Function Approximation

Interpolation and curve-fitting

Most of engineering problem depends upon the solution of equation.

In some problem it is required to express a set of observation by an equation of best fit.

There are several cases when we have information or data available at several discrete location.

For example:

Tabulated values of the properties of steam, trigonometric, logarithmic and other function.

Experimental result taken in laboratory through direct measurement or on line measuring device or recorder are also available in similar form .

Sometime may be required to interpolate/extrapolate these data, or compute slopes, or evaluate integrals of function described by them.

Example:

Measure the temperature at several location in an infinite slab ($0 \le x \le L$) across which heat is being transferred at steady state.

Rate of heat transfer:

Across the surface compute the gradient dT/dx from the tabulated data.

Computation of mean temperature:

Compute the integral $\int_0^L T dx$ from the measured information.

Several equation of different types can be obtained to express the given data approximately

Problem is to find the equation of the curve of "best fit" which may be most suitable for predicting the unknown values.

If n pairs of observed values:

Fit the given data to an equation that contain n arbitrary constants and solve n simultaneous equation for n unknown.

If get the n equation but having less than n arbitrary constants-

> Use graphical method or method of moments or method of least square

Graphical method fails to give values of the unknown so accurately as does other method.

Some time least square is probably the best fit of given data.

Graphical Method

When the curve representing the given data is a linear law $y = mx + c$; we proceed as follows:

(i) Plot the given points on the graph paper taking a suitable scale.

(ii) Draw the straight line of best fit such that the points are evenly distributed about the line.

(iii) Taking two suitable points (x_1, y_1) and (x_2, y_2) on the line, calculate m, the slope of the line and c, its intercept on the y-axis.

When the points do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

Laws Reducible to the Linear Law

Some of the laws in common use can be reduced to the linear form by suitable substitutions:

1. When the law is $y = mx^n + c$

Taking $x^n = X$ and $y = Y$, the above law becomes $Y = mX + c$

2. When the law is $y = ax^n$.

Taking logarithms of both sides

It becomes $\log10y = \log10a + n \log10x$, Putting $\log10x = X$ and $\log10y = Y$, it reduces to the form $Y = nX + c$, where $c = \log10a$.

3. When the law is $y = ax^n + b \log x$.

Writing it as $\frac{y}{\log x} = a\frac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$, the given law becomes, $Y = aX + b$.

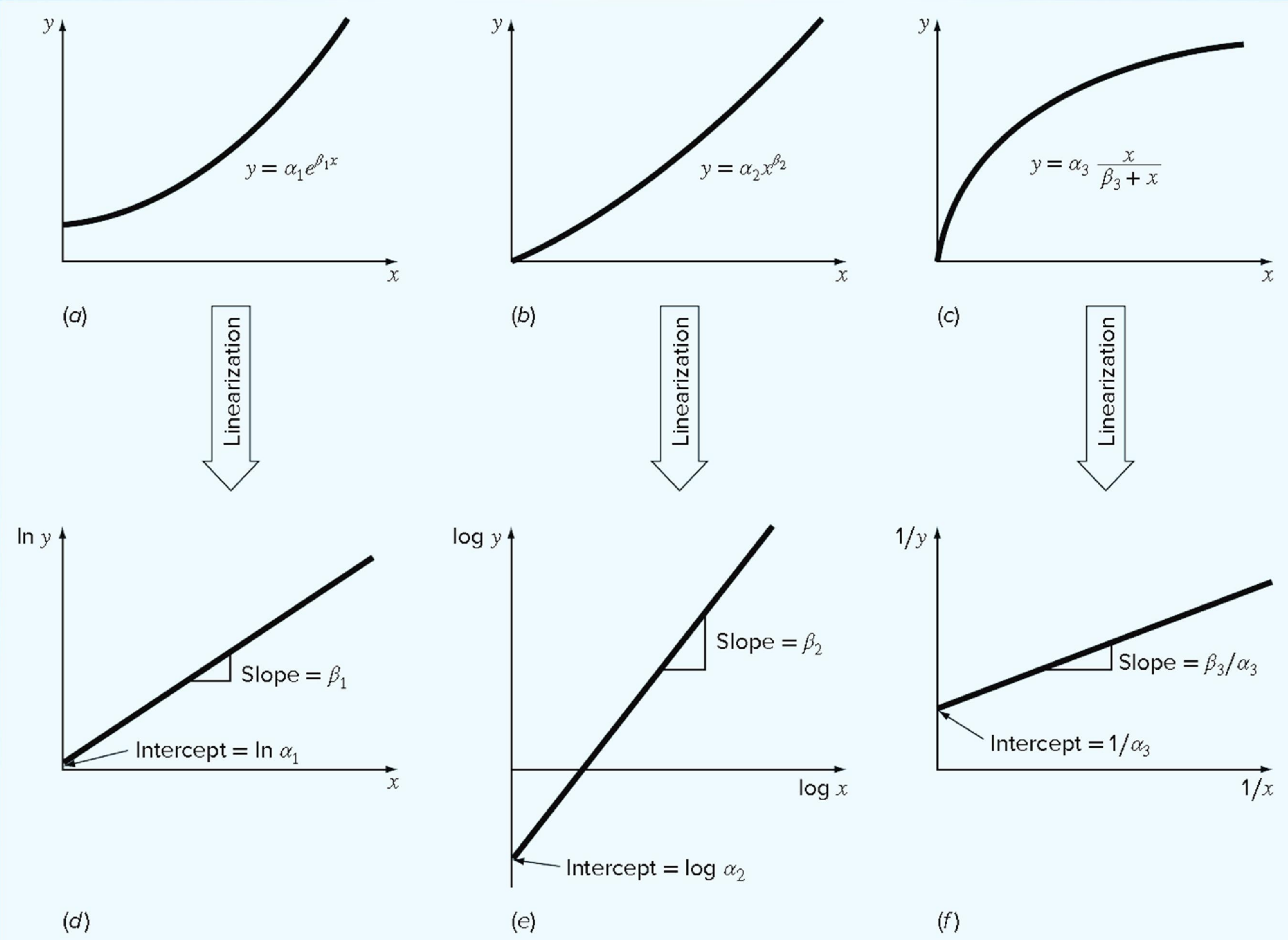
4. When the law is $y = ae^{bx}$.

