Lecture - 15

1.7 Computation of DFT by Linear Filtering Approach :

We have studied that Radix-2 FFT algorithms are used to compute 'N'-point DFT. In this case, $\frac{N}{2}\log_2 N$ complex multiplications and $N\log_2 N$ complex additions are required to compute N-point DFT.

But there are certain applications in which selected number of DFT values are required; but not the entire DFT. In such cases the direct computation of DFT is more efficient than FFT algorithms. The direct computation of DFT for selected values can be realized using linear filtering approach.

This type of computation can be performed using the algorithm, called as Goertzel algorithm.

1.7.1 Goertzel Algorithm :

In this algorithm the periodicity property of twiddle factor (W_N^k) is used.

Now recall the definition of DFT,

$$X(k) = \sum_{m=0}^{N-1} x(m) W_N^{km} \qquad \text{...(1)}$$

Note that here we have used notation x (m) for input sequence.

Now the twiddle factor is given by,

$$W_N = e^{-\frac{j2\pi}{N}}$$

$$\therefore \quad W_N^{-kN} = \left(e^{-\frac{j2\pi}{N}}\right)^{-kN} = e^{+j2\pi k} = \cos 2\pi k + j\sin 2\pi k$$

$$= 1 + j\,0 = 1$$

Since $W_N^{-kN} = 1$ we can multiply Equation (1) by W_N^{-kN}

$$\therefore \quad X(k) = \sum_{m=0}^{N-1} x(m) W_N^{km} \cdot W_N^{-kN}$$

$$\therefore \quad X(k) = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \qquad \text{...(2)}$$

Now according to the equation of linear convolution,

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) h_k(n-m) \qquad \text{...(3)}$$

We know that $h_k(n)$ is an impulse response. Now let,

$$h_k(n) = W_N^{-kn} u(n) \qquad ...(4)$$

Replacing n by 'n – m' we get,

$$h_k(n-m) = W_N^{-k(n-m)} u(n-m) \qquad ...(5)$$

Put this value in Equation (3). Here $u(n-m)$ is unit step. So because of multiplication by $u(n-m)$ the limits of summation will become $m = 0$ to $m = \infty$.

$$\therefore \quad y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)} \qquad ...(6)$$

Now we will calculate $y_k(n)$ at $n = N$.

$$\therefore \quad y_k(n)\Big|_{n=N} = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)} \qquad ...(7)$$

Equations (2) and (7) are identical. Thus we can conclude that DFT, $X(k)$ can be obtained as output of LTI system at $n = N$. ($y_k(n)$). But in this case, we will get $X(k)$ only at one value of K. So we need to use parallel systems to obtain $X(k)$ at different values of K.

Note : Goertzel Algorithm is efficient when $X(k)$ is to be computed at points less than $\log_2^N$.

1.7.2 Computation of DFT of Two Real Sequences using Only One FFT Flow Graph :

Suppose $x_1(n)$ and $x_2(n)$ are two real valued sequences of length N. We will define a complex valued sequence $x(n)$ as,

$$x(n) = x_1(n) + j\,x_2(n), \qquad 0 \le n \le N-1 \qquad ...(1)$$

Since DFT operation is linear, we can write DFT of Equation (1) as,

$$X(k) = X_1(k) + j\,X_2(k) \qquad ...(2)$$

In Equation (1), $x_1(n)$ represents real part of $x(n)$ and $x_2(n)$ represents imaginary part of $x(n)$. Thus we can write,

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \qquad ...(3)$$

$$\text{and} \quad x_2(n) = \frac{x(n) - x^*(n)}{2j} \qquad ...(4)$$

The DFTs of Equations (3) and (4) can be written as,

$$X_1(k) = \frac{1}{2}\left\{ \text{DFT}[x(n)] + \text{DFT}[x^*(n)] \right\} \quad ...(5)$$

and $$X_2(k) = \frac{1}{2j}\left\{ \text{DFT}[x(n)] - \text{DFT}[x^*(n)] \right\} \quad ...(6)$$

According to complex conjugate property we have,

$$x^*(n) \xleftrightarrow[N]{\text{DFT}} X^*((-k))_N = X^*(N-k)$$

$$\therefore \quad X_1(k) = \frac{1}{2}[X(k) + X^*(N-k)] \quad ...(7)$$

and $$X_2(k) = \frac{1}{2j}[X(k) - X^*(N-k)] \quad ...(8)$$

Thus by performing single DFT on complex valued sequence $x(n)$ we can obtain DFT of two real sequences $x_1(n)$ and $x_2(n)$.

Prob. 1 : Find DFT of two real sequences using only one FFT flow graph.

$x_1(n) = \{1, 1, 1, 1\}$

$x_2(n) = \{2, 1, 2, 1\}$

Perform FFT only once.

