force per unit charge when placed in the electric field.

Thus

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q}$$

or simply

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{Q}}$$

The electric field intensity E is obviously in the direction of the force F and is measured in newton/coulomb or volts/meter.

Electric Field Intensity

- The electric field intensity at point r due to a point charge located at r ' is

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}}$$

For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained from eqs. (4.8) and (4.10) as

$$\mathbf{E} = \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r})_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}$$

Example 1

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10-nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$\mathbf{F} = \sum_{k=1,2} \frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2} \frac{QQ_k(\mathbf{r} - \mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_k|^3}$$

$$= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0,3,1) - (3,2,-1)]}{|(0,3,1) - (3,2,-1)|^3} - \frac{2.10^{-3}[(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,4)|^3}\right\}$$

$$= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\dfrac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right]$$

$$= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right]$$

$$\mathbf{F} = -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}$$

Example 1

At that point,

$$\mathbf{E} = \frac{\mathbf{F}}{Q}$$

$$= (-6.507, -3.817, 7.506) \cdot \frac{10^{-3}}{10 \cdot 10^{-9}}$$

$$\mathbf{E} = -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}$$

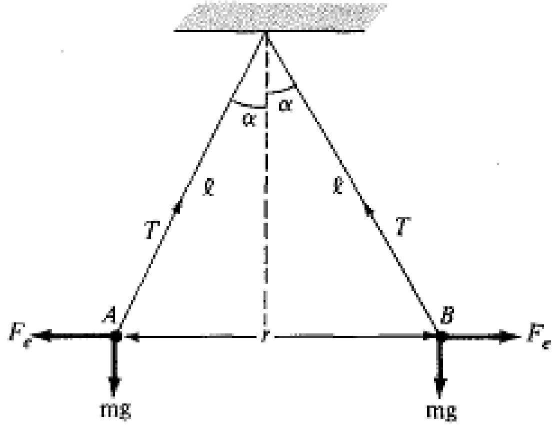
EXAMPLE 2

Two point charges of equal mass m, charge Q are suspended at a common point by two threads of negligible mass and length ℓ. Show that at equilibrium the inclination angle α of each thread to the vertical is given by

$$Q^2 = 16\pi\,\varepsilon_o mg\ell^2 \sin^2\alpha \tan\alpha$$

If α is very small, show that

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_o mg\ell^2}}$$



Example 2

Solution:

Consider the system of charges as shown in Figure 4.3 where F_e is the electric or coulomb force, T is the tension in each thread, and mg is the weight of each charge. At A or B

$$T \sin \alpha = F_e$$

$$T \cos \alpha = mg$$

Hence,

$$\frac{\sin \alpha}{\cos \alpha} = \frac{F_e}{mg} = \frac{1}{mg} \cdot \frac{Q^2}{4\pi\varepsilon_o r^2}$$

But

$$r = 2\ell \sin \alpha$$

Hence,

$$Q^2 \cos \alpha = 16\pi\varepsilon_o mg\ell^2 \sin^3 \alpha$$

or

$$Q^2 = 16\pi\varepsilon_o mg\ell^2 \sin^2 \alpha \tan \alpha$$

as required. When α is very small

$$\tan \alpha \simeq \alpha \simeq \sin \alpha$$

Example 2

and so

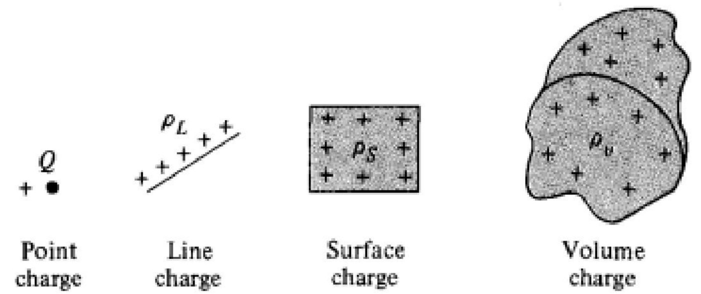
$$Q^2 = 16\pi\varepsilon_0 mg\ell^2\alpha^3$$

or

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_0 mg\ell^2}}$$

ELECTRIC FIELD DUE TO CONTINOUS CHARGE DISTRIBUTION

- So far we have only considered forces and electric fields due to point charges, which are essentially charges occupying very small physical space.
- It is also possible to have continuous charge distribution along a line, on a surface, or in a volume.
  - Line charge density by *pL* (in C/m),
  - surface charge density by *ps* (in C/m2), and
  - volume charge density and *pv* (in C/m3),



