Lecture - 20

Digital Filter-FIR Filter Structures

1.1.2 Transfer Function of FIR Filters :

We know that FIR stands for finite impulse response. The difference equation of FIR system is,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \qquad ...(1)$$

Equation (1) shows that the system has length M as the limits of summation are from 0 to M – 1. These limits also indicates that the system is causal.

Taking Z-transform of Equation (1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} X(k) \qquad ...(2)$$

Here we have used time shifting property of Z-transform.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} X(k)$$

Now the system transfer function is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (2) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} b_k Z^{-k} \quad ...(3)$$

Equation (3) is called as system transfer function of FIR filter.

1.2 Direct Form Structure :

The direct form realization of FIR filter can be obtained by using the equation of linear convolution. It is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad ...(1)$$

If we consider that there are 'M' samples then Equation (1) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \quad ...(2)$$

Expanding Equation (2) we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(M-1)\,x(n-M+1) \quad ...(3)$$

How to draw the structure ?

Now we will study how to draw structure (block diagram) for this filter. The first term in Equation (3) is $h(0)\,x(n)$. Here $x(n)$ is input and $h(0)$ is first sample of $h(n)$. So $h(0)$ is constant. That means $h(0)\,x(n)$ indicates that input $x(n)$ is multiplied by constant $h(0)$. It is represented as shown in Fig. H-2(a).

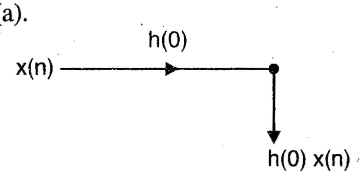


Fig. H-2(a) : h (0) x (n)

The second term in Equation (3) is $h(1)\,x(n-1)$. Here $x(n-1)$ indicates delay of input $x(n)$ by 1 sample. The term $h(1)\,x(n-1)$ is represented as shown in Fig. H-2(b).

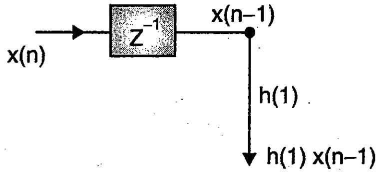


Fig. H-2(b) : h (1) x (n – 1)

Now, the third term is h (2) x (n – 2). Here x (n – 2) indicates delay of input x (n) by two samples. It is represented as shown in Fig. H-2(c).

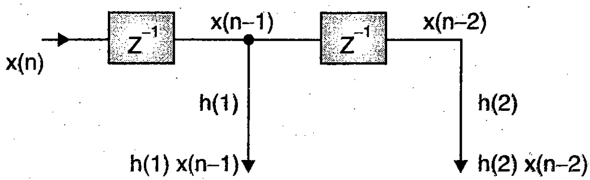


Fig. H-2(c) : h (2) x (n – 2)

Similarly we can draw the block diagram for remaining terms of Equation (3). The total structure for Equation (3) can be drawn by adding the block diagrams for all the terms. This structure is shown in Fig. H-3(a). It is called as direct form realization of FIR system.

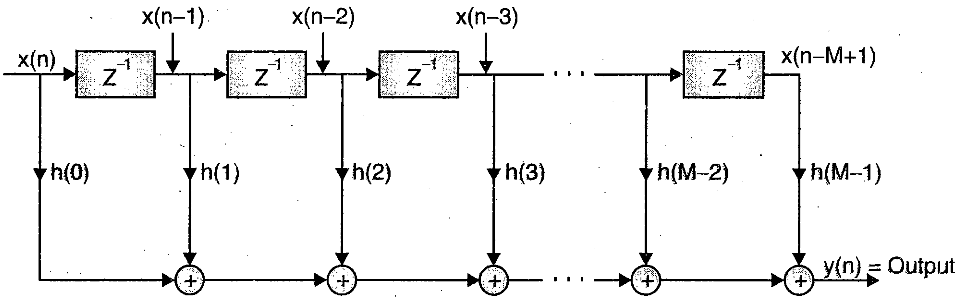


Fig. H-3(a) : Direct form realization of FIR system

Observations :

1. There are 'M – 1' delay blocks. So this is canonic structure.
2. Input signal is delayed 'M – 1' times. So to store this delayed input signal 'M – 1' memory locations are required.
3. This structure has 'M – 1' additions and 'M' multiplications.
4. Equation (3) shows that present input x (n) and past inputs (delayed inputs) e.g. x (n – 1), x (n – 2) etc. are multiplied by the corresponding sample of h (n) that is h (0), h (1)... etc. Thus output y (n) is weighted linear combination of present input and past inputs.

If the structure looks like, Fig. H-3(b) then it is called as pick-off node or tapped line. Now we can draw the same Fig. H-3(a) as shown in Fig. H-3(c). In this case it looks like tapped delay line. Therefore the direct form realization is also called as tapped or transversal delay line filter.