Soln. :

Consider a sequence $x(n)$ such that,

$$x(n) = x_1(n) + j\,x_2(n)$$

$$\therefore \quad x(n) = \{1 + j2,\ 1 + j1,\ 1 + j2,\ 1 + j1\}$$

Here $x(0) = 1 + j2,\ x(1) = 1 + j1,\ x(2) = 1 + j2,\ x(3) = 1 + j1.$

We will obtain DFT using DITFFT, the flow graph is as shown in Fig. G-23(a).

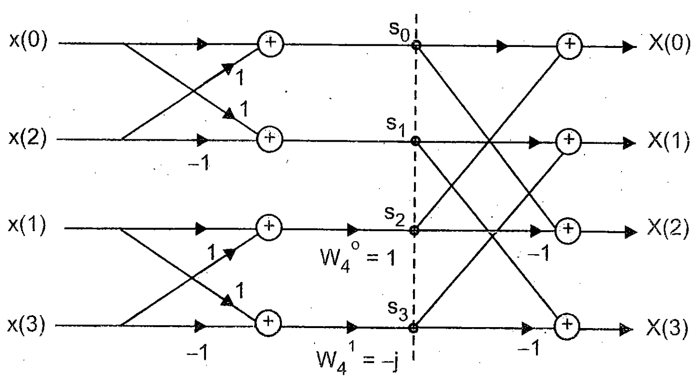

Fig. G-23(a)

$$s_0 = x(0)+x(2) = 1+j2+1+j2 = 2+j4$$

$$s_1 = x(0)-x(2) = 1+j2-1-j2 = 0$$

$$s_2 = [x(1)+x(3)]W_4^0 = [1+j1+1+j1]\cdot 1 = 2+j2$$

$$s_3 = [x(1)-x(3)]W_4^1 = [1+j1-1-j1](-j) = 0$$

The final output is,

$$X(0) = s_0+s_2 = 2+j4+2+j2 = 4+j6$$

$$X(1) = s_1+s_3 = 0$$

$$X(2) = s_0-s_2 = 2+j4-2-j2 = j2$$

$$X(3) = s_1-s_3 = 0$$

$$\therefore \quad X(k) = \{4+j6, 0, j2, 0\} \qquad \text{...(1)}$$

Calculation of $X_1(k)$:

Now we have, $$X_1(k) = \frac{1}{2}[X(k)+X^*(N-k)] \qquad \text{...(2)}$$

Here $N = 4$ and $X^*(k) = \{4-j6, 0, -j2, 0\}$ ...(3)

In Equation (2), $X^*(N-k) = X^*((-k))$ which represents circular folding of $X^*(k)$. It is shown in Fig. G-23(b).

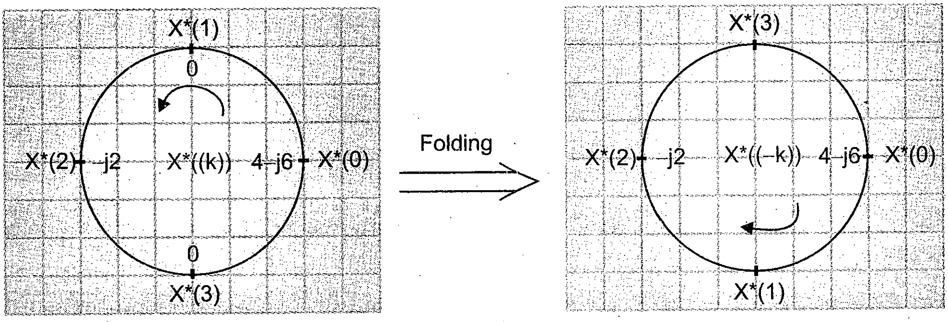

Fig. G-23(b)

$$\therefore \quad X^*(-k) = \{4-j6, 0, -j2, 0\} \qquad \text{...(4)}$$

Using Equation (2) we can obtain DFT $X_1(k)$ as follows :

We have, $$X_1(k) = \frac{1}{2}[X(k)+X^*((-k))]$$

For k = 0 $\Rightarrow$ $X_1(0) = \frac{1}{2}[X(0) + X^*(0)] = \frac{1}{2}[4 + j6 + 4 - j6] = 4$

For k = 1 $\Rightarrow$ $X_1(1) = \frac{1}{2}[X(1) + X^*(1)] = \frac{1}{2}[0 + 0] = 0$

For k = 2 $\Rightarrow$ $X_1(2) = \frac{1}{2}[X(2) + X^*((-2))] = \frac{1}{2}[+j2 - j2] = 0$

For k = 3 $\Rightarrow$ $X_1(3) = \frac{1}{2}[X(3) + X^*((-3))] = \frac{1}{2}[0 + 0] = 0$

$$\therefore\ X_1(k) = \{4, 0, 0, 0\}$$

Calculation of $X_2(k)$:

We have,

$$X_2(k) = \frac{1}{2j}[X(k) - X^*(N-k)] = \frac{1}{2j}[X(k) - X^*((-k))]$$

For k = 0 $\Rightarrow$ $X_2(0) = \frac{1}{2j}[X(0) - X^*(0)] = \frac{1}{2j}[4 + j6 - 4 + j6] = 6$