Taking logarithms, it becomes $\log_{10}y = (b \log_{10}e)x + \log_{10}a$. Putting $x = X$ and $\log_{10}y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10}e$ and $c = \log_{10}a$.

5. When the law is $xy = ax + by$.

Dividing by x, we have $y = b\frac{y}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.



y
$y = \alpha_1 e^{\beta_1 x}$
x
(a)
Linearization
y
$y = \alpha_2 x^{\beta_2}$
x
(b)
Linearization
y
$y = \alpha_3 \frac{x}{\beta_3 + x}$
x
(c)
Linearization
ln y
Slope = β_1
Intercept = ln α_1
x
(d)
log y
Slope = β_2
log x
Intercept = log α_2
(e)
1/y
Slope = β_3/α_3
Intercept = $1/\alpha_3$
1/x
(f)

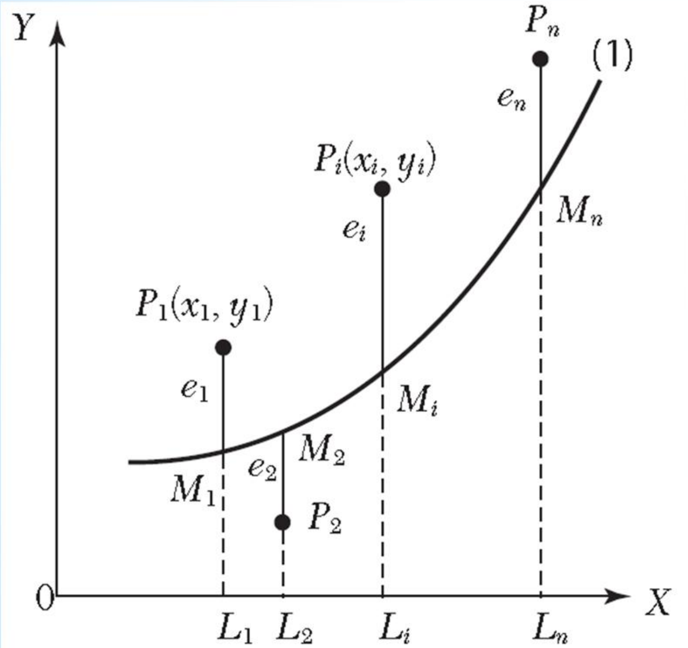
Principle of Least Squares

The graphical method has the obvious drawback of being unable to give a unique curve of fit.

The principle of least squares, provides an elegant procedure of fitting a unique curve to a given data.

Let the curve $y = a + bx + cx^2 + \cdots\cdots + kx^m$

be fitted to the set of data points $(x_1, y_1), (x_2, y_2), \cdots\cdots, (x_n, y_n)$.



Determine the constants *a*, *b*, *c*,... *k* such that they represents the curve of best fit.

In the case of $n = m$, when substituting the values (x_i, y_i) in equ., we get n equations from which a unique set of n constants can be found.

But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants.

So we try to determine the values of $a, b, c, \cdots\cdots k$ which satisfy all the equations as nearly as possible and thus may give the best fit.

In such cases, we apply the *principle of least squares.*

At $x = x_i$, the observed (experimental) value of the ordinate is y_i and the corresponding value on the fitting curve is $a + bx_i + cx_i^2 \cdots\cdots kx_i^m$ ($=\eta_i$) which is the expected (or calculated) value.

The difference of the observed and the expected values, i.e., $y_i - \eta_i$($= e_i$) is called the error (or residual) at $x = x_i$.

Clearly some of the errors $e_1, e_2, \cdots\cdots, e_n$ will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum, i.e., $E = e_1^2 + e_2^2 \cdots\cdots e_n^2$.

The curve of best fit is that for which e's are as small as possible, i.e., the sum of the squares of the errors is a minimum. This is known as the principle of least squares (French mathematician Adrien Marie Legendre in 1806).

Method of Least Squares

Suppose we want to fit the curve $y = a + bx + cx^2$ to a given set of observations $(x_1, y_1), (x_2, y_2), \cdots\cdots (x_5, y_5)$.