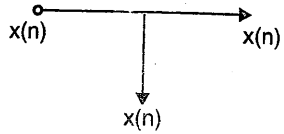


Fig. H-3(b) : Pick-off node

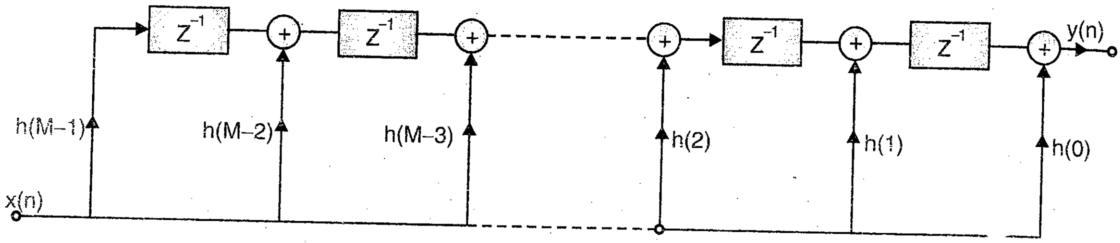


Fig. H-3(c) : Tapped delay line filter

1.3 Cascade Form Structure :

Cascade means the number of stages are connected in series. Now we have the system transfer function of the FIR system is given by,

$$H(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} \qquad ...(1)$$

Generally the higher order FIR filter is realised by using a series connection of different FIR sections (cascade connection). Here each section is characterized by the second order transfer function. Then due to cascade connection ; the total transfer function [H(Z)] will be multiplication of all second order transfer functions.

$$\therefore \quad H(Z) = H_1(Z) \cdot H_2(Z) \cdot H_3(Z) \,.....\, H_k(Z) \qquad ...(2)$$

Here $H_k(Z)$ is the second order transfer function and it is given by,

$$H_k(Z) = b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2} \qquad ...(3)$$

But we know that,

$$H_k(Z) = \frac{Y_k(Z)}{X_k(Z)}$$

$$\therefore \quad \frac{Y_k(Z)}{X_k(Z)} = b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2}$$

$$\therefore \quad Y_k(Z) = b_{k0} X_k(Z) + b_{k1} Z^{-1} X_k(Z) + b_{k2} Z^{-2} X_k(Z) \qquad ...(4)$$

Taking inverse Z-transform (IZT) of both sides we get,

$$Y_k(n) = b_{k0}\, x_k(n) + b_{k1}\, x_k(n-1) + b_{k2}\, x_k(n-2) \qquad ...(5)$$

From Equation (5), we can draw the direct form realization of second order section as shown in Fig. H-4(a).

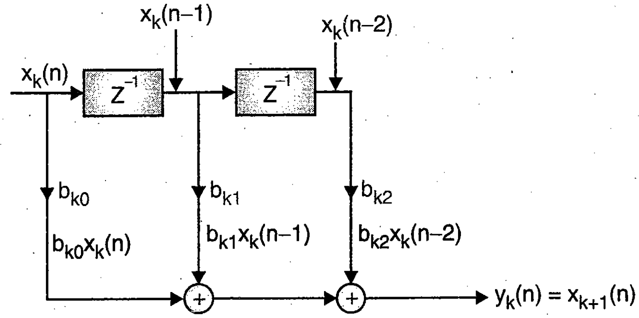


Fig. H-4(a) : Direct-form realization of second order section

Now the total structure is obtained by cascading all second order sections. The block schematic of cascade structure is shown in Fig. H-4(b).

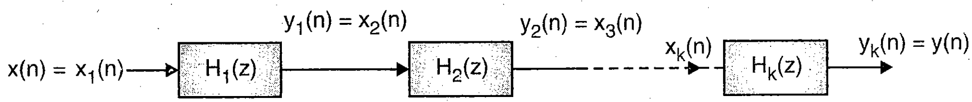


Fig. H-4(b) : Block diagram of cascade structure

Thus using Figs. H-4(a) and H-4(b), we can draw the cascade realization of FIR system as shown in Fig. H-4(c).

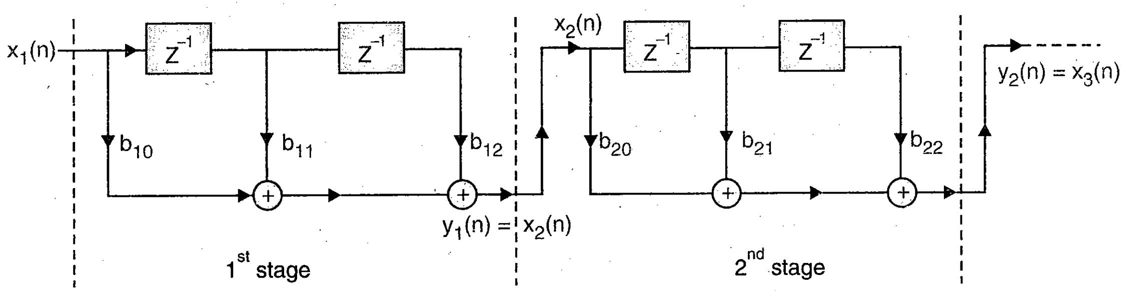


Fig. H-4(c) : Cascade realization of FIR system

Note : This cascade realization is obtained by putting K = 0 in first stage and K = 1 in second stage.

Observations :

1. This cascade form is canonic with respect to delay.
2. It has M – 1 adders and M multipliers for $(M-1)^{th}$ order FIR transfer function.

