1.1 Introduction :

FIR stands for Finite Impulse Response. FIR filters are called as non-recursive filters because they do not use the feedback. Before studying the design of FIR filters; we will discuss one important characteristic of FIR filter.

1.1.1 FIR Filters are Inherently Stable :

We know that LSI system is said to be stable if bounded input produces bounded output (BIBO). We have the difference equation of FIR filter,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \qquad ...(1)$$

Taking Z transform of Equation (1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} b_k Z^{-k} X(Z) \qquad ...(2)$$

Now transfer function, $H(Z) = \frac{Y(Z)}{X(Z)}$

Thus from Equation (2) we get,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} b_k Z^{-k} \qquad ...(3)$$

Taking IZT of Equation (3) we get

$$h(n) = \begin{cases} b_n & \text{For } 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad ...(4)$$

Expanding Equation (1) we get,

$$y(n) = b_0 x(n) + b_1 x(n-1) + ... + b_{M-1} x(n-M+1) \qquad ...(5)$$

Using Equation (4) we get,

$$y(n) = h(0) x(n) + h(1) x(n-1) + ... + h(M-1) x(h-M+1) \qquad ...(6)$$

Here h (0), h (1) ... are constants that means they are bounded. Now from Equation (6); the output will be bounded if we apply bounded input. That means for every bounded input; the output of FIR filter is bounded. Thus FIR filters are inherently stable.

1.2 Symmetric and Antisymmetric FIR Filters :

We will discuss the symmetry and antisymmetry of FIR filters. These conditions are related to their unit sample response h (n).

The unit sample response of FIR filter is symmetric if it satisfies the condition.

$$h(n) = h(M-1-n) \quad ..... \; n = 0, 1 \; ..... \; M-1 \qquad ...(1)$$

Here M = Number of samples; so if M = 8 we get,

for n = 0 $\Rightarrow h(0) = h(8-1-0) = h(7)$

For n = 1 $\Rightarrow h(1) = h(8-1-1) = h(6)$ etc.

If h (n) is symmetric then, the filter is symmetric.

Now unit sample response of FIR filter is antisymmetric if it satisfies the condition,

$$h(n) = -h(M-1-n), \quad n = 0, 1 \; .... \; M-1 \qquad ...(2)$$

If this condition is satisfied then the filter is antisymmetric

Now the phase of FIR filter is given by,

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & \text{for } |H(\omega)| > 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi & \text{for } |H(\omega)| < 0 \end{cases} \qquad ...(3)$$

This equation shows that the phase of FIR filter is piecewise linear. Thus for the symmetric and antisymmetric FIR filters; the condition for linear phase is,

$$\boxed{h(n) = \pm h(M-1-n)}$$

FIR filler can be characterized by,

$$H(Z) = \sum_{k=0}^{M-1} h(k) Z^{-k} \qquad ...(4)$$

Here M is the length of filter and M – 1 is the order of filter. The frequency response for different conditions is as follows :

(1) If M is odd and symmetric then,

$$H(e^{j\omega}) = e^{\frac{-j\omega(M-1)}{2}} \left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2}-n\right) \right]$$

(2) If M is odd and antisymmetric then,

$$H(e^{j\omega}) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\left(\frac{M-1}{2}-n\right)\omega.$$

(3) If M is even and symmetric then,

$$H(e^{j\omega}) = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\left(\frac{M-1}{2}-n\right)\omega$$

(4) If M is even and antisymmetric then,

$$H(e^{j\omega}) = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\sin\left(\frac{M-1}{2}-n\right)\omega$$

1.2.1 Magnitude Specifications :

The magnitude specifications of FIR filter are as shown in Fig. J-1.

Here δ_p = Peak passband deviation

δ_s = Stopband deviation

ω_p = passband edge frequency

ω_s = Stopband edge frequency

$$\Delta f = \text{Transition width} = \frac{\omega_s - \omega_p}{2\pi}$$

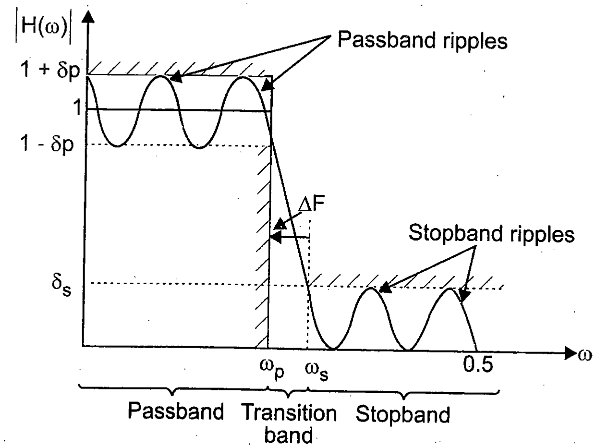


Fig. J -1 : Magnitude specifications

Thus the magnitude specifications of FIR filter can be written as,

$$1-\delta_p \leq |H(\omega)| \leq 1+\delta_p \quad \text{for} \quad 0 \leq \omega \leq \omega_p$$

and $$0 \leq |H(\omega)| \leq \delta_s \quad \text{for} \quad \omega_s \leq \omega \leq \pi$$

These are the magnitude specifications. For the phase response, we have assumed a linear phase. Therefore while designing the FIR filter only the symmetry of the filter is indicated.