For any x_i, the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

The sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2$$

$$= [y_1-(a+bx_1+cx_1^2)]^2 + [y_2-(a+bx_2+cx_2^2)]^2 + \cdots\cdots + [y_5-(a+bx_5+cx_5^2)]^2$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = -2[y_1-(a+bx_1+cx_1^2)]^2 - 2[y_2-(a+bx_2+cx_2^2)]^2 - \cdots\cdots - 2[y_5-(a+bx_5+cx_5^2)]^2 \qquad (1)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a+bx_1+cx_1^2)]^2 - 2x_2[y_2 - (a+bx_2+cx_2^2)]^2 -\cdots - 2x_5[y_5 - (a+bx_5+cx_5^2)]^2 \quad (2)$$

$$\frac{\partial E}{\partial c} = 0 = -2x_1^2[y_1 - (a+bx_1+cx_1^2)]^2 - 2x_2^2[y_2 - (a+bx_2+cx_2^2)]^2 -\cdots - 2x_5^2[y_5 - (a+bx_5+cx_5^2)]^2 \quad (3)$$

Equation (1) simplifies to

$$y_1 + y_2 + \cdots + y_5 = 5a + b(x_1 + x_2 + \cdots + x_5) + c\,(x_1^2 + x_2^2 + \cdots + x_5^2)$$

i.e., $\sum y_i = 5a + b \sum x_i + c \sum x_i^2$ (4)

Equation (2) becomes

$$x_1y_1 + x_2y_2 + \cdots + x_5y_5 = a(x_1 + x_2 + \cdots + x_5) + b\,(x_1^2 + x_2^2 + \cdots + x_5^2) + c\,(x_1^3 + x_2^3 + \cdots + x_5^3)$$

i.e., $\sum x_iy_i = a \sum x_i + b \sum x_i^2 + c \sum x_i^3$ (5)

Similarly (3) simplifies to

$$\sum x_i^2y_i = a \sum x_i^2 + b \sum x_i^3 + c \sum x_i^4 \quad (6)$$

The equations (4), (5) and (6) are known as normal equations and can be solved as simultaneous equations in a, b, c. The values of these constants when substituted in (1) give the desired curve of best fit.

Method of Moments

Let (x_1, y_1), (x_2, y_2), …… (x_n, y_n) be the set of n observations such that

$x_2 - x_1 = x_3 - x_2$ …… $x_n - x_{n-1} = h$ (say)

We define the moments of the observed values of y as follows:

m_1, the 1st moment = $h\sum y$

m_2, the 2nd moment = $h\sum xy$

m_3, the third moment = $h\sum x^2y$ and so on.

Let the curve fitting the given data be $y = f(x)$. Then the moments of the calculated values of y are

μ_1, the 1st moment $\int y\,dx$

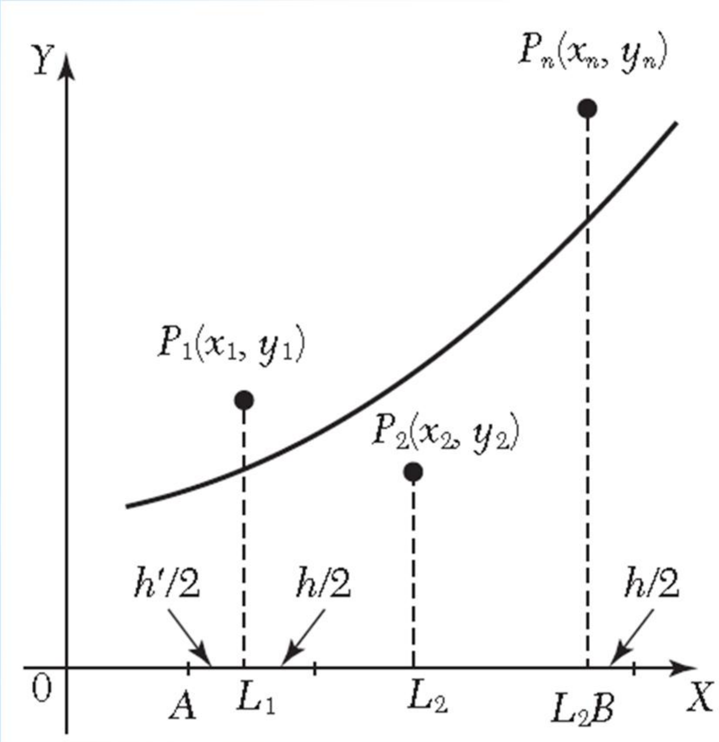
μ_2, the 2nd moment $\int xy\,dx$

μ_3, the 3rd moment $\int x^2y\,dx$ and so on.

This method is based on the assumption that the moments of the observed values of y are respectively equal to the moments of the calculated values of y,

i.e., $m_1 = \mu_1$, $m_2 = \mu_2$, $m_3 = \mu_3$ etc.

These equations (known as *observation equations*) are used to determine the constants in f(x).



In Fig., y_1 the ordinate of $P_1(x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1-h/2, x_1+h/2)$.

Similarly y_n, the ordinate of $P_n(x=x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n-h/2, x_n+h/2)$.

If A and B be the points such that

$OA = x_1 - h/2$ and $OB = x_n + h/2$,

$$\mu_1 = \int y\, dx = \int_{x_1-h/2}^{x_1+h/2} f(x)dx$$

then $\mu_2 = \int_{x_1-h/2}^{x_1+h/2} xf(x)dx$ $\mu_3 = \int_{x_1-h/2}^{x_1+h/2} x^2 f(x)dx$

Interpolation

The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found.

Conversely, if the form of $f(x)$ is not known it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (xi, yi).

In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which is known as the interpolating function or smoothing function.

If $\phi(x)$ is a polynomial, then it called the interpolating polynomial and the process is called the polynomial interpolation.

Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation.

But we shall confine ourselves to polynomial interpolation only.