1.4 Frequency Sampling Structure :

The major advantage of designing FIR filters using frequency sampling structure is that; we can design FIR filter at the frequencies of interest and not at all frequencies. So the design complexity is reduced.

Now in frequency sampling structure; the desired frequency response is specified at the set of equally spaced frequencies as follows :

$$\omega = \omega_k = \frac{2\pi}{M}(k+\alpha) \qquad \text{...(1)}$$

Here $k = 0, 1, 2, \ldots \frac{M-1}{2}$

M is an integer and $\alpha = 0$ or $\frac{1}{2}$

According to the definition of fourier transform we have,

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n} \qquad \text{...(2)}$$

This equation gives the frequency response of input signal.

Putting Equation (1) in Equation (2) we get,

$$H(\omega_k) = \sum_{n=0}^{M-1} h(n)e^{-j\frac{2\pi}{M}(k+\alpha)\cdot n}$$

$$\therefore \quad H(\omega_k) = \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} \qquad \text{...(3)}$$

This set given by Equation (3) are called as frequency samples of $H(\omega_k)$. Here $H(\omega_k)$ is same as DFT $H(k+\alpha)$. Now if $\alpha = 0$, then $H(\omega_k) = H(k)$ which corresponds to M point DFT.

$$\therefore \quad H(k+\alpha) = \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} \qquad \text{...(4)}$$

Using the definition of IDFT we can write,

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} \qquad \text{...(5)}$$

Now we will obtain the equations in Z domain. According to the definition of Z-transform we have,

$$H(Z) = \sum_{n=0}^{M-1} h(n) Z^{-n} \quad \text{...(6)}$$

Putting Equation (5) in Equation (6),

$$H(Z) = \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \right] Z^{-n}$$

Interchanging the order of summation we get,

$$H(Z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \sum_{n=0}^{M-1} e^{j2\pi(k+\alpha)/M} \cdot Z^{-1} \right]^{n} \quad \text{...(7)}$$

Now we have the standard summation formula,

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

Thus Equation (7) becomes,

$$H(Z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \left\{ \frac{1-Z^{-M} e^{j2\pi(k+\alpha)}}{1-e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \right\} \right] \quad \text{...(8)}$$

Here the term $e^{j2\pi(k+\alpha)}$ can be written as,

$$e^{j2\pi(k+\alpha)} = e^{j2\pi k} \cdot e^{j2\pi\alpha}$$

$$\text{And} \quad e^{j2\pi k} = \cos 2\pi k + j \sin 2\pi k = 1$$

Thus Equation (8) can be simplified as,

$$H(Z) = \frac{1-Z^{-M} e^{j2\pi\alpha}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1-e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \quad \text{...(9)}$$

This equation shows that $H(Z)$ is characterised by set of frequency samples, $H(k+\alpha)$. Now we can write $H(Z)$ as the function of two cascade filters as,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

$$\text{Let} \quad H_1(Z) = \frac{1}{M}\left(1-Z^{-M} e^{j2\pi\alpha}\right) \quad \text{...(10)}$$

This is called as All-Zero system (filter) and zeros are located as,

$$Z_k = e^{j2\pi(k+\alpha)/M}$$

Similarly, $$H_2(Z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} \cdot Z^{-1}} \quad \text{...(11)}$$

This consists of parallel banks of single pole.

Now $H_1(Z)$ and $H_2(Z)$ can be realised separately. Then $H(Z)$ is obtained by connecting $H_1(Z)$ and $H_2(Z)$ is cascade form as shown in Fig. H-5.