Electric field due to continous charge distribution

- The charge element dQ and total charge Q due to these continuous charge distribution are

$$dQ = \rho_L \, dl \rightarrow Q = \int_L \rho_L \, dl \qquad \text{(line charge)}$$

$$dQ = \rho_S \, dS \rightarrow Q = \int_S \rho_S \, dS \qquad \text{(surface charge)}$$

$$dQ = \rho_v \, dv \rightarrow Q = \int \rho_v \, dv \qquad \text{(volume charge)}$$

Electric field due to continous charge distribution

- Electric field due to this charge distribution will be

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(line charge)}$$

$$\mathbf{E} = \int \frac{\rho_S \, dS}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(surface charge)}$$

$$\mathbf{E} = \int \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(volume charge)}$$

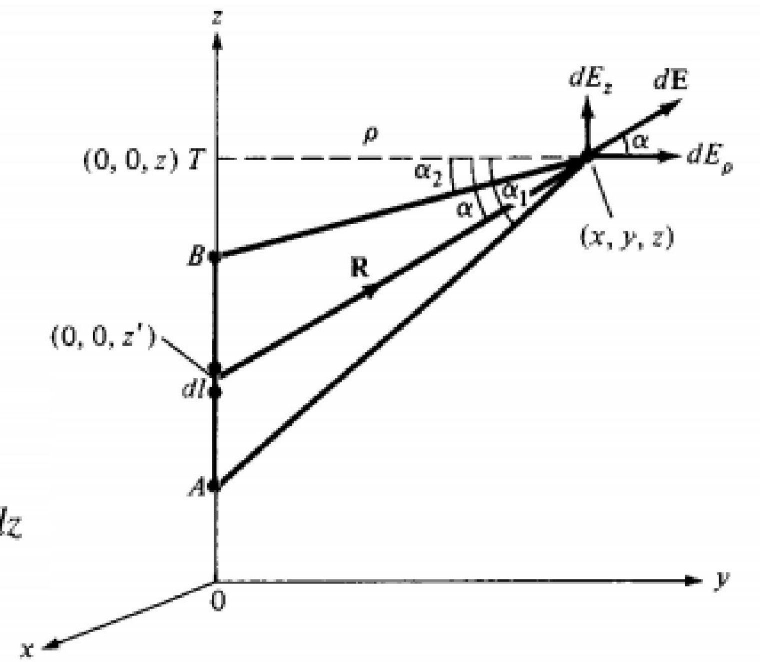
A LINE CHARGE

- Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in Figure.
- The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

and hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz$$



A LINE CHARGE

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(line charge)}$$

$$dl = dz'$$

$$\mathbf{R} = (x, y, z) - (0, 0, z') = x\mathbf{a}_x + y\mathbf{a}_y + (z - z')\mathbf{a}_z$$

or

$$\mathbf{R} = \rho\mathbf{a}_\rho + (z - z')\,\mathbf{a}_z$$

$$R^2 = |\mathbf{R}|^2 = x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{\rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

A Line charge

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int \frac{\rho \mathbf{a}_\rho + (z - z')\,\mathbf{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}\, dz'$$

To evaluate this, it is convenient that we define α, α_1, and α_2 as in Figure

$$R = [\rho^2 + (z - z')^2]^{1/2} = \rho \sec \alpha$$

$$z' = OT - \rho \tan \alpha, \qquad dz' = -\rho \sec^2 \alpha \, d\alpha$$

Hence, eq. becomes

$$\mathbf{E} = \frac{-\rho_L}{4\pi\varepsilon_o} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \,[\cos \alpha\, \mathbf{a}_\rho + \sin \alpha\, \mathbf{a}_z]\, d\alpha}{\rho^2 \sec^2 \alpha}$$

$$= -\frac{\rho_L}{4\pi\varepsilon_o \rho} \int_{\alpha_1}^{\alpha_2} [\cos \alpha\, \mathbf{a}_\rho + \sin \alpha\, \mathbf{a}_z]\, d\alpha$$

Thus for a *finite line charge,*

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o \rho} [-(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\rho + (\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_z]$$

