## 1.1 Introduction :

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). While digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like 'C' or assembly language. We can easily change the algorithm; so we can easily change the filter characteristics according to our requirement.

Basically there are two types of filters as follows :

1. FIR (Finite impulse response) filter.
2. IIR (Infinite impulse response) filter.

We will study each type in detail; later in this chapter. Presently we will compare analog and digital filters by studying advantages and disadvantages of digital filters.

### 1.1.1 Advantages of Digital Filters :

1. Many input signals can be filtered by one digital filter without replacing the hardware.
2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters, does not vary with environmental parameters. But the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. So it is required to calibrate analog filters periodically.
4. In case of digital filters; since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.
5. Unlike analog filters; the digital filters are portable.
6. From unit to unit the performance of digital filters is repeatable.
7. The digital filters are highly flexible.
8. Using VLSI technology; the hardware of digital filters can be reduced. Similarly the power consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example in Biomedical applications.
10. In case of analog filters; maintenance is frequently required. But for digital filters it is not required.

### 1.1.2 Disadvantages of Digital Filters :

#### 1. Speed limitation :

In case of digital filters, ADC and DAC are used. So the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Similarly the speed of operation of digital filter depends on the speed of digital processor. Thus the bandwidth of input signal processed is limited by ADC and DAC. In real time applications, the bandwidth of digital filter is much lower than analog filters.

2. **Finite wordlength effect :**

The accuracy of digital filter depends on the wordlength used to encode them in binary form. Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization of continuous signals. Similarly the accuracy of digital filters is also affected by the roundoff noise occurred during computation.

3. **Long design and development time :**

An initial design and development time for digital hardware is more than analog filters.

## 1.2 Filter Design Methods :

In order to design the digital IIR filter; analog IIR filter is designed first. Then analog filter is converted into the digital filter. Here you may ask a question, why to design digital filter from analog filter ?

The reasons are as follows :

(1) The procedure to design analog filter is readily available and it is highly advanced.

(2) When we design digital filter using analog filter then the implementation becomes simple.

The different methods use to design IIR filter are as follows :

(1) Approximation of derivatives. (2) Impulse invariance. (3) Bilinear transformation.

(4) Matched Z transform. (5) Least square filter design.

### 1.2.1 Approximation of Derivatives :

Consider an analog differentiator with transfer function $H_a(s)$. The function of analog differentiator is to take the derivative of analog input signal. Let $x(t)$ be the input signal applied to analog differentiator. Then its output can be written as,

$$y(t) = \frac{d}{dt} x(t) \qquad ...(1)$$

Consider analog signal as shown in Fig. K-1(a).

Since the output is the differentiation of input, then from Fig. K-1(a) we can write,

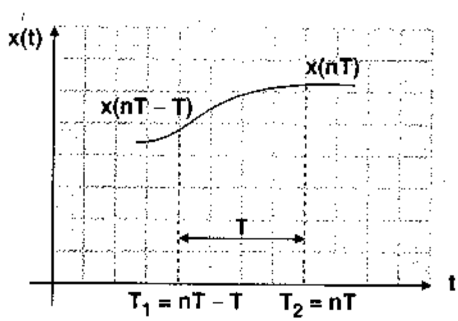


**Fig. K-1(a) : Analog signal**

$$y(nT) = \frac{x(nT) - x(nT-T)}{T}$$

$$\text{Thus } y(n) = \frac{x(n) - x(n-1)}{T} \quad \text{...(2)}$$

Taking Z transform of both sides we get,

$$Y(Z) = \frac{X(Z) - Z^{-1}X(Z)}{T}$$

$$\therefore \; Y(Z) = X(Z)\left[\frac{1-Z^{-1}}{T}\right]$$

$$\therefore \; \frac{Y(Z)}{X(Z)} = H(Z) = \left[\frac{1-Z^{-1}}{T}\right] \quad \text{...(3)}$$

Equation (3) gives the transfer function of digital filter.

Now we will obtain the transfer function of analog filter.

We have,

$$y(t) = \frac{d}{dt}x(t)$$

Taking Laplace transform we get,

$$Y(s) = sX(s)$$

$$\therefore \; \frac{Y(s)}{X(s)} = H(s) = s \quad \text{...(4)}$$

Equation (4) gives the transfer function of analog filter.