Methods for interpolation:

Interpolation with equal intervals

- Newton's forward interpolation formula
- Newton's backward interpolation formula
- Central difference interpolation formulae
- Stirling's formula

Interpolation with unequal intervals

- Lagrange's interpolation formula
- Divided differences
- Newton's divided difference formula

Spline interpolation

- Cubic spline

Newton's Forward Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots\ldots, y_n$ corresponding to the values $x_0, x_1, \ldots\ldots, x_n$ of x.

Let these values of x be equispaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots.)$.

Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots\ldots, y(x_n) = y_n$.

Differences $y_1 - y_0, y_2 - y_1, y_3 - y_2$ --------- $y_n - y_{n-1}$ denoted by $\Delta y_0, \Delta y_1, \Delta y_2$ --------- Δy_{n-1} respectively called first forward differences.

Δ -- forward difference operator

Second forward differences

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ p^{th} forward difference

Define a dimensionless value of x by

$$\alpha = \frac{(x - x_0)}{\Delta x}$$

Write $y(x) = y(x_0 + \alpha \, \Delta x)$ and expand through Taylor series

$y(x_0+\alpha\Delta x) = \mathbf{a_0 + a_1\,\alpha + a_2\,\alpha^2 + a_2\,\alpha^3 \,\ldots\ldots\ldots\ldots\ldots\ldots\, a_n\,\alpha^n}$

This involves n+1 unknown coefficient which can be determined by equating the analytical values at the base points x_i to known values of y

$$
\begin{array}{lll}
Y(x_0) & = \mathbf{a_0} & = y_0 \\
Y(x_0 + \Delta x) & = \mathbf{a_0 + a_1 + a_2 + a_2 \,\ldots\ldots\ldots\ldots\ldots\ldots\, a_n} & = y_1 \\
Y(x_0 + 2\Delta x) & = \mathbf{a_0 + 2a_1 + 2^2 a_2 + 2^3 a_2 \,\ldots\ldots\ldots\ldots\ldots\ldots\, 2^n a_n} & = y_2 \\
. & & \\
. & & \\
Y(x_0 + n\Delta x) & = \mathbf{a_0 + na_1 + n^2 a_2 + n^3 a_2 \,\ldots\ldots\ldots\ldots\ldots\ldots\, n^n a_n} & = y_n
\end{array}
$$

We have a unique solution for a values of coefficient.

Polynomial can be written in slightly different form using the forward difference to give Newton forward difference formula

$$Y(x_0 + \alpha\Delta x) = y_0 + (\alpha\Delta y_0) + \frac{\alpha(\alpha-1)}{2|}(\Delta^2 y_0) + \frac{\alpha(\alpha-1)(\alpha-2)}{3|}(\Delta^3 y_0) \ldots\ldots\ldots$$

$$\frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots\ldots(\alpha-n+1)}{n|}(\Delta^n y_0)$$

$$R = \frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots\ldots(\alpha-n)}{n+1|}(\Delta^{n+1} y[\zeta(\alpha)])$$

Difference Table for Newton Forward Interpolation

x_i	y_i	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		$\Delta y_0 = y_1 - y_0$					
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$				
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$			
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$		
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$		$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$		$\Delta^6 y_0 = \Delta^5 y_1 - \Delta^5 y_0$
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$		$\Delta^5 y_1 = \Delta^4 y_2 - \Delta^4 y_1$	
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$		$\Delta^4 y_2 = \Delta^3 y_3 - \Delta^3 y_2$		
		$\Delta y_4 = y_5 - y_4$		$\Delta^3 y_3 = \Delta^2 y_4 - \Delta^2 y_3$			
x_5	y_5		$\Delta^2 y_4 = \Delta y_5 - \Delta y_4$				
		$\Delta y_5 = y_6 - y_5$					
x_6	y_6						

Newton's Backward Interpolation Formula

Let n number of data point say $y_0, y_1, \ldots.., y_n$ corresponding to the values $x_0, x_1, \ldots\ldots, x_n$.

Let these values of x be equispaced such that $x_i = x_0 + ih$ (i = 0, 1, ….).

Assuming y(x) to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots\ldots, y(x_n) = y_n$.

The differences $y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2$ --------- $y_n - y_{n-1}$ denoted by $\nabla y_1,\ \nabla y_2,\ \nabla y_3,$ --------- ∇y_n respectively called first backward differences.

∇ -- backward difference operator

Second backward differences

$$\nabla^2 y_{r+1} = \nabla y_{r+1} - \nabla y_r$$

In general $\nabla^p y_{r+1} = \nabla^{p-1} y_{r+1} - \nabla^{p-1} y_r$ p^{th} backward difference

Dimensionless value of x is defined by

$$\alpha = \frac{(x - x_n)}{\Delta x}$$

Write $y(x) = y(x_n + \alpha \Delta x)$ and expand through Taylor series

$y(x_n + \alpha\Delta x) = a_0 + a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^3 \ldots\ldots\ldots\ldots\ldots\ldots\ a_n \alpha^n$

This involves n+1 unknown coefficient which can be determined by equating the observed values at the base points x_i to known values of y

$Y(x_n) \qquad = a_0 \qquad = y_n$

$Y(x_n - \Delta x) \qquad = a_0 - a_1 + a_2 - a_3 \ldots\ldots\ldots\ldots\ldots\ldots\ a_n \qquad = y_{n-1}$

$Y(x_n - 2\Delta x) \qquad = a_0 - 2a_1 + 2^2a_2 - 2^3a_3 \ldots\ldots\ldots\ldots\ldots\ldots\ 2^na_n \qquad = y_{n-2}$

.