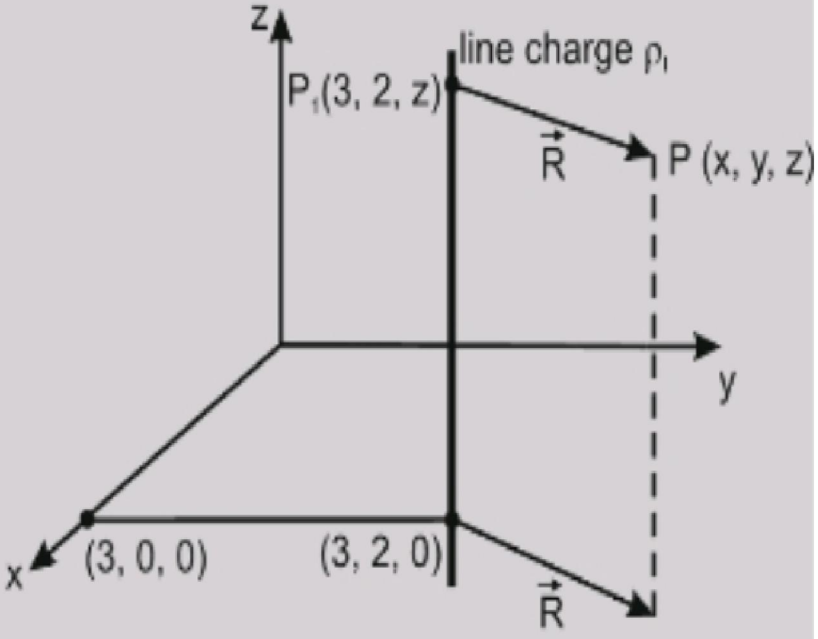
A LINE CHARGE

As a special case, for an *infinite line charge,* point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$; the z-component vanishes and eq. (4.20) becomes

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\,\mathbf{a}_\rho}$$

Example

An infinite uniform line charge is placed parallel to the z-axis at $x = 3$ and $y = 2$. A point $P(x, y, z)$ is located at a distance R from the line charge as shown in the following figure. Find the electric field intensity $\vec{E}$ at point P.



Example

Solution We know that the electric field intensity $\vec{E}$ for the infinite line charge along z-axis is:

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_o\rho}\hat{a}_\rho$$

Replacing ρ in the above equation by the radial distance R between the line charge and the point P and assuming that $\hat{a}_\rho = \hat{a}_R$, we can write from the figure that:

$$\vec{R} = (x, y, z) - (3, 2, z) = (x-3)\,\hat{a}_x + (y-2)\,\hat{a}_y$$

$$|\vec{R}| = \sqrt{(x-3)^2 + (y-2)^2}$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{(x-3)\,\hat{a}_x + (y-2)\,\hat{a}_y}{\sqrt{(x-3)^2 + (y-2)^2}}$$

Thus,

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_o}\,\frac{(x-3)\,\hat{a}_x + (y-2)\,\hat{a}_y}{(x-3)^2 + (y-2)^2}$$

EXAMPLE

A uniform line charge density of 10 nC/m is located at $x = 0$, $y = 2$, while another uniform line charge density of -10 nC/m is located at $x = 0$, $y = -2$. Determine the electric field at the origin.

Solution As we are given two infinitely long charged lines, the total electric field $\vec{E}$ at any point is the vector sum of the electric field at that point due to individual line charges. That is,

$$\vec{E} = \frac{\rho_{l_1}}{2\pi\varepsilon_o\rho_1}\hat{a}_{\rho l_1} + \frac{\rho_{l_2}}{2\pi\varepsilon_o\rho_2}\hat{a}_{\rho l_2}$$

Given that $\rho_{l_1} = 10$ nC/m and $\rho_{l_2} = -10$ nC/m. Also, the location of ρ_{l_1} is $x = 0$, $y = 2$ and the location of ρ_{l_2} is $x = 0$, $y = -2$, and we have to evaluate electric field at the origin, that is, at (0, 0, 0). Therefore:

EXAMPLE

$$\rho_1 = \sqrt{0^2 + 2^2} = 2 \quad \text{and} \quad \rho_2 = \sqrt{0^2 + (-2)^2} = 2$$

$$\hat{a}_{\rho l_1} = \frac{\vec{\rho}_{l_1}}{\rho_1} = \frac{2\hat{a}_y}{2} = \hat{a}_y \quad \text{and} \quad \hat{a}_{\rho l_2} = \frac{\rho_{l_2}}{\rho_2} = \frac{-2\hat{a}_y}{2} = -\hat{a}_y$$

As we know that $\dfrac{1}{4\pi\varepsilon_o} = 9 \times 10^9$ m/F. This implies that $\dfrac{1}{2\pi\varepsilon_o} = 18 \times 10^9$ m/F.