1.2.2 General Filter Coefficient Equation :

The FIR and IIR filters are basically Linear shift Invariant (LSI) systems which are characterized by unit sample response. The FIR system has finite duration unit sample response, as follows,

$$h(n) = 0 \qquad \text{for } n < 0 \text{ and } n \geq M.$$

FIR system is a nonrecursive system i.e. it depends only on past and present input. The difference equation of LSI system is given as,

$$y(n) = -\sum_{K=1}^{N} a_K\, y(n-K) + \sum_{K=0}^{M} b_k\, x(n-K) \qquad \text{...(1)}$$

But, the first term represents past outputs and second term represents past and present input. Hence, the difference equation for FIR system is,

$$y(n) = \sum_{K=0}^{M} b_K\, x(n-K) \qquad \text{...(2)}$$

If we consider 'M' coefficients, then

$$y(n) = \sum_{K=0}^{M-1} b_K\, x(n-K) \qquad \text{...(3)}$$

Taking 'Z' transform of both sides,

$$Y(Z) = \sum_{K=0}^{M-1} b_K\, Z^{-K} X(Z)$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \sum_{K=0}^{M-1} b_K\, Z^{-K} \qquad \text{...(4)}$$

$$\text{Let,} \quad \frac{Y(Z)}{X(Z)} = H(Z) \qquad \text{...(5)}$$

which is a system function of FIR system. Hence, from Equation (4) and (5),

$$H(Z) = \sum_{K=0}^{M-1} b_K\, Z^{-K} \qquad \text{...(6)}$$

Taking inverse 'Z' transform of Equation (6) we get, unit sample response of FIR system as,

$$h(n) = \begin{cases} b_n & \text{for } 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Which gives filter coefficient that means it represents general filter coefficient equation.

1.3 Design of FIR Filters using Windows :

Different types of windows are used to design FIR filter. First we will discuss the design of FIR filter using rectangular window. The rectangular window is as shown in Fig. J-2

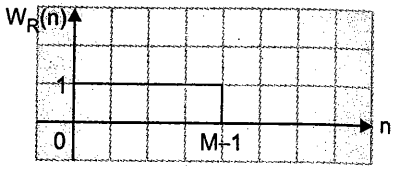


Fig. J-2 : Rectangular window

It is denoted by $W_R(n)$. Its magnitude is 1 for the range, n = 0 to M − 1. Now let $h_d(n)$ be the impulse response having infinite duration. If $h_d(n)$ is multiplied by $W_R(n)$ then a finite impulse response is obtained as shown in Fig. J-3. That means we will get only limited pulse of $h_d(n)$; not all (∞) pulse. Since we are truncating the input sequence by using a window, this process is called as truncation process. Since the shape of window function is rectangular; it is called as rectangular window.

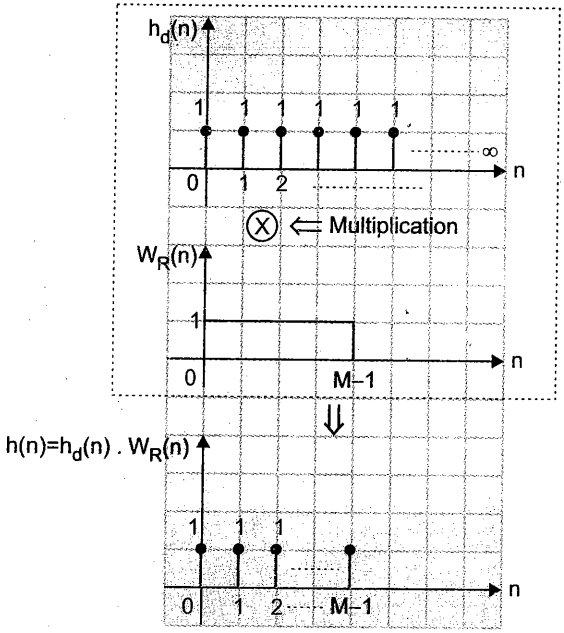


Fig. J-3 : Truncation process

Magnitude response of rectangular window :

The rectangular window is defined as,

$$W_R(n) = \begin{cases} 1 & \text{for } n = 0, 1, 2 \ldots M-1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(1)$$

Let $h_d(n)$ be infinite duration impulse response. We know that the finite duration impulse response $h(n)$ is obtained by multiplying $h_d(n)$ by $W_R(n)$.

$$\therefore \quad h(n) = h_d(n) \cdot W_R(n) \qquad \text{...(2)}$$

We will denote fourier transform of $W_R(n)$ by $W_R(\omega)$. Thus using the definition of fourier transform we can write,

$$W_R(\omega) = \sum_{n=0}^{M-1} W_R(n)\, e^{-j\omega n} \qquad \text{...(3)}$$

But the value of $W_R(n)$ is 1 for the range $n = 0$ to $M-1$.

$$\therefore \quad W_R(\omega) = \sum_{n=0}^{M-1} 1 \cdot e^{-j\omega n} \qquad \text{...(4)}$$

We can represent the window sequence as,

$$W_R(n) = u(n) - u(n-M)$$

here $u(n)$ is unit step having duration $n = 0$ to $n = \infty$ and $u(n-M)$ is delay unit step.