.

$Y(x_n - n\Delta x) \qquad = a_0 - na_1 + n^2a_2 - n^3a_3 \ldots\ldots\ldots\ldots\ldots\ldots\ n^na_n \qquad = y_0$

We have a unique solution for a values of coefficient.

Polynomial can be written in slightly different form using the backward difference to give Newton backward difference formula

$$Y(x_n + \alpha\Delta x) = y_n + \alpha(\nabla y_n) + \frac{\alpha(\alpha+1)}{2|}(\nabla^2 y_n) + \frac{\alpha(\alpha+1)(\alpha+2)}{3|}(\nabla^3 y_n) \ldots\ldots\ldots$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots\ldots\ldots(\alpha+n-1)}{n|}(\nabla^n y_n)$$

$$R = \frac{\alpha(\alpha+1)(\alpha+2)\ldots\ldots\ldots(\alpha+n)}{n+1|}(\nabla^{n+1} y[\zeta(\alpha)])$$

Difference Table for Newton Backward Interpolation

x_i	y_i	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		$\nabla y_1 = y_1 - y_0$					
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$				
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$			
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$		
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$		$\nabla^5 y_5 = \nabla^4 y_5 - \nabla^4 y_4$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		$\nabla^4 y_5 = \nabla^3 y_5 - \nabla^3 y_4$		$\nabla^6 y_6 = \nabla^5 y_6 - \nabla^5 y_5$
		$\nabla y_4 = y_4 - y_3$		$\nabla^3 y_5 = \nabla^2 y_5 - \nabla^2 y_4$		$\nabla^5 y_6 = \nabla^4 y_6 - \nabla^4 y_5$	
x_4	y_4		$\nabla^2 y_5 = \nabla y_5 - \nabla y_4$		$\nabla^4 y_6 = \nabla^3 y_6 - \nabla^3 y_5$		
		$\nabla y_5 = y_5 - y_4$		$\nabla^3 y_6 = \nabla^2 y_6 - \nabla^2 y_5$			
x_5	y_5		$\nabla^2 y_6 = \nabla y_6 - \nabla y_5$				
		$\nabla y_6 = y_6 - y_5$					
x_6	y_6						

Linear difference operator E and Δ

We have already introduced the operators Δ, and ∇. Besides these, there are the operators E, δ, and μ, which we define below:

Shift operator E is the operation of increasing the argument x by Δx so that

$$Ey(x) = y(x + \Delta x),$$

$$E^2y(x) = E[y(x + \Delta x)] = y(x + 2\Delta x),$$

$$E^3y(x) = y(x + 3\Delta x) \text{ etc.}$$

The inverse operator E^{-1} is defined by $E^{-1}y(x) = y(x - \Delta x)$

$\Delta y(x) = Ey(x) – y(x) = (E – 1)y(x)$

This lead to following relationship between the two operator

$E = 1 + \Delta = e^{hD}$

$$E^\alpha = (1 + \Delta)^\alpha = 1 + \alpha\,\Delta + \frac{\alpha(\alpha-1)}{2|}\Delta^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3|}\Delta^3 \text{ ....}$$

$$\frac{\alpha(\alpha-1)(\alpha-2)\text{.........}(\alpha-n+1)}{n|}\Delta^n \text{ ....}$$

Thus

$$Y(x_0 + \alpha\Delta x) = E^{\alpha}y(x) = y_0 + (\alpha\Delta y_0) + \frac{\alpha(\alpha-1)}{2|}(\Delta^2 y_0) + \frac{\alpha(\alpha-1)(\alpha-2)}{3|}(\Delta^3 y_0) \ldots\ldots\ldots \frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots(\alpha-n+1)}{n|}(\Delta^n y_0)$$

$$R = \frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots(\alpha-n)}{n+1|}(\Delta^{n+1} y[\zeta(\alpha)])$$

Similarly

$$\nabla y(x + \Delta x) = y(x + \Delta x) - y(x)$$

$$\nabla Ey(x) = Ey(x) - y(x)$$

$$Y(x) = Ey(x)\,[1 - \nabla]$$

$$1 = E\,[1 - \nabla]$$

$$E = 1/\,[1 - \nabla] = [1 - \nabla]^{-1}$$

$$E^{\alpha} = [1 - \nabla]^{-\alpha}$$

Central Difference Interpolation Formulae

In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values.

Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations in next slide:

In this system, the central difference operator δ is defined by the relations:

$y_{-1} - y_{-2} = \delta y_{-3/2},\quad y_0 - y_{-1} = \delta y_{-1/2},\ \ldots\ldots\ldots\ldots\ y_2 - y_1 = \delta y_{3/2}$

Similarly, higher order central differences are defined as

$\delta y_{-1/2} - \delta y_{-3/2} = \delta^2 y_{-1},\quad \delta^2 y_0 - \delta^2 y_{-1} = \delta^3 y_{-1/2}$ and so on.

These differences are shown in Table.