Thus, $\vec{E}$ at the origin is computed as follows.

$$\vec{E} = \frac{10 \times 10^{-9} \times 18 \times 10^9}{2}\hat{a}_y + \frac{-10 \times 10^{-9} \times 18 \times 10^9}{2}(-\hat{a}_y) = 180\hat{a}_y \ \text{V/m}$$

Surface Charge

- Consider an infinite sheet of charge in the xy-plane with uniform charge density *ps*. The charge associated with an elemental area *dS* is

$$dQ = \rho_S \, dS$$

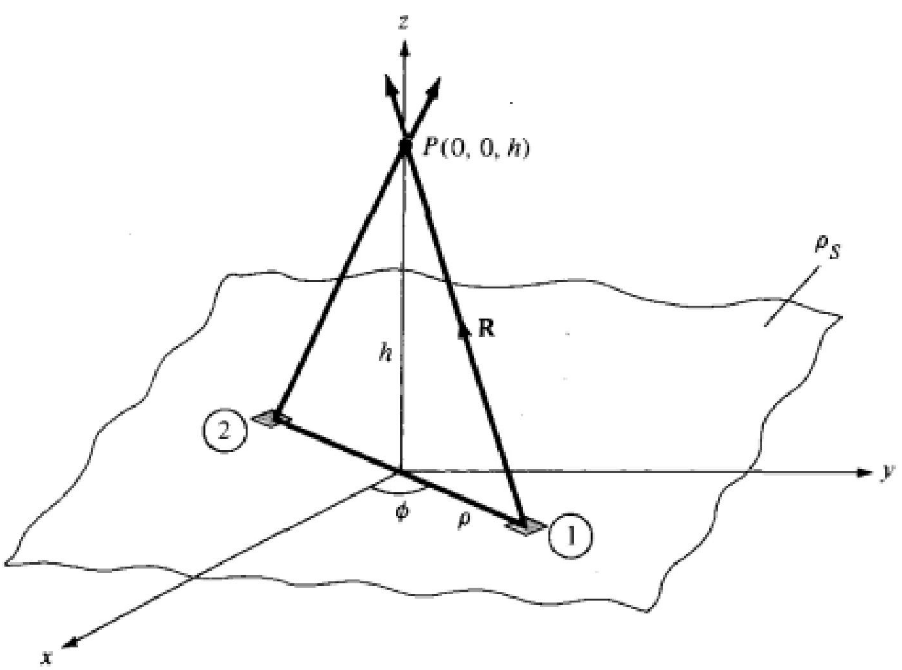
and hence the total charge is

$$Q = \int \rho_S \, dS$$

SURFACE CHARGE

- The contribution to the E field at point P(0, 0, *h)* by the elemental surface 1 shown in Figure

$$d\mathbf{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R$$



SURFACE CHARGE

$$\mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z, \qquad R = |\mathbf{R}| = \left[\rho^2 + h^2\right]^{1/2}$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{R}, \qquad dQ = \rho_S\, dS = \rho_S\, \rho\, d\phi\, d\rho$$

Substitution of these terms gives

$$d\mathbf{E} = \frac{\rho_S\, \rho\, d\phi\, d\rho\left[-\rho\mathbf{a}_\rho + h\mathbf{a}_z\right]}{4\pi\varepsilon_o\left[\rho^2 + h^2\right]^{3/2}}$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\mathbf{a}_\rho$ cancels that of element 1, as illustrated in Figure . Thus the contributions to E_ρ add up to zero so that E has only z-component.

SURFACE CHARGE

$$\mathbf{E} = \int d\mathbf{E}_z = \frac{\rho_S}{4\pi\varepsilon_o} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}}\, \mathbf{a}_z$$

$$= \frac{\rho_S h}{4\pi\varepsilon_o} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} d(\rho^2)\, \mathbf{a}_z$$

$$= \frac{\rho_S h}{2\varepsilon_o} \left\{ -[\rho^2 + h^2]^{-1/2} \right\}_0^{\infty} \mathbf{a}_z$$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\, \mathbf{a}_z$$

$\angle\varepsilon_o$

that is, **E** has only z-component if the charge is in the xy-plane. In general, for an *infinite sheet* of charge

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\, \mathbf{a}_n}$$

Surface Charge

- In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_n + \frac{-\rho_S}{2\varepsilon_o}(-\mathbf{a}_n) = \frac{\rho_S}{\varepsilon_o}\mathbf{a}_n$$

EXAMPLE

An infinite sheet in xy-plane extending from $-\infty$ to ∞ in both directions has a uniform charge density of 5 nC/m^2. Find the electric field at $z = 10$ cm.