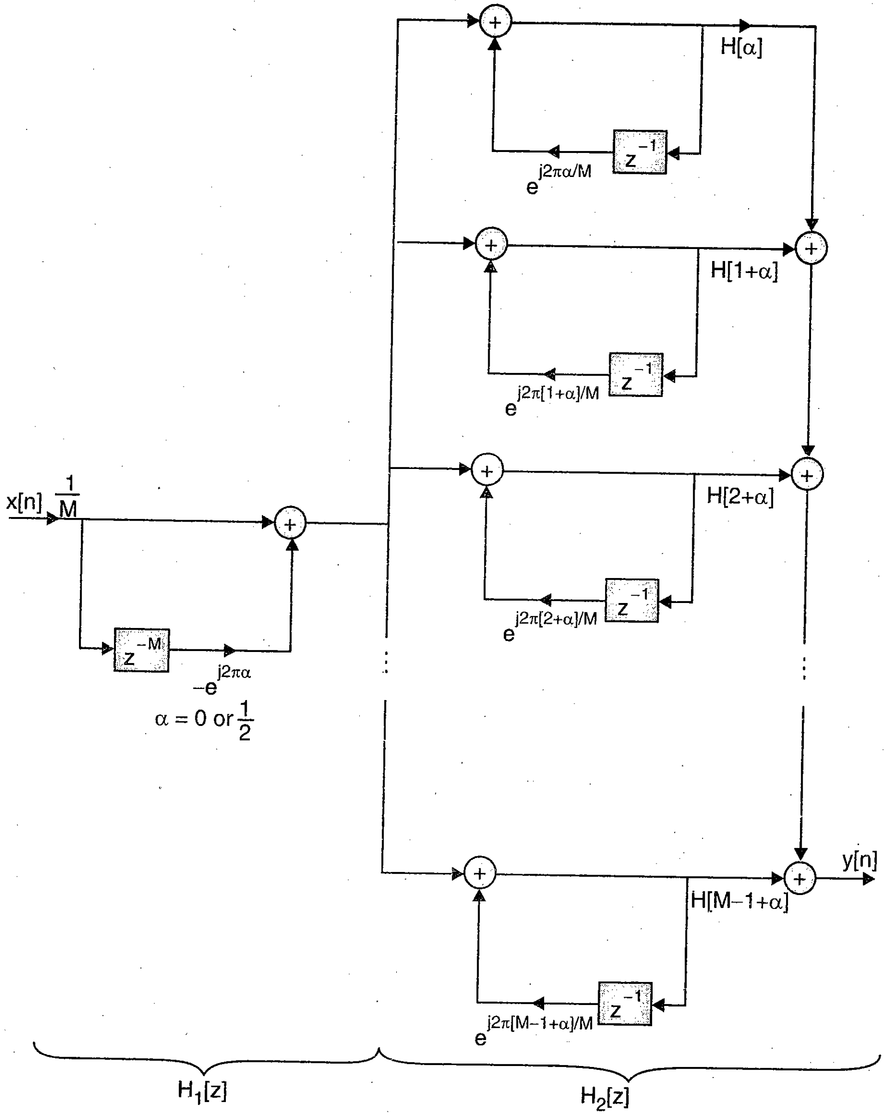


Fig. H-5 : Frequency sampling realization

1.5 Lattice Ladder Structure for FIR Filters :

Consider M^{th} order FIR system with the transfer function,

$$H_M(Z) = 1 + \sum_{k=1}^{M} \beta_M(k) Z^{-k}, \; M \geq 1 \qquad \text{...(1)}$$

Here M denotes the degree of polynomial and β_M is coefficient.

Thus when M = 0 we get,

$$H_o(Z) = 1 \qquad \text{...(2)}$$

Basically $H_M(Z)$ is the transfer function which can be written as,

$$H_M(Z) = \frac{\text{output}}{\text{input}} = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad Y(Z) = X(Z) H_M(Z) \qquad \text{...(3)}$$

Putting Equation (1) in Equation (3) we get,

$$Y(Z) = X(Z) \left[1 + \sum_{k=1}^{M} \beta_M(k) Z^{-k} \right]$$

$$\therefore \quad Y(Z) = X(Z) + X(Z) \sum_{k=1}^{M} \beta_M(k) Z^{-k} \qquad \text{...(4)}$$

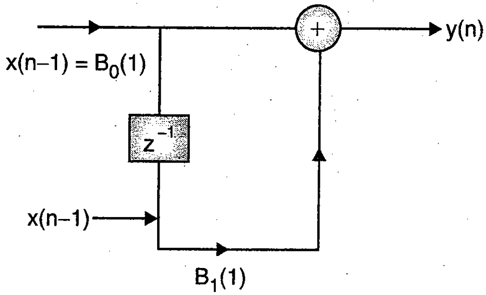
Taking IZT of both sides we get,

$$y(n) = x(n) + \sum_{k=1}^{M} \beta_M(k) \cdot x(n-k) \qquad \text{...(5)}$$

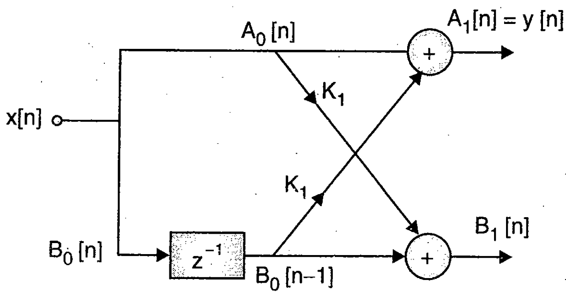
Let $M = 1$ then Equation (5) becomes,

$$y(n) = x(n) + \beta_1(1) x(n-1) \text{ ....(Here } k = M = 1) \qquad \text{...(6)}$$

In the simplest way Equation (6) can be realized as shown in Fig. H-6(a).



(a)



(b) Single stage lattice structure
Fig. H-6

Now the same output can be obtained by using the structure shown in Fig. H-6(b). This structure is called as single stage lattice structure. This structure provides two outputs namely $A_1(n)$ and $B_1(n)$.

Here $$A_1(n) = A_0(n) + k_1 B_0(n-1) \qquad ...(7(a))$$

A_0 and B_0 are constant multipliers.

In terms of x (n) we can write,

$$A_1(n) = x(n) + k_1 x(n-1) \qquad ...(7(b))$$

Similarly, the other output can be written as,

$$B_1(n) = k_1 A_0(n) + B_0(n-1) \qquad ...(8(a))$$

In terms of x (n) we can write,

$$B_1(n) = k_1 x(n) + x(n-1) \qquad ...(8(b))$$

We know that Equation (7(b)) is obtained by using single stage lattic structure. Equations (6) and (7(b)) are same if,

$$A_0(n) = x(n) \quad \text{and} \quad k_1 = \beta_1(1)$$

Similarly, Equations (6) and (7(a)) are matching if,

$$A_0(n) = x(n),\ \beta_1(1) = k_1 \text{ and } B_0(n-1) = x(n-1)$$

Here 'k' is called as reflection coefficient.