Comparing Equations (3) and (4) we get,

$$s = \frac{1-Z^{-1}}{T} \quad \text{...(5)}$$

Thus transfer function of digital filter is obtained by putting $s = \frac{1-Z^{-1}}{T}$ in the equation of the transfer function of digital filter.

That means,

$$H(Z) = H_a(s)\Big|_{s=\frac{1-Z^{-1}}{T}} \quad \text{...(6)}$$

**Mapping between s and Z plane :**

We have,

$$s = \frac{1-Z^{-1}}{T}$$

Multiplying numerator and denominator by Z we get,

$$s = \frac{Z-1}{ZT}$$

$$\therefore \quad sZT = Z - 1$$

$$\therefore \quad sZT - Z = -1$$

$$\therefore \quad -sZT + Z = 1$$

$$\therefore \quad Z(1 - sT) = 1$$

$$\therefore \quad Z = \frac{1}{1 - sT} \quad ...(7)$$

We know that 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\,\Omega$$

Putting this value in Equation (7) we get,

$$Z = \frac{1}{1 - T(\sigma + j\,\Omega)}$$

$$\therefore \quad Z = \frac{1}{1 - \sigma T - j\,\Omega T}$$

$$\therefore \quad Z = \frac{1}{1 - \sigma T - j\,\Omega T} \times \frac{1 - \sigma T + j\,\Omega T}{1 - \sigma T + j\,\Omega T}$$

$$\therefore \quad Z = \frac{1 - \sigma T + j\,\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2}$$

$$\therefore \quad Z = \frac{1 - \sigma T}{(1 - \sigma T)^2 + (\Omega T)^2} + j\,\frac{\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2} \quad ...(8)$$

Putting $\sigma = 0$ in Equation (8) we get,

$$Z = \frac{1}{1 + (\Omega T)^2} + j\,\frac{\Omega T}{1 + (\Omega T)^2} \quad ...(9)$$

Now as $\Omega$ varies from $-\infty$ to $+\infty$, the corresponding locus of points in Z plane is a circle of radius $\frac{1}{2}$ and its centre at $Z = \frac{1}{2}$. This mapping is shown in Fig. K-1(b).

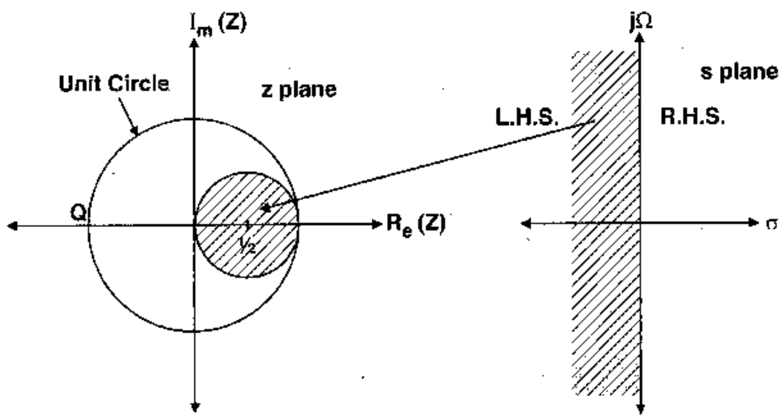


**Fig. K-1(b) : Mapping between s and Z plane**

This mapping shows that,

i) L.H.S. of s plane is mapped onto the points inside the circle in Z plane havin, radius $= \frac{1}{2}$ and centre at $Z = \frac{1}{2}$.

ii) R.H.S. of s plane is mapped onto the points outside the circle in Z plane.

iii) The stable analog filter is converted into stable digital filter.

**Limitation of approximation of derivatives method :**

This method is suitable only for designing of low pass and bandpass IIR digital filters with relatively small resonant frequencies.

## 1.2.2 Impulse Invariance Method :

In this method, the design starts from the specifications of analog filter. Here we have to replace analog filter by digital filter. This is achieved if impulse response of digital filter resembles the sampled version of impulse response of analog filter. If impulse response of both, analog and digital filter matches then, both filters perform in a similar manner.

Before studying this method we will list out the different notations, we are going to use.

$h(t)$ = Impulse response in time domain

$H_a(s)$ = Transfer function of analog filter; here 's' is laplace operator

$h(nT_s)$ = Sampled version of $h(t)$, obtained by replacing t by $nT_s$.