Solution: Given that $\rho_s = 5 \text{ nC/m}^2 = 5 \times 10^{-9} \text{ C/m}^2$

As the given sheet lies in xy-plane and the field point is on z-axis, the electric field $\vec{E}$ can be given as:

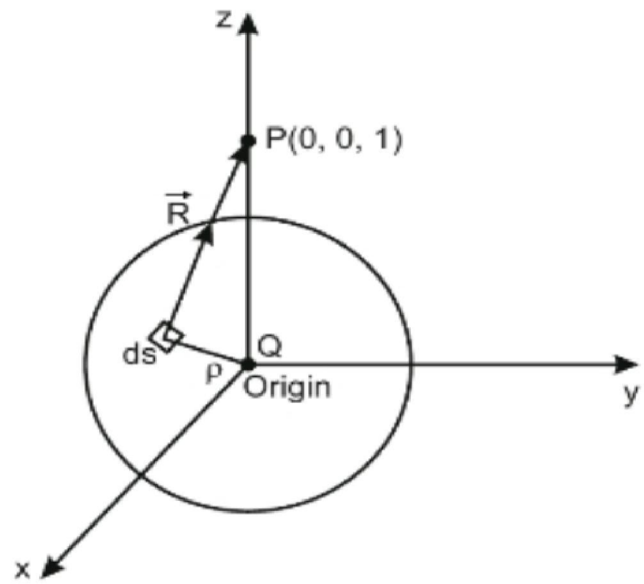
$$\vec{E} = \frac{\rho_s}{2\varepsilon_o} \hat{a}_z$$

$$\Rightarrow \quad \vec{E} = \frac{5 \times 10^{-9}}{2 \times 8.854 \times 10^{-12}} \hat{a}_z = 282.3 \hat{a}_z \text{ V/m}$$

EXAMPLE

A disk located at $0 < \rho < 1$, $z = 1$ has a uniform charge distribution $\rho_s = 200$ pC/m^2. A 30 μC point charge Q is located at (0, 0, 0). Find the force on Q due to electric field produced by the disk.

Solution Consider small elemental area ds of the finite uniformly charged disk as shown in the following figure



If the field point P on z-axis ($z = 1$) is at a distance R from ds, then observe from the figure that:

EXAMPLE

$$\vec{R} = -\rho\hat{a}_\rho + 1\hat{a}_z = -\rho\hat{a}_\rho + \hat{a}_z$$

$\Rightarrow$

$$|\vec{R}| = \sqrt{(-\rho)^2 + 1} = \sqrt{1+\rho^2}$$

the differential surface in cylindrical coordinate system along z-direction is:

$$ds = \rho d\phi d\rho$$

Now, the electric field at a point due to sheet charge distribution can be obtained as:

$$\vec{E} = \int_s \frac{\rho_s ds}{4\pi\varepsilon_o R}\hat{a}_R = \int_s \frac{\rho_s ds\,\vec{R}}{4\pi\varepsilon_o R^3}$$

On putting values of ds, R, and $\vec{R}$, we have:

$$\vec{E} = \int_s \frac{\rho_s(-\rho\hat{a}_\rho + \hat{a}_z)\,\rho d\phi d\rho}{4\pi\varepsilon_o(1+\rho^2)^{3/2}}$$

EXAMPLE

Due to symmetry, the electric field along ρ-direction contributes to zero and we have electric field only along the z-direction. Thus, ignoring the $\hat{a}_\rho$ component in the above equation and taking only $\hat{a}_z$ component, we get E_z as shown below.

$$E_z = \frac{\rho_s}{4\pi\varepsilon_o} \int_s \frac{\rho d\phi d\rho}{(1+\rho^2)^{3/2}}$$

$$= \frac{\rho_s}{4\pi\varepsilon_o} \int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{1} \frac{\rho}{(1+\rho^2)^{3/2}} d\rho = \frac{\rho_s}{4\pi\varepsilon_o} [\phi]_0^{2\pi} \int_{\rho=0}^{1} \frac{\rho}{(1+\rho^2)^{3/2}} d\rho \quad [\text{Given } 0 < \rho < 1]$$