For k = 1 $\Rightarrow$ $X_2(1) = \frac{1}{2j}[X(1) - X^*((-1))] = \frac{1}{2j}[0 + 0] = 0$

For k = 2 $\Rightarrow$ $X_2(2) = \frac{1}{2j}[X(2) - X^*((-2))] = \frac{1}{2j}[j2 + j2] = 2$

For k = 3 $\Rightarrow$ $X_2(3) = \frac{1}{2j}[X(3) - X^*((-3))] = \frac{1}{2j}[0 - 0] = 0$

$$\therefore\ X_2(k) = \{6, 0, 2, 0\}$$

1.9 Chirp-Z Algorithm :

The chirp-Z algorithm is used to compute $X(Z)$ on the other countours; in the Z-plane. It may include the unit circle.

If we want to calculate $X(Z)$ at a set of points $[Z_k]$ then the equation of Z-transform can be written as,

$$X[Z_k] = \sum_{n=0}^{N-1} x(n) Z_k^{-n} \qquad ...(1)$$

Here let $k = 0, 1, ..., L-1$

and $n = 0, 1, ... N-1$

Now Z can be represented in the polar form with radius of countour 'r' and at N equally spaced points then

$$Z_k = r e^{\frac{j 2\pi kn}{N}}, \qquad k = 0, 1, ..., N-1 \qquad ...(2)$$

Putting this value in Equation (1) we get,

$$X[Z_k] = \sum_{n=0}^{N-1} [x(n) r^{-n}] e^{-j 2\pi kn / N} \qquad ...(3)$$

Now let us say that Z_k fall on an arc which starts at some point $Z_o = r_o e^{j\theta_o}$ and spirals either in; towards the origin or out, away from the origin. Then Z_k can be defined as,

$$Z_k = r_o e^{j\theta_o} (R_o e^{j\phi_o})^k, \qquad k = 0, 1, ..., L-1 \qquad ...(4)$$

Here if $R_o < 1$; the points fall on a contour that spirals towards the origin and if $R_o > 1$ then points fall on a countour that spirals away from the origin. If $R = 1$ then the countour is a circular arc of radius r_o. And when $r_o = R_o = 1$ then the countour is an arc of unit circle.

Now if $r_o = R_o = 1$, $\theta = 0$, $\phi = \frac{2\pi}{N}$ and $L = N$ then the countour is the entire unit circle and frequencies are those of the DFT.

Putting Equation (4) in Equation (1) we get,

$$X[Z^k] = \sum_{n=0}^{N-1} x(n)\left[r_o e^{j\theta_o}\right]^{-n} \cdot \left[R_o e^{j\phi_o}\right]^{-nk}$$

$$\text{Let } V = R_o e^{j\phi_o}$$

$$\therefore \quad X[Z_k] = \sum_{n=0}^{N-1} x(n)[r_o e^{j\theta_o}]^{-n} \cdot V^{-nk} \qquad ...(5)$$

Now the term nk can be expressed as,

$$nk = \frac{1}{2}[n^2 + k^2 - (k-n)^2] \qquad ...(6)$$

Putting this value in Equation (5) we get,

$$X[Z_k] = V^{-k^2/2} \sum_{n=0}^{N-1} \left[x(n)[r_o e^{j\theta_o}]^{-n} \cdot V^{-n^2/2} \cdot V^{(k-n)^2/2}\right] \qquad ...(7)$$

Let us define, $$g(n) = x(n)[r_o e^{j\theta_o}]^{-n} V^{-n^2/2}$$

$$\therefore X[Z_k] = V^{-k^2/2} \cdot \sum_{n=0}^{N-1} g(n) V^{(k-n)^2/2} \qquad ...(8)$$

$$\text{Let } g(n) = \text{Input sequence}$$

and impulse response $$h(n) = V^{n^2/2} \qquad ...(9)$$

$$\therefore \quad h(k-n) = V^{(k-n)^2/2} \qquad ...(10)$$

Thus Equation (8) becomes,

$$X[Z_k] = V^{-k^2/2} \sum_{n=0}^{N-1} g(n) h(k-n) \qquad ...(11)$$

Here the summation term represents the equation of linear convolution. That means it represents the output of digital filter $y(k)$.

$$\therefore \quad X[Z_k] = V^{-k^2/2} \cdot y(k) \qquad \text{...(12)}$$

Now we have $V^{n^2/2} = h(n)$ (from Equation (9))

$$\therefore \quad V^{-k^2/2} = [h(k)]^{-1} = \frac{1}{h(k)}$$

Thus Equation (12) becomes,

$$X[Z_k] = \frac{y(k)}{h(k)}$$

Thus the DFT at discrete point Z_k is the ratio of $\frac{y(k)}{h(k)}$. Now the Z-transform evaluated as in Equation (8) is called as, **Chirp-Z transform.**