$H(Z)$ = Z transform of $h(nT_s)$. This is response of digital filter.

$\Omega$ = Analog frequency

$\omega$ = Digital frequency

**Transformation of analog system function $H_a(s)$ to digital system function $H(Z)$ :**

Now let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion. That means,

$$H_a(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \frac{A_3}{s-P_3} \dots$$

$$\therefore \quad H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-P_k} \qquad \dots(1)$$

Here $A_k = A_1, A_2 \dots A_N$ are the coefficients of partial fraction expansion.

and $P_k = P_1, P_2 \dots P_N$ are the poles.

Here 's' is the laplace operator. So we can obtain impulse response of analog filter, h ( t ) from $H_a(s)$ by taking inverse laplace of $H_a(s)$. So using standard relation of inverse laplace we get,

$$h(t) = \sum_{k=1}^{N} A_k e^{P_k t} \qquad ...(2)$$

Now unit impulse response for discrete structure is obtained by sampling h ( t ). That means, h ( n ) can be obtained from h ( t ) by replacing 't' by $n T_s$ in Equation (2).

$$\therefore \quad h(n) = \sum_{k=1}^{N} A_k e^{P_k n T_s} \qquad ...(3)$$

Here $T_s$ is the sampling time.

The system transfer function of digital filter is denoted by H ( Z ). It is obtained by taking Z-transform of h ( n ). According to the definition of Z-transform for causal system,

$$H(Z) = \sum_{n=0}^{\infty} h(n) Z^{-n} \qquad ...(4)$$

Putting Equation (3) in Equation (4) we get,

$$H(Z) = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{N} A_k e^{P_k n T_s} \right] \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} e^{P_k T_s n} \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left( e^{P_k T_s} \cdot Z^{-1} \right)^n \qquad ...(5)$$

Using the standard summation formula,

$$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a};$$

Equation (5) becomes,

$$\boxed{H(Z) = \sum_{k=1}^{N} A_k \cdot \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}}} \qquad ...(6)$$

This is the required transfer function of digital filter.

Thus comparing Equation (1) and Equation (6), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the transformation.

$$\frac{1}{s - P_k} \longrightarrow \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}} \quad \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}} \qquad ...(7)$$

Equation (7) shows, how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as mapping of poles.

**Relationship of s-plane to Z plane :**

We know that the poles of analog filters are located at $s = P_k$. Now from Equation (7) we can say that the poles of digital filter, $H(Z)$ are located at,

$$Z = e^{P_k T_s} \qquad ...(8)$$

This equation indicates that the poles of analog filter at $s = P_k$ are transformed into the poles of digital filter at $Z = e^{P_k T_s}$. Thus the relationship between laplace ('s' domain) and Z domain is given by,

$$Z = e^{s T_s} \qquad ...(9)$$

Here $s = P_k$ and $T_s$ is the sampling time.

Now 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\Omega \qquad ...(10)$$

Here $\sigma$ = Attenuation factor

and $\Omega$ = Analog frequency

We know the 'Z' can be expressed in polar form as,

$$Z = r e^{j\omega} \qquad ...(11)$$

Here 'r' is magnitude and '$\omega$' is the digital frequency.

Putting Equations (10) and (11) in Equation (9) we get,

$$r e^{j\omega} = e^{(\sigma + j\Omega) T_s}$$

$$\therefore \quad r e^{j\omega} = e^{\sigma T_s} \cdot e^{j\Omega T_s} \qquad ...(12)$$

Separating real and imaginary parts of Equation (12) we get,

$$r = e^{\sigma T_s} \qquad ...(13)$$

$$\text{and} \quad e^{j\omega} = e^{j\Omega T_s}$$

$$\therefore \quad \omega = \Omega T_s \qquad ...(14)$$

Now we will find the relationship between s plane and Z plane. Basically plot in 's'-domain means, $\sigma$ is plotted on X-axis and $j\Omega$ is plotted on Y- axis. And Z-domain representation means real Z is plotted on X- axis and imaginary Z is plotted on Y-axis.

Now consider Equation (13), it is

$$r = e^{\sigma T_s}$$

We will discuss the following conditions :

(i) If $\sigma < 0$, then r is equal to reciprocal of 'e' raise to some constant. Thus range of r will be 0 to 1.

$$\sigma < 0 \Rightarrow 0 < r < 1 \quad ...(15)$$

Now $\sigma < 0$ means negative values of $\sigma$. That is L.H.S. of s plane. We know that 'r' is the radius of circle is Z plane.