To solve the integral, let us take $1 + \rho^2 = v$. This gives $\frac{dv}{d\rho} = 2\rho \Rightarrow dv = 2\rho d\rho$. Thus, E_z becomes:

$$E_z = \frac{\rho_s}{4\pi\varepsilon_o}(2\pi)\frac{1}{2}\int v^{-3/2}\,dv = -\frac{\rho_s}{4\varepsilon_o}[2v^{-1/2}] = -\frac{\rho_s}{2\varepsilon_o}[v^{-1/2}]$$

Replacing v with $(1 + \rho^2)$ in the above equation and using limits $0 < \rho < 1$, we get:

$$E_z = -\frac{\rho_s}{2\varepsilon_o}\left[\frac{1}{\sqrt{1+\rho^2}}\right]_0^1 = \frac{\rho_s}{2\varepsilon_o}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Hence, $$\vec{E} = \frac{\rho_s}{2\varepsilon_o}\left(1 - \frac{1}{\sqrt{2}}\right)\hat{a}_z$$

EXAMPLE

Now, $\vec{F} = Q\vec{E}$. Given that $Q = 30$ μC and $\rho_s = 200$ pC/m^2. Therefore:

$$\vec{F} = \frac{30\times10^{-6}\times200\times10^{-12}}{2\times\dfrac{10^{-9}}{36\pi}}\left(1-\frac{1}{\sqrt{2}}\right)\hat{a}_z = 99.24\,\hat{a}_z \ \mu\text{N}$$

EXAMPLE

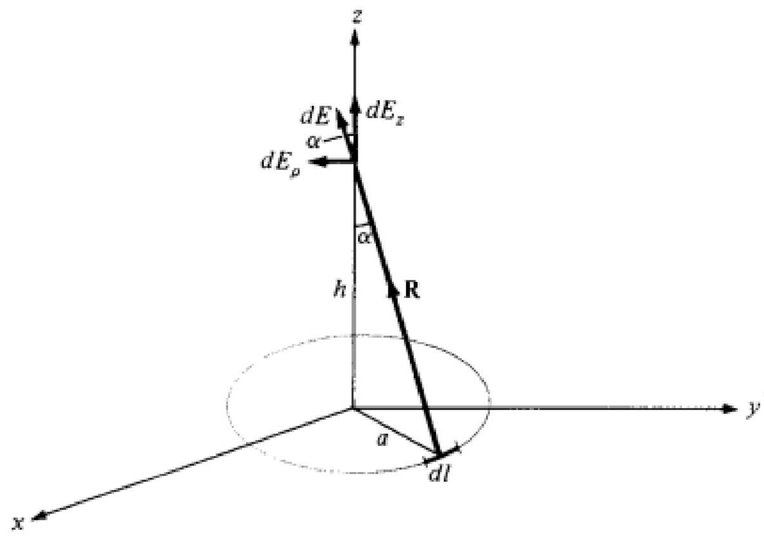
A circular ring of radius a carries a uniform charge ρ_L C/m and is placed on the xy-plane with axis the same as the z-axis.

(a) Show that

$$\mathbf{E}(0, 0, h) = \frac{\rho_L a h}{2\varepsilon_o[h^2 + a^2]^{3/2}}\,\mathbf{a}_z$$

(b) What values of h gives the maximum value of $\mathbf{E}$?

(c) If the total charge on the ring is Q, find $\mathbf{E}$ as $a \to 0$.