Thus Equation (4) becomes,

$$W_R(\omega) = \sum_{n=0}^{\infty} [u(n) - u(n-M)]\, e^{-j\omega n}$$

$$\therefore \quad W_R(\omega) = \sum_{n=0}^{\infty} u(n)\, e^{-j\omega n} - \sum_{n=0}^{\infty} u(n-M)\, e^{-j\omega n} \qquad \text{...(5)}$$

Consider the first term at R.H.S. It represents fourier transform (FT) of unit step. And we have,

$$\text{F.T. of } u(n) = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \left(e^{-j\omega}\right)^n = \frac{1}{1 - e^{-j\omega}}$$

Now consider the second term. It represents the fourier tranform of delayed unit step.

$$\therefore \quad \text{F.T. of } u(n-M) \longleftrightarrow e^{-j\omega M}\, F\{u(n)\}$$

$$= e^{-j\omega M} \frac{1}{1 - e^{-j\omega}} = \frac{e^{-j\omega M}}{1 - e^{-j\omega}}$$

Thus Equation (5) becomes,

$$W_R(\omega) = \frac{1}{1-e^{-j\omega}} - \frac{e^{-j\omega M}}{1-e^{-j\omega}}$$

$$\therefore \quad W_R(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \qquad \text{...(6)}$$

We will rearrange Equation (6) as follows,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}} \cdot e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}} \cdot e^{-j\omega\frac{M}{2}}}{e^{-j\frac{\omega}{2}} \cdot e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \cdot e^{-j\frac{\omega}{2}}}$$

$$\therefore \quad W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)} \qquad \text{...(7)}$$

We have the trignometric identity,

$$e^{j\theta} - e^{-j\theta} = 2\sin\theta$$

Thus Equation (7) becomes,

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}\left[2\sin\left(\frac{\omega M}{2}\right)\right]}{e^{-j\frac{\omega}{2}}\left[2\sin\left(\frac{\omega}{2}\right)\right]}$$

$$\therefore \quad W_R(\omega) = e^{-j\omega\frac{M}{2}} \cdot e^{j\frac{\omega}{2}} \cdot \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore \quad W_R(\omega) = \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} \qquad \text{...(8)}$$

Now $W_R(\omega)$ can be expressed in terms of magnitude and angle as;

$$W_R(\omega) = |W_R(\omega)| \cdot e^{j\angle W_R(\omega)} \qquad \text{...(9)}$$

Comparing Equations (8) and (9) we can write,

$$|W_R(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|} \qquad \text{...(10)}$$

1.3.1 Properties of Commenly used Windows :

Some other types of window functions are as follows :

1) Hamming window
2) Hanning window
3) Triangular (Bartlett) window
4) Blackman window
5) Kaiser window

We will consider that the range of each window is from $-Q$ to Q. Here Q is positive integer number.

(1) Hamming window :

Hamming window function is given by,

$$W_{H_m}[n] = \begin{cases} 0.54 + 0.46\cos\dfrac{2\pi n}{M-1} & \\ 0 & \text{elesewhere} \end{cases}$$

Fig. J-4 shows shape of Hamming window.

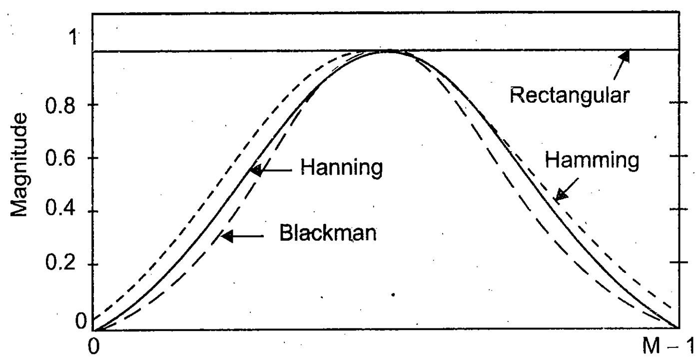


Fig. J-4

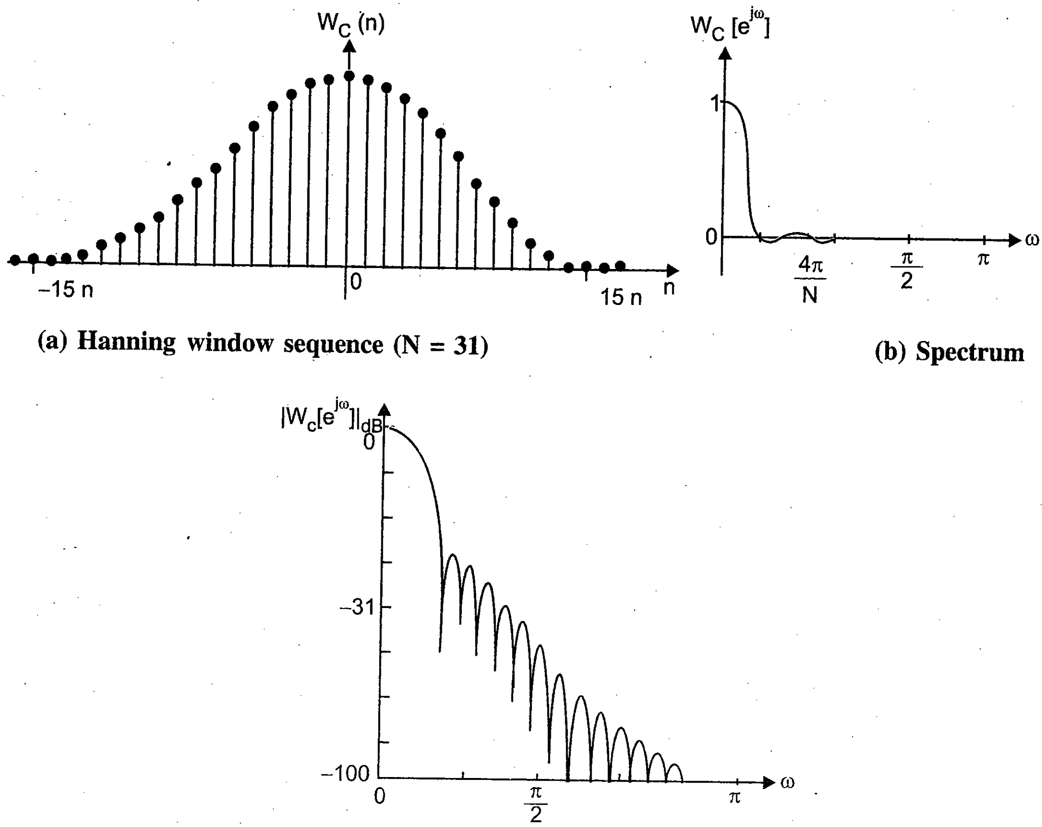
$W_{H_m}[n]$ is a bell-shaped sequence that is symmetric about $n = 0$.