So '$0 < r < 1$' indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of 's' plane is mapped inside the unit circle.

(ii) If $\sigma = 0$ then $r = e^0 = 1$

$$\sigma = 0 \Rightarrow r = 1$$

Now $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus,

$j\Omega$ axis in 's' plane is mapped on the unit circle.

(iii) If $\sigma > 0$ then, r is equal to 'e' raise to some constant. That means $r > 1$.

$$\sigma > 0 \Rightarrow r > 1$$

Now $\sigma > 0$ indicates R.H.S. of 's' plane and 'r' $> 1$ indicates exterior part of unit circle. Thus,

R.H.S. of 's' plane is mapped outside the unit circle.

Combining all conditions; this mapping is shown in Fig. K-2.

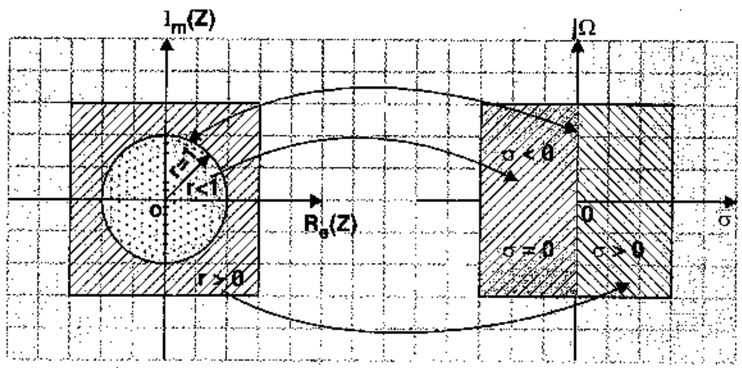


**Fig. K-2 : Relationship of s plane to Z-plane**

## Disadvantages of impulse invariance method :

(1) We know that '$\Omega$' is analog frequency and its range is from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$. While the digital frequency '$\omega$' varies from $-\pi$ to $\pi$. That means from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$ '$\omega$' maps from $-\pi$ to $\pi$. Let k be any integer. Then, we can write the general range of $\Omega$ as $(k-1)\frac{\pi}{T_s}$ to $(k+1)\frac{\pi}{T_s}$; but

for this range also; 'ω' maps from $-\pi$ to $\pi$. Thus mapping from analog frequency 'Ω' to digital frequency 'ω' is many to one. This mapping is not one to one.

(2) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(3) The change in the value of sampling time ( $T_s$ ) has no effect on the amount of aliasing.

Some standard formulae for transformation in impulse invariance method are as follows :

(i) $$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s}\cdot Z^{-1}}$$

(ii) $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT_s}[\cos bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

(iii) $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT_s}[\sin bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s}\cdot Z^{-2}}$$

**Design steps for impulse invariance method :**

**Step I :** Analog frequency transfer function H ( s ) will be given. If it is not given then, obtain expression of H ( s ) from the given specifications.

**Step II :** If required expand H ( s ) by using partial fraction expansion (PFE).

**Step III :** Obtain Z transform of each PFE term using impulse invariance transformation equation.

**Step IV :** Obtain H ( Z ), this is required digital IIR filter.

**Solved Problems :**

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**Prob. 1 :** Find out H ( Z ) using impulse invariance method at 5 Hz sampling frequency from H ( s ) as given below :

$$H(s) = \frac{2}{(s+1)(s+2)}$$

**Soln. :**

**Step I :** Given analog transfer function is,

$$H(s) = \frac{2}{(s+1)(s+2)} \quad ...(1)$$

**Step II :** We will expand H ( s ) using partial fraction expansion as :

$$\therefore \quad H(s) = \frac{A_1}{(s+1)}+\frac{A_2}{(s+2)} \quad ...(2)$$

Thus poles are at $P_1 = -1$ and $P_2 = -2$.