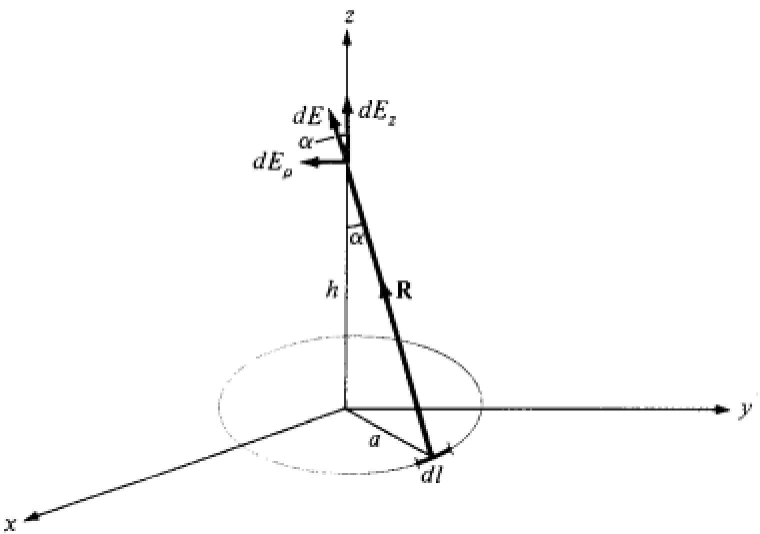
SOLUTION

$$dl = a\,d\phi, \qquad \mathbf{R} = a(-\mathbf{a}_\rho) + h\mathbf{a}_z$$

$$R = |\mathbf{R}| = [a^2 + h^2]^{1/2}, \qquad \mathbf{a}_R = \frac{\mathbf{R}}{R}$$

or

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{-a\mathbf{a}_\rho + h\mathbf{a}_z}{[a^2 + h^2]^{3/2}}$$



$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\frac{(-a\mathbf{a}_\rho + h\mathbf{a}_z)}{[a^2 + h^2]^{3/2}}\,a\,d\phi$$

SOLUTION

- By symmetry radial components are cancelled out and field is only due to Z axis

$$\mathbf{E} = \frac{\rho_L a h \mathbf{a}_z}{4\pi\varepsilon_o[h^2 + a^2]^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h \mathbf{a}_z}{2\varepsilon_o[h^2 + a^2]^{3/2}}$$

(b)

$$\frac{d|\mathbf{E}|}{dh} = \frac{\rho_L a}{2\varepsilon_o}\left\{\frac{[h^2 + a^2]^{3/2}(1) - \frac{3}{2}(h)2h[h^2 + a^2]^{1/2}}{[h^2 + a^2]^3}\right\}$$

For maximum $\mathbf{E}$, $\frac{d|\mathbf{E}|}{dh} = 0$, which implies that

$$[h^2 + a^2]^{1/2}[h^2 + a^2 - 3h^2] = 0$$

$$a^2 - 2h^2 = 0 \qquad \text{or} \qquad h = \pm\frac{a}{\sqrt{2}}$$

SOLUTION

(c) Since the charge is uniformly distributed, the line charge density is

$$\rho_L = \frac{Q}{2\pi a}$$

so that

$$\mathbf{E} = \frac{Qh}{4\pi\varepsilon_o\left[h^2 + a^2\right]^{3/2}}\,\mathbf{a}_z$$

As $a \to 0$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o h^2}\,\mathbf{a}_z$$

or in general

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_R$$

which is the same as that of a point charge as one would expect.

Example

The finite sheet $0 \leq x \leq 1$, $0 \leq y \leq 1$ on the $z = 0$ plane has a charge density $\rho_S = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2. Find

(a) The total charge on the sheet

(b) The electric field at (0, 0, 5)

(c) The force experienced by a -1 mC charge located at (0, 0, 5)

Solution:

(a) $$Q = \int \rho_S \, dS = \int_0^1 \int_0^1 xy(x^2 + y^2 + 25)^{3/2} \, dx \, dy \text{ nC}$$

Since $x\,dx = 1/2\, d(x^2)$, we now integrate with respect to x^2

SOLUTION

$$Q = \frac{1}{2}\int_0^1 y \int_0^1 (x^2 + y^2 + 25)^{3/2}\, d(x^2)\, dy \text{ nC}$$

$$= \frac{1}{2}\int_0^1 y \frac{2}{5}(x^2 + y^2 + 25)^{5/2}\Big|_0^1 dy$$

$$= \frac{1}{5}\int_0^1 \frac{1}{2}\left[(y^2 + 26)^{5/2} - (y^2 + 25)^{5/2}\right] d(y^2)$$

$$= \frac{1}{10}\cdot\frac{2}{7}\left[(y^2 + 26)^{7/2} - (y^2 + 25)^{7/2}\right]\Big|_0^1$$

$$= \frac{1}{35}\left[(27)^{7/2} + (25)^{7/2} - 2(26)^{7/2}\right]$$

$$Q = 33.15 \text{ nC}$$

SOLUTION

(b) $\mathbf{E} = \int \frac{\rho_S \, dS \, \mathbf{a}_R}{4\pi\varepsilon_o r^2} = \int \frac{\rho_S \, dS \, (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o |\mathbf{r} - \mathbf{r}'|^3}$

