Gaussian Elimination Method

This is the Elementry Elimination method and it reduces the System of Equation to an Equivalent upper triangular System which can be Solved by $\boxed{\text{back Substitution}}$.

Let the System be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad -(1)$$

Step 1. # To Eliminate x_1 from the second Equation, we multiply the first Equation by $-\frac{a_{21}}{a_{11}}$

i.e Take a multiplier $m_{21} = -\frac{a_{21}}{a_{11}}$

multiply 1^{st} row of (1) by m_{21} and add to the second 2^{nd}

i.e $R_2 \leftarrow R_2 + m_{21} R_1$

~~Take~~ Similarly, to eliminate x_1 from the third equation, we multiply the first equation by $-\frac{a_{31}}{a_{11}}$ and add it to the third.

i.e. Take a multiplier $m_{31} = -\frac{a_{31}}{a_{11}}$

Multiply 1'st row by m_{31} and add to 3rd

i.e $R_3 \leftarrow R_3 + m_{31} \cdot R_1$

\# Take a multiplier $m_{41} = \frac{-a_{41}}{a_{11}}$

multiply 1st row by m_{41} and add to 4th

i.e $R_4 \leftarrow R_4 + m_{41} \cdot R_1$

Note: First row is being multiplied by appropriate multiplier and added to different rows. The rows which is being multiplied (first in this case) is known as "pivotal" row and the element a_{11}, the divisor, is called "pivoltal element" or simply 'pivot'.

After execution of the first stage, the system will have the following form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & a_{34}^{(1)} \\ 0 & a_{42}^{(1)} & a_{43}^{(1)} & a_{44}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ b_4^{(1)} \end{bmatrix}. \quad \text{— (1)}$$

The elements with superscript 1 idicate that they have changed after state 1.

Stage 2: we omit the first row and first column of the coefficient matrix (1), thereby dealing with 3×3 system of Equations. Thus Stage 2 has two steps

(i) Take multiplier $m_2 = -\frac{a_{32}^{(1)}}{a_{22}^{(1)}}$

multiply 2^{nd} row of (1) by m_{32} and add to the 3^{rd}

i.e $R_3 \leftarrow R_3 + m_{32} \cdot R_2$

(ii) take multiplier $m_{42} = \frac{-a_{42}^{(1)}}{a_{22}^{(1)}}$

multiply 2^{nd} row by m_{42} and add to 4^{th}

i.e $R_4 \Leftarrow R_4 + m_{42} \cdot R_2$

Here second row is the $\boxed{\text{pivotal row and } a_{22}^{(1)} \text{, the}}$ $\boxed{\text{pivot.}}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \\ 0 & 0 & a_{43}^{(2)} & a_{44}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ b_4^{(2)} \end{bmatrix} \quad -(2)$$

Stage 4: we ignore first two rows and first two column in (2) and deal with a 2×2 system having superscript 2. This stage has only one step.

Take the multiplier $m_{43} = \dfrac{-a_{43}^{(2)}}{a_{33}^{(2)}}$

multiply by 3rd row of (2) by m_{43} and add to 4th.

$$R_4 \leftarrow R_4 + m_{43} \cdot R_1$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ b_4^{(3)} \end{bmatrix}$$

upper triangular form.

By back substitution.

$$x_4 = b_4^{(3)} / a_{44}^{(3)}.$$

Substituting x_4 in third equation gives the value of x_3

x_3 and x_4 in second equation we get x_2.

x_2, x_3, x_4 in first equation we get x_1.

Example Solve the following system of Equation by Gauss elimination method.

$$3x_1 + 2x_2 + x_3 - 4x_4 = 5$$
$$x_1 - 5x_2 + 2x_3 + x_4 = 18$$
$$5x_1 + x_2 - 3x_3 + 2x_4 = -4$$
$$2x_1 + 3x_2 + x_3 + 5x_4 = 1$$

Sol.

$$\begin{bmatrix} 3 & 2 & 1 & -4 \\ 1 & -5 & 2 & 1 \\ 5 & 1 & -3 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ -4 \\ 11 \end{bmatrix}$$

$$m_{21} = \frac{-a_{21}}{a_{11}} = -\frac{1}{3} = -0.333$$

$$R_2 \leftarrow R_2 - 0.333 R_1$$

$$m_{31} = -\frac{a_{31}}{a_{11}} = -1.667, \quad R_3 \leftarrow R_3 - 1.667 R_1$$

$$m_{41} = \frac{-a_{41}}{a_{11}} = -\frac{2}{3} = -0.667$$

$$R_4 \leftarrow R_4 - 0.667 R_1$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & -4 & 5 \\ 0 & -5.66 & 1.667 & 2.332 & 16.335 \\ 0 & -2.334 & -4.667 & 8.668 & -19.335 \\ 0 & 1.666 & 0.333 & 7.668 & 7.665 \end{array}\right]$$

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$$m_{32} = \frac{-a_{32}^{(1)}}{a_{22}^{(1)}} = -\frac{2.334}{5.666} = -0.412$$

$R_3 \leftarrow R_3 - 0.412 R_2$

$$m_{42} = \frac{1.666}{5.666} = -\frac{a_{42}^{(1)}}{a_{22}^{(1)}} = 0.294;$$

$R_4 \leftarrow R_4 + 0.294 R_2$

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & -4 & 5 \\ 0 & -5.666 & 1.667 & 2.332 & 16.335 \\ 0 & 0 & -5.354 & 7.707 & -19.065 \\ 0 & 0 & 0.823 & 8.354 & 12.467 \end{array}\right]$$

$$m_{43} = -\frac{a_{43}^{(2)}}{a_{33}^{(2)}} = \frac{0.823}{5.354} = 0.154$$

$R_4 \leftarrow R_4 + 0.154 R_3.$

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & -4 & 5 \\ 0 & -5.666 & 1.667 & 2.335 & 16.335 \\ 0 & 0 & -5.354 & 7.707 & -19.065 \\ 0 & 0 & 0 & 9.541 & 9.531 \end{array}\right]$$

Back Substitution

$$x_4 = \frac{9.531}{9.541} = 0.999$$

$x_3 = 4.999$

$x_2 = -1.001.$

$x_1 = 2.000.$ A.