Now values of $A_1$ and $A_2$ are calculated as follows :

$$A_1 = (s-P_1)H(s)\Big|_{s=P_1}$$

$$\therefore \quad A_1 = (s+1)\cdot\frac{2}{(s+1)(s+2)}\bigg|_{s=-1}$$

$$\therefore \quad A_1 = \frac{2}{-1+2} = 2$$

and $$A_2 = (s-P_2)H(s)\Big|_{s=P_2} = (s+2)\cdot\frac{2}{(s+1)(s+2)}\bigg|_{s=-2}$$

$$\therefore \quad A_2 = \frac{2}{-2+1} = -2$$

Putting values of $A_1$ and $A_2$ in Equation (2) we get,

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \qquad \text{...(3)}$$

**Step III :** Now we will obtain the Z-transform using impulse invariance transformation equation. It is,

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s}\cdot Z^{-1}} \qquad \text{...(4)}$$

Here $T_s$ = Sampling time. Now given sampling frequency is $F_s = 5$ Hz.

$$\therefore \quad T_s = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

we have poles at $P_1 = -1$ and $P_2 = -2$

So using Equation (4) we get,

$$\frac{1}{s+1} \longrightarrow \frac{1}{1-e^{-1(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.2}\cdot Z^{-1}} \qquad \text{...(5)}$$

and $$\frac{1}{s+2} \longrightarrow \frac{1}{1-e^{-2(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.4}Z^{-1}} \qquad \text{...(6)}$$

**Step IV :** The transfer function of digital filter is given by,

$$H(Z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{P_k T_s}\cdot Z^{-1}}$$

In this case we get,

$$H(Z) = \frac{A_1}{1-e^{P_1 T_s}\cdot Z^{-1}} + \frac{A_2}{1-e^{P_2 T_s}\cdot Z^{-1}} \qquad \text{...(7)}$$

Using Equations (5) and (6) we get,

$$H(Z) = \frac{2}{1-e^{-0.2}\cdot Z^{-1}} - \frac{2}{1-e^{-0.4}\cdot Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{2}{1-0.818\,Z^{-1}} - \frac{2}{1-0.67\,Z^{-1}}$$

To convert each term into positive powers of Z; multiplying numerator and denominator of each term by Z we get,

$$H(Z) = \frac{2Z}{Z-0.818} - \frac{2Z}{Z-0.67}$$

$$\therefore \quad H(Z) = \frac{2Z(Z-0.67) - 2Z(Z-0.818)}{(Z-0.818)(Z-0.67)}$$

$$\therefore \quad H(Z) = \frac{2Z^2 - 1.34\,Z - 2Z^2 + 1.636\,Z}{Z^2 - 0.67\,Z - 0.818\,Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29\,Z}{Z^2 - 1.488\,Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29\,Z^{-1}}{1 - 1.488\,Z^{-1} + 0.54\,Z^{-2}}$$

This is the required transfer function for digital IIR filter.

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**Prob. 2 :** Determine H ( Z ) using impulse invariance method for the system function,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5\,s+2)}$$

**Soln. :**

**Step I :** The given transfer function is,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5\,s+2)} \qquad ...(1)$$

**Step II :** In the partial fraction expansion form H ( s ) can be written as,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5\,s+2)} = \frac{A}{(s+0.5)} + \frac{B\,s+C}{(s^2+0.5\,s+2)} \qquad ...(2)$$

Let us obtain the values of A, B and C.

$$\frac{1}{(s+0.5)(s^2+0.5\,s+2)} = \frac{A(s^2+0.5\,s+2)+(Bs+C)(s+0.5)}{(s+0.5)(s^2+0.5\,s+2)}$$

$$\therefore \quad A(s^2+0.5\,s+2)+(Bs+C)(s+0.5) = 1$$

$$\therefore \quad As^2 + A\,0.5\,s + 2A + Bs^2 + B\,0.5\,s + C.s + 0.5\,C = 1$$

$$s^2(A+B) + s(0.5\,A + 0.5\,B + C) + (2A + 0.5\,C) = 1 \qquad ...(3)$$

Now $s^2$ term is absent in R.H.S.

$$\therefore \quad A + B = 0 \qquad ...(4)$$

Similarly 's' term is absent in R.H.S.

$$\therefore \quad 0.5\,A + 0.5\,B + C = 0 \qquad ...(5)$$

$$\text{And} \quad 2\,A + 0.5\,C = 1 \qquad ...(6)$$

Now we will solve Equations (4), (5) and (6) to obtain the values of A, B and C.