That means the same output can be obtained using single stage lattice structure.

Now for M = 2; Equation (5) becomes,

$$y(n) = x(n)+\beta_2(1)x(n-1)+\beta_2(2)x(n-2) \quad ...(9)$$

As M = 2 we have to cascade two stages to obtain two stage lattice structure as shown in Fig. H-6(c).

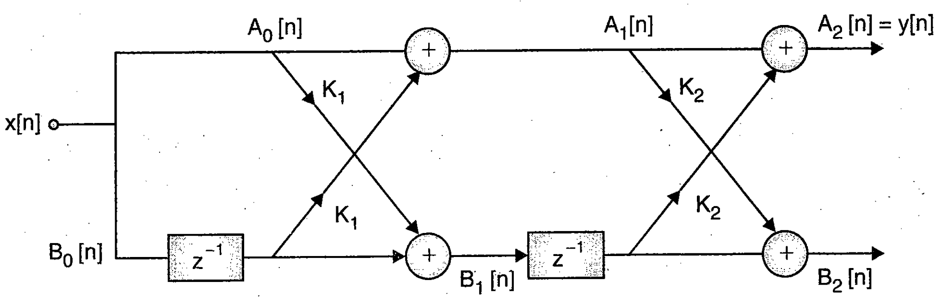


Fig. H-6(c) : Two stage lattice structure

From Fig. H-6(c) we can write,

Output of first stage :

$$A_1(n) = x(n)+k_1 x(n-1) \quad ...(10)$$

and $$B_1(n) = k_1 x(n)+x(n-1) \quad ...(11)$$

And the output of second stage is,

$$A_2(n) = A_1(n)+k_2 B_1(n-1) \quad ...(12)$$

and $$B_2(n) = k_2 A_1(n)+B_1(n-1) \quad ...(13)$$

Now putting Equation (10) in Equation (12) we get,

$$A_2(n) = x(n)+k_1 x(n-1)+k_2 B_1(n-1) \quad ...(14)$$

From Equation (11) we can write,

$$B_1(n-1) = k_1 x(n-1)+x(n-2)$$

Putting this value in Equation (14) we get,

$$A_2(n) = x(n)+k_1 x(n-1)+k_2[k_1 x(n-1)+x(n-2)]$$

$$\therefore \quad A_2(n) = x(n)+k_1(1+k_2)x(n-1)+k_2 x(n-2) \quad ...(15)$$

Observe that Equations (9) and (15) are matching.

Here, $\beta_2(2) = k_2$, $\beta_2(1) = k_1(1+k_2)$

or $k_2 = \beta_2(2)$, or $k_1 = \dfrac{\beta_2(1)}{1+\beta_2(2)}$

Here k_1 and k_2 are reflection coefficients. The same way we can increase the number of stages. Finally M^{th} stage lattice structure is obtained as shown in Fig. H-6(d).

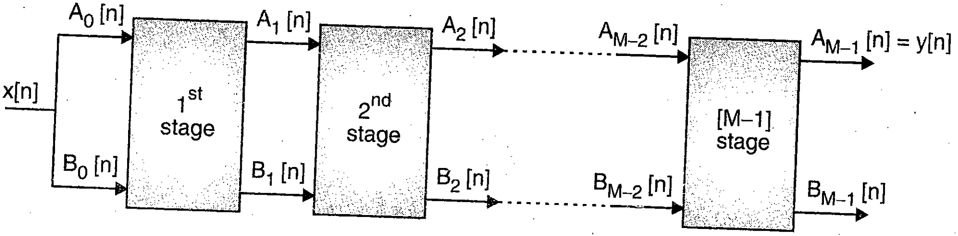


Fig. H-6(d) : M^{th} order lattice structure

1.7 Structures for Linear Phase FIR Filters :

FIR filter is said to be having a linear phase structure if its unit impulse sequence is either symmetric or antisymmetric about some point in time. That means FIR filter has linear phase if it satisfies the condition.

$$h(n) = h(M-1-n), \qquad n = 0, 1 \ldots M-1 \qquad \ldots(1)$$

Here M = Number of samples

The transfer function of FIR filter is given by,

$$H(Z) = \sum_{n=0}^{M-1} h(n) Z^{-n} \qquad \ldots(2)$$

Let us split the summation into two parts.

$$\therefore \quad H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) Z^{-n} + \sum_{n=\frac{M}{2}}^{M-1} h(n) Z^{-n} \qquad \ldots(3)$$

But for linear phase we have,

$$h(n) = h(M-1-n) \Rightarrow n = M-1-n$$

Putting this condition in the second summation of Equation (3) we get,

$$H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)Z^{-n} + \sum_{n=0}^{\frac{M}{2}-1} h(M-1-n)Z^{-(M-1-n)} \qquad ...(4)$$