(2) Hanning window function :

Hanning window function is also called as *Raised-cosine window*. The function is denoted by,

$$W_{hn} = \frac{1}{2}\left[1 - \frac{\cos 2\pi n}{M-1}\right]$$

Refer Figs. J-5(a), (b) and (c) to have a look to the spectrum.



(a) Hanning window sequence (N = 31)

(b) Spectrum

(c) Log-magnitude spectrum

Fig. J-5

Magnitude of first side-lobe level is -31 dB relative to maximum value.

(3) Triangular window function OR Bartlett window :

Triangular function is like ***tapering the rectangular window sequence linearly from the middle to the ends***. Triangular window function can be given by

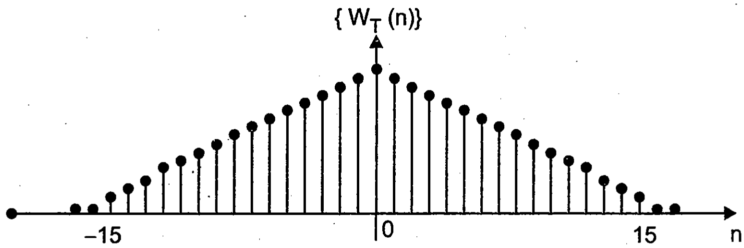
$$W_T[n] = 1 - \frac{2|n|}{M-1} \text{ for } |n| \leq M-1$$

Window and its spectrum is shown in Fig. J-6.

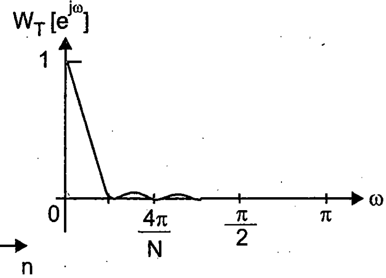
Sidelobe level is smaller than that of rectangular window.

Triangular window produces smoother magnitude response than that of rectangular window function.

The transition from passband to stopband is not as steep as that for the rectangular window. In the stopband, the response is smoother, but attenuation is less than that produced by rectangular window, therefore because of this characteristic, triangular window is not usually good choice.



(a)

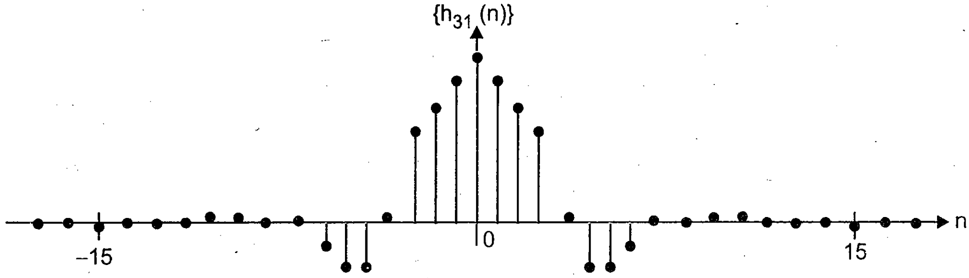


(b)

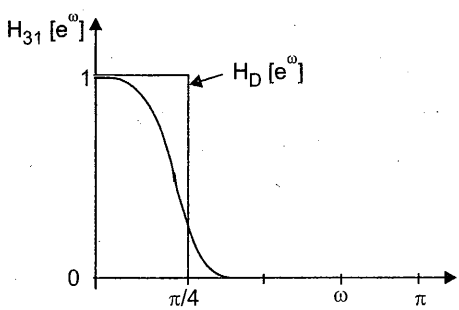
Fig. J-6

If we compare hanning window with triangular then hanning window function is smoother at the ends. Smoother ends reduces sidelobe level, while broaden middle section.