From Equation (4),

$$B = -A$$

Putting this value in Equation (5) we get,

$$0.5\,A - 0.5\,A + C = 0$$

$$\therefore \quad C = 0$$

From Equation (6), $\quad 2A + 0 = 1$

$$\therefore \quad A = 0.5$$

Since B = – A we get,

$$\therefore \quad B = -0.5$$

Putting these values in Equation (2) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{s^2 + 0.5\,s + 2} \qquad ...(7)$$

To use the standard transformation formulae we will convert the second term on R.H.S. in the form $\frac{s+a}{(s+a)^2 + b^2}$ and $\frac{b}{(s+a)^2 + b^2}$

Consider the term $s^2 + 0.5\,s + 2$. It can be expressed as,

$$s^2 + 0.5\,s + 2 = (s^2 + 0.5\,s + 0.0625) + (1.9375)$$

$$\therefore \quad s^2 + 0.5\,s + 2 = (s+0.25)^2 + (1.39)^2$$

Putting this value in Equation (7) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{(s+0.25)^2 + (1.39)^2}$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2 + (1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] - 0.5\left[\frac{-0.25}{(s+0.25)^2 + (1.39)^2}\right]$$

Now we want the numerator of third term equal to 1.39. It is arranged as follows,

$$H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + \frac{0.5 \times 0.25}{1.39}\left[\frac{1.39}{(s+0.25)^2 + (1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + 0.089\left[\frac{1.39}{(s+0.25)^2 + (1.39)^2}\right] \qquad ...(8)$$

Now recall the standard transformation formulae,

(i) $\frac{1}{s - P_k} \longrightarrow \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}}$

(ii) $\frac{s + a}{(s + a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT_s} [\cos bT_s] Z^{-1}}{1 - 2e^{-aT_s} [\cos bT_s] Z^{-1} + e^{-2aT_s} \cdot Z^{-2}}$

(iii) $\frac{b}{(s + a)^2 + b^2} \longrightarrow \frac{e^{-aT_s} [\sin bT_s] Z^{-1}}{1 - 2e^{-aT_s} [\cos bT_s] Z^{-1} + e^{-2aT_s} \cdot Z^{-2}}$

**Step III :** Using these formulae, equation of H ( Z ) can be obtained from Equation (8) as follows :

$$H(Z) = \frac{0.5}{1 - e^{0.5 T_s} \cdot Z^{-1}} - 0.5 \left[ \frac{1 - e^{-0.25 T_s} [\cos 1.39 T_s] Z^{-1}}{1 - 2 e^{-0.25 T_s} [\cos 1.39 T_s] Z^{-1} + e^{-0.5 T_s} \cdot Z^{-2}} \right]$$

$$+ 0.089 \left[ \frac{e^{-0.25 T_s} (\sin 1.39 T_s) Z^{-1}}{1 - 2e^{-0.25 T_s} (\cos 1.39 T_s) Z^{-1} + e^{-0.5 T_s} \cdot Z^{-2}} \right]$$

This is the required transfer function. Note that the value of $T_s$ is not given in the problem; so $T_s$ is kept as it is.

### 1.2.3 Bilinear Transformation Method :

In case of impulse invariance method, we have studied that the mapping is many to one. So this method is not suitable to design high-pass filter and band reject filter.

In case of bilinear transformation; the mapping is one to one from s domain to the Z domain. So there is no aliasing effect. The limitations of impulse invariance method are overcome by using BLT method.

Consider an analog integrator as shown in Fig. K-2(A)

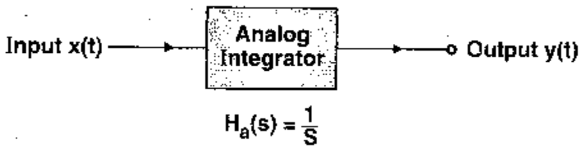


**Fig. K-2(A)**

The response of analog integrator is,

$$H_a(s) = \frac{1}{s} \quad ...(1)$$

Input in laplace domain is X (s) and output in laplace domain is Y (s).

For time period T; the difference in output is given by,

$$Y(t_2) - Y(t_1) = \int_{t_1}^{t} x(n T_s) \, dT \quad ...(2)$$

Consider input signal as shown in Fig. K-2(B).