where $\mathbf{r} - \mathbf{r}' = (0, 0, 5) - (x, y, 0) = (-x, -y, 5)$. Hence,

$$\mathbf{E} = \int_0^1 \int_0^1 \frac{10^{-9} xy(x^2 + y^2 + 25)^{3/2}(-x\mathbf{a}_x - y\mathbf{a}_y + 5\mathbf{a}_z)dx\, dy}{4\pi \cdot \frac{10^{-9}}{36\pi}(x^2 + y^2 + 25)^{3/2}}$$

$$= 9\left[-\int_0^1 x^2\, dx \int_0^1 y\, dy\, \mathbf{a}_x - \int_0^1 x\, dx \int_0^1 y^2 dy\, \mathbf{a}_y + 5\int_0^1 x\, dx \int_0^1 y\, dy\, \mathbf{a}_z\right]$$

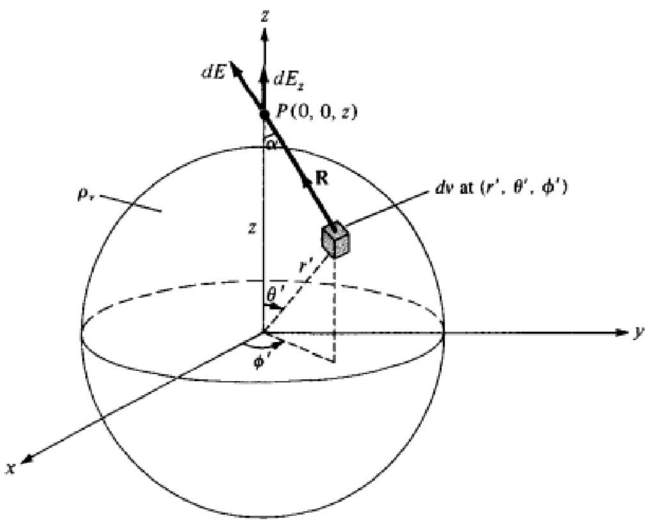
$$= 9\left(\frac{-1}{6}, \frac{-1}{6}, \frac{5}{4}\right)$$

$$= (-1.5, -1.5, 11.25) \text{ V/m}$$

(c) $\mathbf{F} = q\mathbf{E} = (1.5, 1.5, -11.25)$ mN

Volume Charge

- Let the volume charge distribution with uniform charge density *pv* be as shown in Figure
- The charge dQ associated with the elemental volume dv is



$$dQ = \rho_v \, dv$$

and hence the total charge in a sphere of radius a is

$$Q = \int \rho_v \, dv = \rho_v \int dv$$

$$= \rho_v \frac{4\pi a^3}{3}$$

VOLUME CHARGE

The electric field $d\mathbf{E}$ at $P(0, 0, z)$ due to the elementary volume charge is

$$d\mathbf{E} = \frac{\rho_v\, dv}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R$$

where $\mathbf{a}_R = \cos\alpha\,\mathbf{a}_z + \sin\alpha\,\mathbf{a}_\rho$. Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero. We are left with only E_z, given by

$$E_z = \mathbf{E}\cdot\mathbf{a}_z = \int dE\cos\alpha = \frac{\rho_v}{4\pi\varepsilon_o}\int\frac{dv\cos\alpha}{R^2}$$

Again, we need to derive expressions for dv, R^2, and $\cos\alpha$.

$$dv = r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

Applying the cosine rule to Figure we have

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

Volume Charge

Substituting

$$\cos \alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

$$\cos \theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

$$\sin \theta' \, d\theta' = \frac{R \, dR}{z \, r'}$$

$$\begin{aligned} E_z &= \frac{\rho_v}{4\pi\varepsilon_0} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{R\,dR}{zr'} \, dr' \, \frac{z^2 + R^2 - r'^2}{2zR} \frac{1}{R^2} \\ &= \frac{\rho_v 2\pi}{8\pi\varepsilon_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 - r'^2}{R^2} \right] dR \, dr' \\ &= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a r' \left[R - \frac{(z^2 - r'^2)}{R} \right]_{z-r'}^{z+r'} dr' \\ &= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a 4r'^2 \, dr' = \frac{1}{4\pi\varepsilon_0} \frac{1}{z^2} \left(\frac{4}{3} \pi a^3 \rho_v \right) \end{aligned}$$

or

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 z^2} \mathbf{a}_z$$