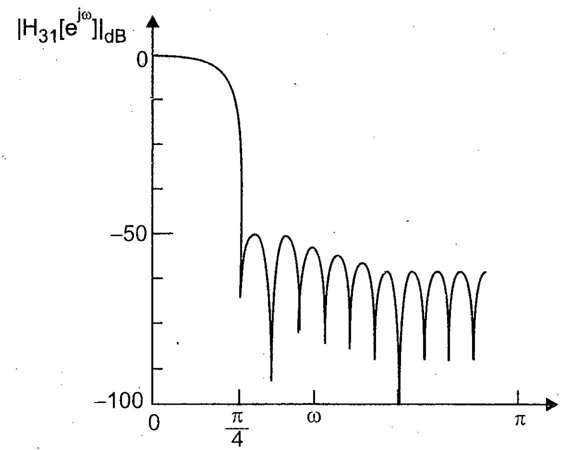
(4) Blackman window function :



(a) : Window sequence (N = 31)



(b) Seectrum



(c) Log-magnitude

Fig. J-7 : Blackman window function

In blackman window function, we will find one more additional term in comparision with hamming and hanning window. Because of additional cosine term, sidelobs are reduced further.

Window function for blackman can be given by ,

$$W_B[n] = 0.42 + 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$$

Window and its spectrum are shown in Fig. J-7.

(5) Kaiser window function :

Kaiser window function can be defined as,

$$W_k[n] = \begin{cases} \dfrac{I_o(\beta)}{I_o(\alpha)} & |n| \le Q \\ 0 & \text{elsewhere} \end{cases}$$

Where $I_o(x)$ is modified bessel function of the first kind and zero order. The tradeoff between main lobe width and side lobe level can be adjusted by varying parameter α.

Here α is independent variable.

β can be expressed as

$$\beta = \alpha\sqrt{1-(n/Q)^2}$$

$$W_k[n] = \begin{cases} I_0\left(2\sqrt{1-(n/Q)^2}\right)/I_0(\alpha) & |n| \le Q \\ 0 & \text{elsewhere} \end{cases}$$

The spectrum of Kaiser window is as shown in Fig. J-8.

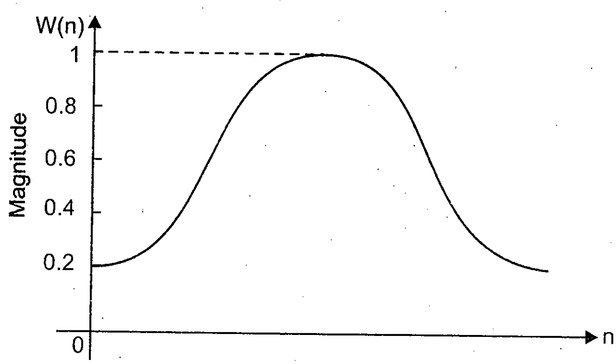


Fig. J-8 : Kaiser window function

Table J-1. shows the summary of window functions.

Table J-1

Sr. No.	Window	Function
1.	Triangular Bartlett	$W_T[n] = 1 - \frac{2\lvert n \rvert}{M-1}$ for $\lvert n \rvert \le M-1$
2.	Hanning	$W_{hn} = \frac{1}{2}\left[1 - \frac{\cos 2\pi n}{M-1}\right]$
3.	Hamming	$W_{Hm}[n] = \begin{cases} 0.54 + 0.46 \frac{\cos 2\pi n}{M-1} & \lvert n \rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$
4.	Blackman	$W_B[n] = 0.42 + 0.5 \cos \frac{2\pi n}{M-1} + 0.08 \cos \frac{4\pi n}{(M-1)}$
5.	Kaiser	$W_k[n] = \begin{cases} \frac{I_o(\beta)}{I_o(\alpha)} & \lvert n \rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$ OR
		$W_k[n] = \begin{cases} \frac{I_0\left(\alpha\sqrt{1-(n/Q)^2}\right)}{I_0[\alpha]} & \lvert n \rvert \le Q \\ 0 & \text{elsewhere} \end{cases}$

1.3.2 Gibb's Phenomenon :

The impulse response of FIR filter in terms of rectangular window is given by,

$$h(n) = h_d(n) \cdot W_R(n) \quad ...(1)$$

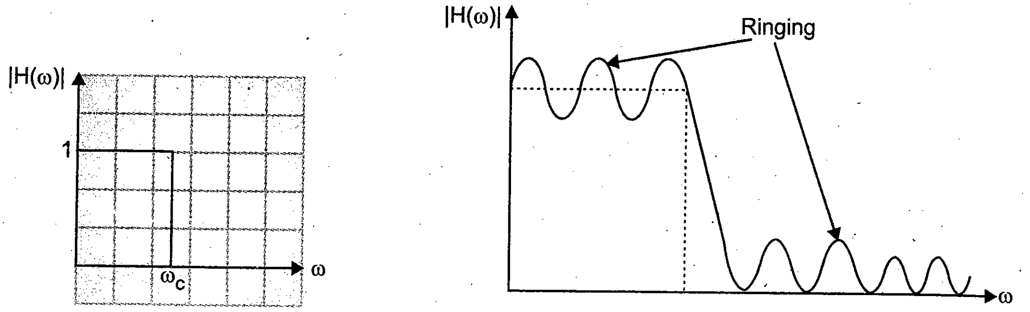
The frequency response of filter is obtained by taking fourier transform of Equation (1).

$$\therefore \quad H(\omega) = FT\left\{h_d(n) \cdot W_R(n)\right\}$$

$$\therefore \quad H(\omega) = H_d(\omega) * W_R(\omega) \quad ...(2)$$

This shows that the frequency response of FIR filter is equal to the convolution of desired frequency response, $H_d(\omega)$ and the fourier transform of window function.