Note : The limits of summation are changed as follows :

We have $n = M - 1 - \underline{\underline{n}}$

When $n = \frac{M}{2} \Rightarrow \frac{M}{2} = M - 1 - \underline{\underline{n}} \Rightarrow \underline{\underline{n}} = M - 1 - \frac{M}{2} = \frac{M}{2} - 1$

When $n = M - 1 \Rightarrow M - 1 = M - 1 - \underline{\underline{n}} \Rightarrow \underline{\underline{n}} = 0$

Equation (4) can be written as,

$$H(Z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[Z^{-n} + Z^{-(M-1-n)}\right] \qquad ...(5)$$

Case (i) For even M :

We know that $H(Z) = \frac{Y(Z)}{X(Z)}$; thus Equation (5) becomes,

$$\frac{Y(Z)}{X(Z)} = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[Z^{-n} + Z^{-(M-1-n)}\right]$$

$$\therefore \quad Y(Z) = \left\{ \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[Z^{-n} + Z^{-(M-1-n)}\right] \right\} X(Z)$$

Now expanding the summation we get,

$$Y(Z) = h(0)\left[1 + Z^{-(M-1)}\right]X(Z) + h(1)\left[Z^{-1} + Z^{-(M-2)}\right]X(Z)$$

$$+ \; .... + h\left(\frac{M}{2}-1\right)\left[Z^{-\left(\frac{M}{2}-1\right)} + Z^{-\frac{M}{2}}\right]X(Z)$$

$$\therefore \quad Y(Z) = h(0)\left[X[Z]+Z^{-(M-1)}\cdot X(Z)\right]+h(1)\left[Z^{-1}X(Z)+Z^{-(M-2)}X(Z)\right]$$

$$+ \ldots + h\left(\frac{M}{2}-1\right)\left[Z^{-\left(\frac{M}{2}-1\right)}X(Z)+Z^{-\frac{M}{2}}X(Z)\right] \quad \ldots(6)$$

Taking IZT of both sides we get,

$$y(n) = h(0)\{x(n)+x[n-(M-1)]\}+h(1)\{x(n-1)+x[n-(M-2)]\}$$

$$+ \ldots + h\left(\frac{M}{2}-1\right)\left\{x\left[\left(n-\left(\frac{M}{2}-1\right)\right)\right]+x\left(n-\frac{M}{2}\right)\right\} \quad \ldots(7)$$

Let M = 6. Thus Equation (7) becomes,

$$y(n) = h(0)\{x(n)+x(n-5)\}+h(1)[x(n-1)+x(n-4)]+h(2)[x(n-2)+x(n-3)] \quad \ldots(8)$$

The realization of Equation (8) is shown in Fig. H-8(a).

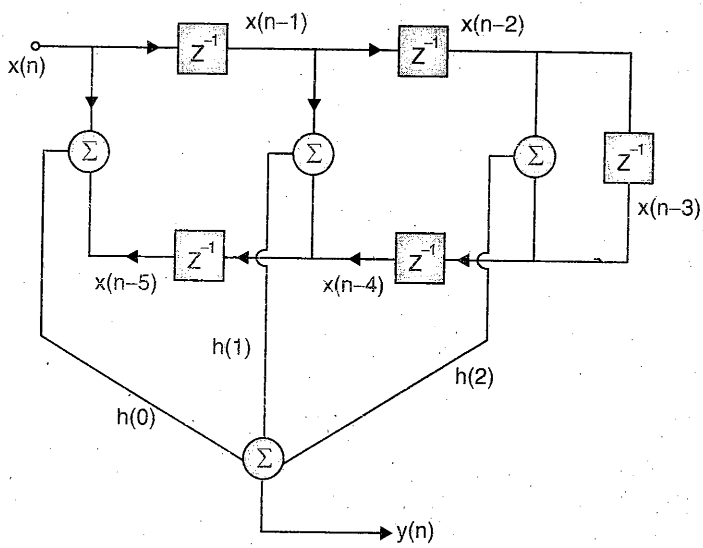


Fig. H-8(a) : Linear phase FIR structure for M = 6 (even)

Case (ii) : For odd M :

When M is odd (let M = 5) we can simply write the difference equation as,

$$y(n) = h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)+h(3)x(n-3)+h(4)x(n-4) \quad \ldots(9)$$

But we have the condition for symmetry,

$$h(n) = h(M-1-n) \quad \ldots(10)$$

Here M = 5. Thus from Equation (10) we get,

For $n = 0 \Rightarrow h(0) = h(4)$

For $n = 1 \Rightarrow h(1) = h(3)$

For $n = 2 \Rightarrow h(2) = h(2)$

For $n = 3 \Rightarrow h(3) = h(1)$

For $n = 4 \Rightarrow h(4) = h(0)$

Thus Equation (9) becomes,

$$y(n) = h(0)x(n)+h(1)x(n-1)+h(2)x(n-2)+h(1)x(n-3)+h(0)x(n-4)$$

$$\therefore \quad y(n) = h(0)[x(n)+x(n-4)]+h(1)[x(n-1)+x(n-3)]+h(2)x(n-2) \quad \text{...(11)}$$

The realization of Equation (11) is shown in Fig. H-8(b).