Difference Table for Central Interpolation

x_i	y_i	δ	δ^2	δ^3	δ^4
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2}=\delta y_{-3/2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2}=\delta^2 y_{-1}$		
		$\Delta y_{-1}=\delta y_{-1/2}$		$\Delta^3 y_{-2}=\delta^3 y_{-1/2}$	
x_0	y_0		$\Delta^2 y_{-1}=\delta^2 y_0$		$\Delta^4 y_{-2}=\delta^4 y_0$
		$\Delta y_0= \delta y_{1/2}$		$\Delta^3 y_{-1}=\delta^3 y_{1/2}$	
$x_0 + h$	y_1		$\Delta^2 y_0=\delta^2 y_1$		
		$\Delta y_1=\delta y_{3/2}$			
$x_0 + 2h$	y_2				

Polynomial can be written form using the Newton forward difference formula

$$Y(x_0 + \alpha\Delta x) = y_0 + (\alpha\Delta y_0) + \frac{\alpha(\alpha-1)}{2|}(\Delta^2 y_0) + \frac{\alpha(\alpha-1)(\alpha-2)}{3|}(\Delta^3 y_0) \ldots\ldots\ldots$$

$$\frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots\ldots(\alpha-n+1)}{n|}(\Delta^n y_0)$$

$$R = \frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots\ldots(\alpha-n)}{n+1|}(\Delta^{n+1} y[\zeta(\alpha)])$$

From Stirling's formula

$$Y(x_0 + \alpha\Delta x) = y_0 + (\alpha\delta y_{1/2}) + \frac{\alpha(\alpha-1)}{2|}(\delta^2 y_1) + \frac{\alpha(\alpha-1)(\alpha-2)}{3|}(\delta^3 y_{3/2}) \ldots\ldots\ldots$$

$$\frac{\alpha(\alpha-1)(\alpha-2)\ldots\ldots\ldots(\alpha-n+1)}{n|}(\delta^n y_{n/2})$$

Now define

$$\delta y_{1/2} = ½(\delta y_{-1/2} + \delta y_{1/2}) + ½(\delta y_{1/2} - \delta y_{-1/2})$$

$$= \mu\, \delta y_0 + ½\, \delta^2 y_0$$

$\delta^2 y_1 = \delta^2 y_0 + (\delta^2 y_1 - \delta^2 y_0) = \delta^2 y_0 + \delta^3 y_{1/2}$

$= \delta^2 y_0 + ½ (\delta^3 y_{-1/2} + \delta^3 y_{1/2}) + ½ (\delta^3 y_{1/2} - \delta^3 y_{-1/2})$

$= \delta^2 y_0 + \mu\, \delta^3 y_{0} + ½\, \delta^4 y_0$ and so on.

On substitution and simplification

$$Y(x_0 + \alpha\Delta x) = y_0 + \alpha\,(\mu\delta y_0) + \frac{\alpha^2}{2|}(\delta^2 y_0) + \frac{\alpha(\alpha^2-1^2)}{3|}(\mu\delta^3 y_0) + \frac{\alpha^2(\alpha^2-1^2)}{4|}(\delta^4 y_0) + \frac{\alpha(\alpha^2-1^2)(\alpha^2-2^2)}{5|}(\mu\delta^4 y_0) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Difference Table for Central Interpolation

x_i	y_i	δ	δ^2	δ^3	δ^4
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2}=\delta y_{-3/2}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2}=\delta^2 y_{-1}$		
		$\Delta y_{-1}=\delta y_{-1/2}$		$\Delta^3 y_{-2}=\delta^3 y_{-1/2}$	
x_0	y_0		$\Delta^2 y_{-1}=\delta^2 y_0$		$\Delta^4 y_{-2}=\delta^4 y_0$
		$\Delta y_0= \delta y_{1/2}$		$\Delta^3 y_{-1}=\delta^3 y_{1/2}$	
$x_0 + h$	y_1		$\Delta^2 y_0=\delta^2 y_1$		
		$\Delta y_1=\delta y_{3/2}$			
$x_0 + 2h$	y_2				

Interpolation with Unequal Intervals

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument.

It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae:

(i) Lagrange's interpolation formula

(ii) Newton's general interpolation formula with divided differences.

Lagrange's Interpolation Formula

If $y = f(x)$ takes the value $y_0, y_1, \ldots\ldots, y_n$ corresponding to $x = x_0, x_1, \ldots\ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

This is known as Lagrange's interpolation formula for unequal intervals.

Divided Differences

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated.

This labor of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called "divided differences."

Before deriving this formula, we shall first define these differences.

If $(x_0,y_0),(x_1,y_1),(x_2,y_2),\ldots\ldots\ldots$be given points, then the first divided difference for the arguments x_0, x_1 is defined by the relation $[x_0, x_1]$ or

$$\underset{x_1}{\Delta} y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly $[x_1, x_2]$ or $\underset{x_2}{\Delta} y_0 = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3]$ or $\underset{x_3}{\Delta} y_0 = \dfrac{y_3 - y_2}{x_3 - x_2}$

The second divided difference for x_0, x_1, x_2 is defined as

$[x_0, x_1, x_2]$ or $\underset{x_1, x_2}{\Delta^2} y_0 = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3]$ or $\underset{x_1, x_2, x_3}{\Delta^3} y_0 = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_2 - x_0}$

Newton's Divided Difference Formula

Let $y_0, y_1, \ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots\ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So that $$y = y_0 + (x - x_0)[x, x_0]$$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

Which gives $$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$

Also $$[x, x_0, x_1, x_2] = \frac{[x.x_{0,} x_1] - [x.x_{0,} x_2]}{x - x_2}$$

Which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of [x, x0, x1] in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_n)[x_0, x_1, \cdots x_n]$$

which is called Newton's general interpolation formula with divided differences.