Now the desired frequency response of low pass FIR is shown in Fig. J-9(a); while the frequency response of FIR filter obtained because of windowing is shown in Fig. J-9(b).



(a) **Desired frequency response** (b) **Frequency response obtained because of windowing**

Fig. J-9

The sidelobes are present in the frequency response of window function. Because of these sidelobes; the ringing is observed in the frequency response of FIR filter. This ringing is predominantly present near the bandedge and it is known as Gibb's phenomenon.

Now the question arises, **why the side lobes are present in the frequency response of window function ?** This is because of the sudden discontinuities in the window function. Observe the magnitude response of rectangular window. In this case, the discontinuity is very abrupt. Therefore the sidelobes are of larger amplitude. Thus the ringing effect is maximum in case of rectangular window.

Because of this reason; other window functions are developed which will not have the abrupt discontinuities. That means the window function will change more gradually in the time domain.

1.3.3 Advantages and Disadvantages of Window Method :

Advantages :

1. The windowing method requires minimum amount of computational effort; so window method is simple to implement.
2. For the given window; the maximum amplitude of ripple in the filter response is fixed. Thus the stopband attenuation is fixed in the given window.

Disadvantages :

1. The designing of FIR filters using windows is not flexible.
2. The frequency response of FIR filter shows the convolution of spectrum of window function and desired frequency response. Because of this; the passband and stopband edge frequencies cannot be precisely specified.
3. In many applications the expression for the desired filter response will be too complicated.

Design steps for FIR filter :

1. Get the desired frequency response, $H_d(\omega)$.
2. Take inverse fourier transform (IFT) of $H_d(\omega)$ to obtain $h_d(n)$.
3. Decide the length of FIR filter.
4. Multiply $h_d(n)$ by selected window function to get $h(n)$.
5. From $h(n)$ obtain $H(Z)$ and then realize it, if asked.

Solved Problems :

Prob. 1 : Design a linear phase FIR low pass filter of order seven with cut-off frequency 1 rad/sec using rectangular window.

Soln. :

Step I : The desired frequency response $H_d(\omega)$ for the low pass FIR filter is given by,

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad ...(1)$$

Here M = length of filter = 7 (given)

$$\therefore \quad H_d(\omega) = \begin{cases} e^{-3j\omega} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad ...(2)$$

Step II : We will obtain $h_d(n)$ by taking IFT of Equation (2). Using the definition of IFT we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega) \cdot e^{j\omega n} \, d\omega \quad ..(3)$$

Here given cut-off frequency = $\omega_c = 1$. Now for the symmetric filter we can write the range of integration (ω_c), from $\omega_c = -1$ to $\omega_c = 1$.

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-1}^{1} e^{-3j\omega} \cdot e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^{1} e^{j\omega(n-3)} \, d\omega \quad ...(4)$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-3)}}{j(n-3)} \right]_{-1}^{1}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j(n-3)} - e^{-j(n-3)}}{j(n-3)} \right]$$

Now we have the trignometric identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore \quad h_d(n) = \frac{\sin(n-3)}{\pi(n-3)} \quad \text{...for } n \neq 3 \quad ...(5)$$

Now if n = 3 then Equation (4) becomes,

$$h_d(n) = \frac{1}{2\pi} \int_{-1}^{1} e^{0} \cdot d\omega = \frac{1}{2\pi} \int_{-1}^{1} d\omega = \frac{1}{2\pi}[1-(-1)] = \frac{1}{\pi} \quad \text{...(6)}$$

Thus we can write,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi} & \text{for } n = 3 \end{cases} \quad \text{...(7)}$$

Step III : Here we have to make use of rectangular window of the order 7. We have for rectangular window,

$$\therefore \quad W_R(n) = \begin{cases} 1 & \text{for } n = 0 \text{ to } 6 \\ 0 & \text{otherwise} \end{cases} \qquad (\text{for } n = 0 \text{ to } M-1)$$

Now h (n) is, $h(n) = h_d(n) \cdot W_R(n)$

Thus using Equation (7) we get,

$$h(n) = \begin{cases} h_d(n) & \text{for } n = 0 \text{ to } 6 \\ 0 & \text{otherwise} \end{cases} \quad \text{...(8)}$$

Equation (8) gives unit impulse response of FIR filter. Making use of Equation (7), we can obtain the values of $h_d(n)$ and h (n) as shown in the Table J-2.

Table J-2

Value of n	Value of $h_d(n) = h(n)$
0	$h_d(0) = h(0) = \frac{\sin(-3)}{-3\pi} = 0.015$
1	$h_d(1) = h(1) = \frac{\sin(-2)}{-2\pi} = 0.145$
2	$h_d(2) = h(2) = \frac{\sin(-1)}{-\pi} = 0.268$
3	$h_d(3) = h(3) = \frac{1}{\pi} = 0.318$
4	$h_d(4) = h(4) = \frac{\sin(1)}{\pi} = 0.268$
5	$h_d(5) = h(5) = \frac{\sin(2)}{2\pi} = 0.145$
6	$h_d(6) = h(6) = \frac{\sin(3)}{3\pi} = 0.0149$

Value of n	Value of $h_d(n) = h(n)$
0	$h_d(0) = 0.4$
1	$h_d(1) = h(1) = \frac{\sin 1.257}{\pi} = 0.302$
2	$h(2) = h_d(2) = \frac{\sin 1.257 \times 2}{2\pi} = 0.093$
3	$h(3) = h_d(3) = \frac{\sin 1.257 \times 3}{3\pi} = -0.062$
4	$h(4) = h_d(4) = \frac{\sin 1.257 \times 4}{4\pi} = -0.075$
5	$h(5) = h_d(5) = \frac{\sin 1.257 \times 5}{5\pi} = 1.15 \times 10^{-4}$
6	$h(6) = h_d(6) = \frac{\sin 1.257 \times 6}{6\pi} = 0.05$

1.3.4 FIR Filter Design using Kaiser Window :

By using Kaiser window it is possible to obtain seperate control upon length or order (M) of filter and the transition width of the main lobe.