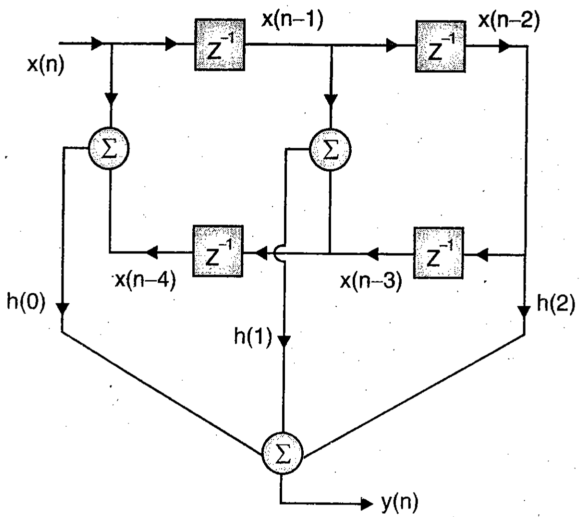


Fig. H-8(b) : Linear phase FIR structure for M = 5 (odd)

Note : Similar to case (i), equation of H (Z) for odd M can be expressed as,

$$H(Z) = h\left(\frac{M-1}{2}\right)Z^{\frac{-(M-1)}{2}} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[Z^{-n}+Z^{-(M-1-n)}\right]$$

Then by using same procedure we can obtain the required realization.

Solved Problems :

Prob. 1 : Obtain linear phase realization of

$$H(Z) = 1+\frac{Z^{-1}}{4}+\frac{Z^{-2}}{4}+Z^{-3}$$

Soln. : Given equation is,

$$H(Z) = 1+\frac{Z^{-1}}{4}+\frac{Z^{-2}}{4}+Z^{-3}$$

$$\therefore \quad h(0) = 1, \quad h(1) = \frac{1}{4}, \quad h(2) = \frac{1}{4}, \quad h(3) = 1$$

Here $M = 4$ that means M is even.

Now we have the condition for symmetric response.

$$h(n) = h(M-1-n), \quad n = 0, 1, \ldots M-1$$

For $n = 0 \Rightarrow h(0) = h(3)$

For $n = 1 \Rightarrow h(1) = h(2)$

This realization is similar to Fig. H-8(a). So for M = 4; the realization is shown in Fig. H-9.

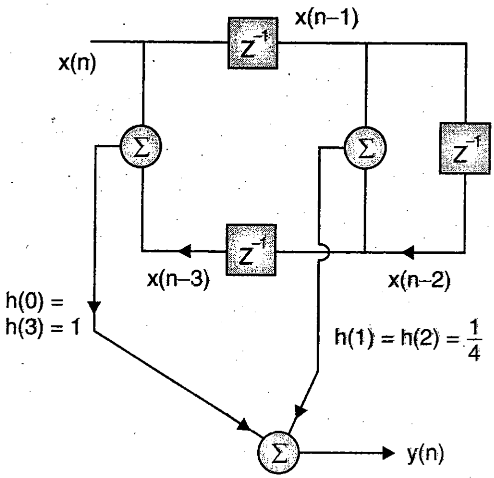


Fig. H-9

Prob. 2 : Realize a linear phase FIR filter with the following impulse response.

$$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) - \frac{1}{4}\delta(n-2) + \delta(n-4) + \frac{1}{2}\delta(n-3)$$

Soln. : We can write the impulse response as,

$$h(n) = \left\{1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 1\right\}$$
$$\uparrow$$

$$\therefore \quad h(0) = 1, h(1) = \frac{1}{2}, \quad h(2) = -\frac{1}{4}, h(3) = \frac{1}{2}, h(4) = 1$$

Here $M = 5$

Now according to the condition of symmetry,

$$h(n) = h(M-1-n), \qquad n = 0, 1, \ldots M-1$$

For $n = 0 \Rightarrow h(0) = h(4)$

For $n = 1 \Rightarrow h(1) = h(3)$

For $n = 2 \Rightarrow h(2) = h(2)$

This is the case for M = 5 (odd). The realization is same as shown in Fig. H-8(b).

Prob. 3 : An FIR filter is given by

$$y(n) = x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Find the lattice structure coefficients.

Soln. :

$$y(n) = x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Taking Z-transform,

$$Y(Z) = X(Z) + \frac{4}{5}Z^{-1}X(Z) + \frac{3}{2}Z^{-2}X(Z) + \frac{2}{3}Z^{-3}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)\left[1 + \frac{4}{5}z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}\right]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = 1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3} \qquad ...(1)$$