Inverse Interpolation

For a given set of values of x and y

- Finding the values of y corresponding to a certain value of x
- On the other hand process of estimating the values of x for a value of y is called inverse interpolation

Lagrange's formula

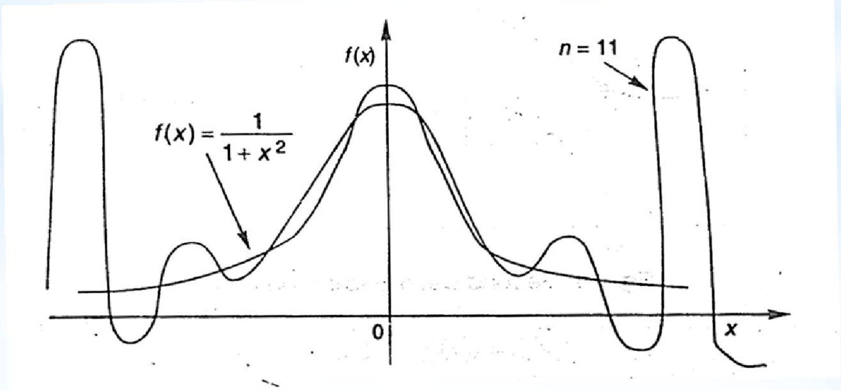
$$\mathrm{x} = \frac{(y-y_1)(y-y_2)\ldots\ldots.(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\ldots\ldots.(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}x_1 + \cdots\ldots\ldots.. \frac{(y-y_0)(y-y_1)\ldots\ldots.(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})}x_n$$

Spline Interpolation

An interpolation polynomial of degree n can be constructed and used a given set of values of functions.

There are situation in which this approach is likely to face problems and produce incorrect estimates. Because interpolation takes a global rather than a local view of data.

It has been proved that when n is large compared to the order of the true function, the interpolation polynomial of degree n does not provide accurate result at the ends of the range.



Interpolation polynomials contains undesireable maxima and minima between the data points

Cubic Spline Approximation

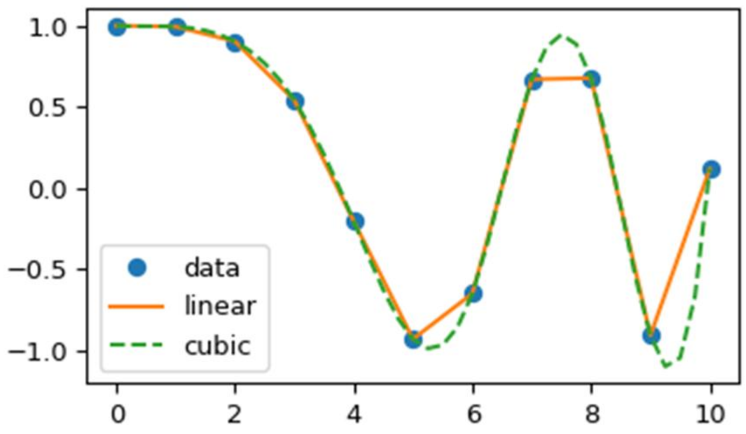
In the polynomial approximation higher the degree of polynomial, the more oscillatory behavior of its.

Pade introduced an approximation method to overcome this type of problem but it is inconvenient to work with.

An alternate technique which simplifies the polynomial expansion and avoid their oscillatory nature, is the (lower order) cubic (or equivalent) spline approximation.

In this method divide the entire range of points into subintervals and use low order polynomial to interpolate each subintervals such polynomials are called piecewise polynomials.

Piecewise polynomial exhibit discontinuity at the interpolating points (which connect these points).
It is possible to construct piecewise polynomials that prevent such discontinuities at the connecting points such piecewise polynomials are called spline function or simply spline.



Connecting points are called as knots or nodes.
A spline function s(x) of degree m must satisfy the following conditions

1. s(x) is a polynomial of degree utmost m in each of the subintervals $[x_i, x_{i+1}]$ i = 0, 1, 2, …. N
2. S(x) and its derivatives of order 1, 2, … m-1 are continuous in the range $[x_0, x_n]$

Construct the cubic spline function which would interpolate the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ .... (x_n, y_n). The cubic spline $s(x)$ consists of (n-1) cubics corresponding to (n-1) subintervals. If we denote such cubic

$$S_i(x_i) = k_1 + k_2(x-x_i) + k_3(x-x_i)^2 + k_4(x-x_i)^3 \quad i = 1, 2, ...n$$

These cubics must satisfy the following conditions

1. $s(x)$ must interpolate y at all the points $x_0, x_1, x_2, \ldots x_n$.

$$s(x_i) = y_i$$

2. The function values must be equal at all the interior knots

$$s_i(x_i) = s_{i+1}(x_i)$$

3. The first derivatives at the interior knots must be equal

$$s'_i(x_i) = s'_{i+1}(x_i)$$

4. The second derivatives at the interior knots must be equal

$$s''_i(x_i) = s''_{i+1}(x_i)$$

The second derivatives at the end points are zero

$$s''(x_0) = s''(x_n) = 0$$

<u>Step 1.</u>

Let us first consider the second derivatives. Since $s(x_i)$ is a cubic function its second derivatives $s''_i(x)$ is a straight line. The straight line can be represented by a first order (linear) Lagrange interpolating polynomial.