The magnitude specifications of required FIR filter is as shown in Fig. J-10.

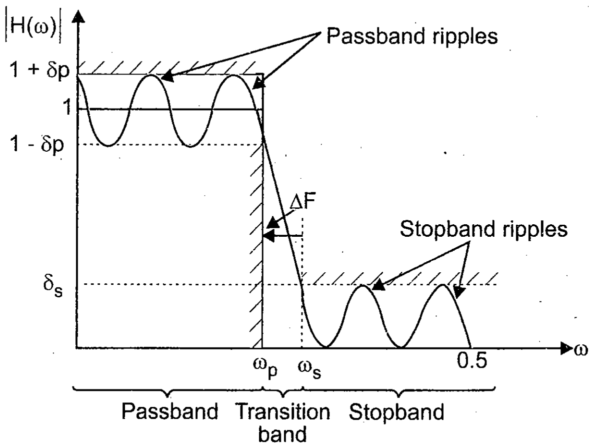


Fig. J-10 : Magnitude specifications

Here δ_p = Peak passband deviation

δ_s = Stopband deviation

ω_p = passband edge frequency

ω_s = Stopband edge frequency

$$\Delta f = \text{Transition width} = \frac{\omega_s - \omega_p}{2\pi}$$

In Kaiser window there are two main parameters. The length $(M+1)$ of wi shape parameter β. The Kaiser window is defined as,

$$\omega(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1-\left(\dfrac{n-\alpha}{\alpha}\right)^2\right]\right\}^{\frac{1}{2}}}{I_0(\beta)} & \text{for } 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Here $\alpha = \dfrac{M}{2}$ and $I_0(\ )$ is the 0^{th} order modified bessel function of first kind. is obtained by using the equation,

$$I_0(x) = 1 + \frac{0.25\,x^2}{(1!)^2} + \frac{(0.25\,x^2)^2}{(2!)^2} + \frac{(0.25\,x^2)^3}{(3!)^2} + \ldots$$

Design steps :

Step I : Choose the optimum value of ripple 'δ' using the equation,

$$\delta = \text{minimum of } \delta_P \text{ and } \delta_S$$

Here $\delta_P = 10^{-0.05\,A_s}$

or obtain δ_P by solving the equation,

$$A_P = 20\log_{10}\left(\frac{1+\delta_P}{1-\delta_P}\right)$$

$$\text{And } \delta_s = \frac{10^{0.05\,A_P} - 1}{10^{0.05\,A_P} + 1}$$

OR obtain δ_s by solving the equation,

$$A_s = -20\log_{10}(\delta_s)$$

Step II : Calculate attenuation 'A' in dB using the equation,

$$A = -20\log_{10}(\delta)$$

Step III : Calculate the parameter β from empirical equations proposed by Kaiser as follows :

$$\beta = \begin{cases} 0.1102\,(A - 8.7) & \text{for } A > 50 \\ 0.5842\,(A - 21)^{0.4} + 0.07886\,(A - 21) & \text{for } 21 \le A \le 50 \\ 0 & \text{for } A < 21 \end{cases}$$

Step IV : The length of filter is $M + 1$. Then calculate value of M using the equation,

$$M = \frac{A - 8}{2.285\,\Delta\omega}$$

Here $\Delta\omega$ = transition width and is given by,

$$\Delta\omega = \omega_s - \omega_P$$

Where ω_s = Stopband edge frequency

and ω_P = Passband edge frequency.

Step V : Select the desired impulse response depending on type of filter e.g. low pass, high pass etc.

Step VI : Calculate the FIR filter coefficients using the relation $h(n) = h_d(n) \cdot W(n)$.

Solved problems :

Prob. 1 : Design an FIR linear phase filter using Kaiser window to meet the following specifications

$0.99 \le |H(e^{j\omega})| \le 1.01, \quad 0 \le \omega \le 0.19\pi$

$|H(e^{j\omega})| \le 0.01, \quad 0.21\pi \le \omega \le \pi.$

Soln. : The magnitude specifications of FIR filter are given by,

$$1 - \delta_P \le |H(\omega)| \le 1 + \delta_P \quad , 0 \le \omega \le \omega_P$$

$$\text{and} \quad 0 \le |H(\omega)| \le \delta_s \quad , \omega_s \le \omega \le \pi \qquad \text{...(1)}$$