As we know

$$A_m(Z) = A_{m-1}(Z) + K_m Z^{-1} B_{m-1}(Z) \qquad ...(2)$$

$$B_m(Z) = K_m A_m(Z) + Z^{-1} B_{m-1}(Z) \qquad ...(3)$$

$$A_{m-1}(Z) = \frac{A_m(Z) - K_m B_m(Z)}{1 - K_m^2} \qquad ...(4)$$

In above example, using step down recursion, as order of filter m = 3,

$$A_3(Z) = 1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}$$

$$\therefore \quad \alpha_3(3) = \frac{2}{3}$$

as K_3 = coefficient of lattice structure

$$\therefore \quad K_3 = \alpha_3(3) = \frac{2}{3} \qquad ...(5)$$

$$A_s \quad B_3(Z) = Z^{-3} A_3(Z^{-1})$$

$$= Z^{-3}\left[1 + \frac{4}{5}Z^{1} + \frac{3}{2}Z^{2} + \frac{2}{3}Z^{3}\right]$$

$$B_3(Z) = Z^{-3} + \frac{4}{5}Z^{-2} + \frac{3}{2}Z^{-1} + \frac{2}{3}$$

$$B_3(Z) = \frac{2}{3} + \frac{3}{2}Z^{-1} + \frac{4}{5}Z^{-2} + Z^{-3} \qquad ....(6)$$

Now, $$A_{m-1}(Z) = \frac{A_m(Z) - K_m B_m(Z)}{1 - K_m^2}$$

$$\therefore \; A_2(Z) = \frac{A_3(Z) - K_3 \; B_3(Z)}{1 - K_3^2}$$

$$= \frac{\left[1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3}\right] - \frac{2}{3}\left[\frac{2}{3} + \frac{3}{2}Z^{-1} + \frac{4}{5}Z^{-2} + 2^{-3}\right]}{1 - \left(\frac{2}{3}\right)^2}$$

$$= \frac{1 + \frac{4}{5}Z^{-1} + \frac{3}{2}Z^{-2} + \frac{2}{3}Z^{-3} - \frac{4}{9} - Z^{-1} - \frac{8}{15}Z^{-2} - \frac{2}{3}Z^{-3}}{1 - \left(\frac{4}{9}\right)}$$

Collecting same terms together

$$A_2(Z) = \frac{\left(1 - \frac{4}{9}\right) + Z^{-1}\left(\frac{4}{5} - 1\right) + Z^{-2}\left(\frac{3}{2} - \frac{8}{15}\right)}{\frac{5}{9}}$$

$$= \frac{\frac{5}{9} - \frac{1}{5}Z^{-1} + \frac{29}{30}Z^{-2}}{\frac{5}{9}}$$

$$A_2(Z) = 1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2}$$

$$\therefore \; \alpha_2(2) = \frac{87}{50}$$

$$\therefore \; K_2 = \alpha_2(2) = \frac{87}{50} \qquad \text{...(7)}$$

Now, $$B_2(Z) = Z^{-2}\left[A_2(Z^{-1})\right]$$

$$= Z^{-2}\left[1 - \frac{9}{25}Z^{1} + \frac{87}{50}Z^{2}\right]$$

$$B_2(Z) = Z^{-2} - \frac{9}{25}Z^{-1} + \frac{87}{50}$$

$$B_2(Z) = \frac{87}{50} - \frac{9}{25}Z^{-1} + Z^{-2} \qquad \text{...(8)}$$

Now,

$$A_1(Z) = \frac{A_2(Z) - K_2 B_2(Z)}{1 - K_2^2}$$

$$= \frac{\left(1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2}\right) - \frac{87}{50}\left(\frac{87}{50} - \frac{9}{25}Z^{-1} + Z^{-2}\right)}{1 - \left(\frac{87}{50}\right)^2}$$

$$A_1(Z) = \frac{1 - \frac{9}{25}Z^{-1} + \frac{87}{50}Z^{-2} - \left(\frac{87}{50}\right)^2 + \left(\frac{87 \times 9}{50 \times 25}\right)Z^{-1} - \frac{87}{50}Z^{-2}}{1 - \left(\frac{87}{5}\right)^2}$$

Collecting same terms together,

$$A_1(Z) = \frac{1 - \left(\frac{87}{50}\right)^2 - Z^{-1}\left(\frac{783}{1250} - \frac{9}{25}\right)}{1 - \frac{7569}{2500}}$$

$$= \frac{(1 - 3.0276) - Z^{-1}(0.6264 - 0.36)}{1 - 3.0276}$$

$$= \frac{-2.0276 - 0.2664\,Z^{-1}}{-2.0276}$$

$$= 1 - \frac{0.2664\,Z^{-1}}{-2.0276}$$

$$A_1(Z) = 1 + 0.1313869\,Z^{-1}$$

$$\therefore \quad \alpha_1(1) = 0.1314$$

$$\therefore \quad K_1 = 0.1314$$

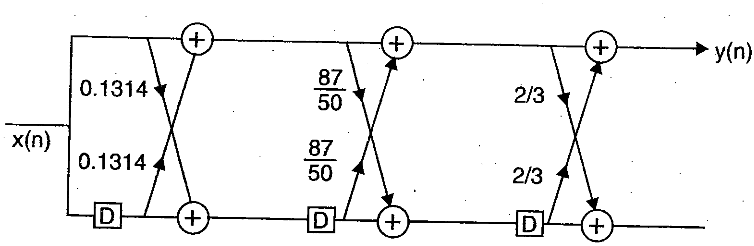


Fig. H-10