Line pass through the points $[x_i, s''_i(x_i)]$ and $[x_{i-1}, s''_i(x_{i-1})]$

$$s''_i(x) = [(x-x_i)/(x_{i-1} - x_i)]\, s''_i(x_{i-1}) + [(x-x_{i-1})/(x_i - x_{i-1})]\, s''_i(x_i) \qquad (1)$$

unknown $s''_i(x_{i-1})$, and $s''_i(x_i)$ are to be determined

Let us denote

$$s''_i(x_{i-1}) = a_{i-1} \quad \text{and} \quad s''_i(x_i) = a_i$$

$$x-x_i = u_i \qquad\qquad x-x_{i-1} = u_{i-1}$$

$$x_i - x_{i-1} = u_{i-1} - u_i = h_i$$

Then equ (1) becomes

$$s''_i(x) = [u_i/-h_i]\, a_{i-1} + [u_{i-1}/h_i]\, a_i$$

Step 2.

Integrating twice the above equation

$$s_i(x) = [u_i^3/-6h_i]\, a_{i-1} + [u_{i-1}^3/6h_i]\, a_i + C_1x+C_2$$

Linear part C_1x+C_2 can be expressed as $b_1(x-x_{i-1}) + b_2(x-x_i)$ with suitable choice of b_1 and b_2

$$C_1x+C_2 = b_1(x-x_{i-1}) + b_2(x-x_i) = b_1u_{i-1} + b_2u_i$$

Equation becomes

$$s_i(x) = [a_i\, u_{i-1}^3 - a_{i-1}\, u_i^3]/6h_i + b_1u_{i-1} + b_2u_i$$

Step 3.

Determine the coefficient b_1 and b_2

We know from condition 1

$$s(x_i) = y_i \text{ and } s(x_{i-1}) = y_{i-1}$$

At $x = x_i$ $u_i = 0$ $u_{i-1} = h_i$

$$y_i = a_i h_i^2/6 + b_1 h_i \qquad b_1 = y_i/h_i - a_i h_i/6$$

Similarly at $x = x_{i-1}$ $u_{i-1} = 0$ $u_i = -h_i$

Therefore $y_{i-1} = a_{i-1} h_i^2/6 + b_2 h_i$ $\qquad b_2 = - y_{i-1}/h_i - a_{i-1} h_i/6$

Substituting b_1 and b_2 and after rearrangement we get

$$s_i(x) = [a_{i-1} /6h_i](h_i^2 u_i - u_i^3) + [a_i/6h_i] (u_{i-1}^3 - h_i^2 u_{i-1}) + (y_i u_{i-1} - y_{i-1}u_i)/h_i \qquad (2)$$

a_{i-1} and a_i are the unknown

Step 4.

Final step is to satisfy all conditions discussed above for evaluation of unknown

$$s'_i(x_i) = s'_{i+1}(x_i)$$

Differntiating equ. (2)

$$s'_i(x)=\frac{a_{i-1}}{6hi}(h_i^2-3u_i^2)+\frac{a_i}{6hi}(3ui_{-1}^2-h_i^2)+\frac{1}{h_i}(y_i-yi_{-1)}$$

Setting $x = x_i$

$$s'_i(xi)=\frac{a_{i-1}}{6}(h_i)+\frac{a_i}{3}(h_i)+\frac{1}{h_i}(y_i-yi_{-1)}$$

Similarly

$$s'_{i+1}(xi)=-\frac{a_i}{3}(h_{i+1})-\frac{a_{i+1}}{6}(h_{i+1})+\frac{1}{h_{i+1}}(y_{i+1}-yi_{)}$$

Since $\qquad s'_i(x_i) = s'_{i+1}(x_i)$

We have

$$h_i\,a_{i-1}+2(h_i+h_{i+1)}\,a_i+h_{i+1}\,a_{i+1}=6\left[\frac{(y_{i+1}-yi_{)}}{h_{i+1}}-\frac{(y_i-yi_{-1)}}{h_i}\right] \qquad (3)$$

Equ. (3) written for all interior knots then we get n-1 simultaneous equation containing n+1 unknown

After applying the 4^{th} condition that the second derivative at the end points are zero then

$$a_0 = a_n = 0 \qquad \{s''(x_0) = s''(x_n) = 0\}$$

[Cubic splines with zero second derivatives at the end points are called natural cubic splines.]

Thus we have n-1 equ. with n-1 unknown which can be solved easily

Type equation here.

System of n-1 equ. can be expressed as

$$\begin{array}{ccccccc}
2(h_1 + h_2) & h_2 & 0 & 0 & 0 & 0 & ----0 \\
h_2 & 2(h_2 + h_3) & h_3 & 0 & 0 & 0 & ----0 \\
0 & h_3 & 2(h_3 + h_4) & h_4 & 0 & 0 & ----0 \\
- & - & - & - & - & 0 & ----0 \\
- & - & - & - & - & 0 & ----0 \\
0 & 0 & 0 & 0 & 0 & h_{n-1} & 2(h_{n-1} + hn)
\end{array}$$

$$= X$$

$$X \begin{bmatrix} a_1 \\ a_2 \\ \\ - \\ - \\ - \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \\ - \\ - \\ - \\ D_{n-1} \end{bmatrix}$$

$$D_i = 6\left[\frac{(y_{i+1} - y_i)}{h_{i+1}} - \frac{(y_i - y_{i-1})}{h_i}\right]$$