Comparing Equation (1) with given specifications we get,

$$1 - \delta_P = 0.99 \Rightarrow \delta_P = 0.01$$

$$\delta_s = 0.01, \ \omega_P = 0.19\pi \text{ and } \omega_s = 0.21\pi$$

Step I : Optimum value of ripple 'δ' is,

$$\delta = \text{Minimum of } \delta_P \text{ and } \delta_s$$

$$\therefore \ \delta = 0.01$$

Step II : Attenuation 'A' is given by,

$$A = -20\log_{10}(\delta) \text{ dB}$$

$$\therefore \quad A = -20 \log_{10}(0.01) \text{ dB}$$

$$\therefore \quad A = 40 \text{ dB}$$

Step III : Here the value of A is 40 dB, that means in the range $21 \le A \le 50$. Thus we will calculate β using the equation,

$$\beta = 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21)$$

$$\therefore \quad \beta = 0.5842\,(40-21)^{0.4} + 0.07886\,(40-21)$$

$$\therefore \quad \beta = 3.395$$

Step IV : The length of filter is M + 1. The value of M is calculated using the equation.

$$M = \frac{A-8}{2.285\,\Delta\omega}$$

$$\text{Here} \quad \Delta\omega = \omega_s - \omega_P = 0.21\,\pi - 0.19\,\pi = 0.02\,\pi$$

$$\therefore \quad M = \frac{40-8}{2.285\,(0.02\,\pi)} = 222.87$$

$$\therefore \quad M \approx 223$$

Step V : We have, the desired frequency response for the low pass filter.

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad \text{...(2)}$$

Remember that Equation (2) is for the filter length M.

For this filter using Kaiser window, the length is M + 1.

Thus putting M = M + 1 in Equation (2) we get,

$$H_d(\omega) = \begin{cases} e^{-j\omega\frac{M}{2}} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad \text{...(3)}$$

Now $h_d(n)$ is calculated by taking inverse fourier of Equation (3).

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega)\, e^{j\omega n}\, d\omega \quad \text{...(4)}$$

Here ω_c = Cutoff frequency. It is obtained by using the equation,

$$\omega_c = \frac{\omega_P + \omega_s}{2}$$

$$\therefore \quad \omega_c = \frac{0.19\,\pi + 0.21\,\pi}{2} = 0.2\,\pi$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} e^{-j\omega\left(\frac{223}{2}\right)} e^{j\omega n}\, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} e^{-j\omega\, 111.5} \cdot e^{j\omega n}\, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} e^{+j\omega(n - 111.5)}\, d\omega \qquad \text{...(5)}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega(n-111.5)}}{j(n-111.5)} \right]_{-0.2\,\pi}^{0.2\,\pi}$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi j(n-111.5)} \left[e^{j\,0.2\,\pi(n-111.5)} - e^{-j\,0.2\,\pi(n-111.5)} \right]$$

We have the trignometric identity.

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore \quad h_d(n) = \frac{\sin[0.2\,\pi(n-111.5)]}{\pi(n-111.5)} \quad \text{... for } n \neq 111.5 \qquad \text{...(6)}$$

Now for n = 111.5, using Equation (5) we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} e^{j\omega(111.5 - 111.5)}\, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} e^{0}\, d\omega = \frac{1}{2\pi} \int_{-0.2\,\pi}^{0.2\,\pi} 1\, d\omega$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi} [0.2\,\pi - (-0.2\,\pi)]$$

$$\therefore \quad h_d(n) = 0.2 \qquad \text{for } n = 111.5 \qquad \text{...(7)}$$

Now combining Equations (6) and (7) we can write,

$$h_d(n) = \begin{cases} \dfrac{\sin[0.2\,\pi(n-111.5)]}{\pi(n-111.5)} & \text{for } n \neq 111.5 \\ 0.2 & \text{for } n = 111.5 \end{cases} \qquad \text{...(8)}$$

Step VI : The filter coefficients are calculated using the equation

$$h(n) = h_d(n) \cdot W(n) \qquad \text{...(9)}$$

We have,
$$\omega(n) = \frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]\right\}^{1/2}}{I_0(\beta)}$$

We have $\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$

$$\therefore\ \omega(n) = \frac{I_0\left\{3.395\left[1-\left(\frac{n-111.5}{111.5}\right)^2\right]\right\}^{1/2}}{I_0(3.395)} \qquad \text{...(10)}$$

Thus using Equations (8) and (10), we can calculate the required filter coefficients.

1.4 Design of Linear Phase FIR Filters using Frequency Sampling Method :

(1) The desired frequency response is denoted by $H_d(\omega)$.

(2) This frequency response is sampled uniformly at M points.

(3) The frequency samples are given by,

$$\omega_k = \frac{2\pi}{M}k, \qquad k = 0, 1 \ldots\ldots M-1$$

(4) A set of samples determined from $H_d(\omega)$ are identified as discrete fourier transform (DFT) coefficients. It is denoted by $H(k)$.

There are two types of design as follows :

(A) Type I :

Here frequency samples are :

$$\omega_k = \frac{2\pi k}{M}, \qquad k = 0, 1 \ldots\ldots M-1 \qquad \text{...(1)}$$

The sampled desired frequency response is denoted by $H(k)$.

$$\text{and } H(k) = H_d(\omega)\Big|_{\omega=\omega_k} \qquad k = 0, 1 \ldots\ldots M-1$$

$$\therefore\ H(k) = H_d\left(\frac{2\pi k}{M}\right) \qquad k = 0, 1 \ldots\ldots M-1 \qquad \text{...(2)}$$

The filter coefficient h(n) can be obtained by taking IDFT of H(k).