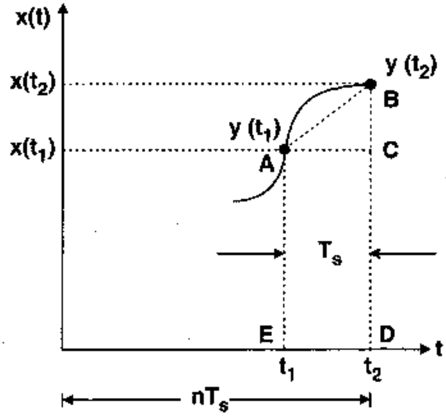


**Fig. K-2(B)**

Here we have assumed two input positions $x(t_1)$ and $x(t_2)$. Corresponding output is denoted by $Y(t_1)$ and $Y(t_2)$ respectively. Now area under the curve is addition of area of triangle ABC and area of rectangle ACDE.

$$\therefore \quad Y(t_2) - Y(t_1) = \frac{1}{2}(t_2 - t_1)\left[x(t_2) - x(t_1)\right] + x(t_1)(t_2 - t_1) \qquad \text{...(3)}$$

As shown in Fig. K-2(B), time period $t_2 = nT_s$. Thus time period $t_1$ is $nT_s - T_s$ That means $t_2 - t_1 = T_s$

Putting these values in Equation (3) we get,

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T\left[x(nT_s) - x(nT_s - T)\right] + x(nT_s - T)T_s$$

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T_s x(nT) - \frac{1}{2}T_s x(nT_s - T_s) + T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_S - T_s) = \frac{1}{2}T_s x(nT_s) + \frac{1}{2}T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_s) - T_s = \frac{T_s}{2}\left[x(nT_s) + x(nT_s - T_s)\right] \qquad \text{...(4)}$$

Taking Z transform of both sides we get,

$$Y(Z) - Z^{-1}Y(Z) = \frac{T_s}{2}\left[X(Z) + Z^{-1}X(Z)\right]$$

$$\therefore Y(Z)\left[1 - Z^{-1}\right] = \frac{T_s}{2}\left[X(Z)(1 + Z^{-1})\right]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{T_s}{2} \frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(5)$$

Now we have transfer function of analog filter,

$$H_a(s) = \frac{1}{s} \qquad ...(6)$$

Equating (5) and (6) we get,

$$\frac{1}{s} = \frac{T_s}{2} \frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(7)$$

Thus relationship between s plane and Z-plane is given by,

$$s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right) \qquad ..(8)$$

Here $T_s$ is the sampling time.

We know that 's' is the laplace operator and it can be expressed as,

$$s = \sigma + j\Omega \qquad ...(9)$$

Now the equation of Z in polar form is,

$$Z = r e^{j\omega} \qquad ...(10)$$

Putting Equations (9) and (10) in Equation (8) and multiplying numerator and denominator by $(re^{-j\omega}+1)$ we get,

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r e^{j\omega}-1}{r e^{j\omega}+1}\right] \times \frac{re^{-j\omega}+1}{re^{-j\omega}+1}$$

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 \cdot e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} - re^{-j\omega} - 1}{r^2 e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} + re^{-j\omega} + 1}\right] \qquad ...(11)$$

But $e^{j\omega} \cdot e^{-j\omega} = e^0 = 1$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 + re^{j\omega} - re^{-j\omega} - 1}{r^2 + re^{j\omega} + re^{-j\omega} + 1}\right] \qquad ...(12)$$

Now we have, $\dfrac{e^{j\omega} - e^{-j\omega}}{2j} = \sin\omega$ and $\dfrac{e^{j\omega} + e^{-j\omega}}{2} = \cos\omega$.

We will rearrange Equation (12) as follows :

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\left(\dfrac{e^{j\omega} - e^{-j\omega}}{2j}\right)}{r^2 + 2r\left(\dfrac{e^{j\omega} + e^{-j\omega}}{2}\right) + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\sin\omega}{r^2 + 2r\cos\omega + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2-1}{r^2+2r\cos\omega+1} + j\frac{2r\sin\omega}{r^2+2r\cos\omega+1}\right] \quad ...(13)$$

Equating real and imaginary parts of Equation (13) we get,

$$\sigma = \frac{2}{T_s} \times \frac{r^2-1}{r^2+2r\cos\omega+1} \quad ...(14)$$

$$\text{and} \quad \Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1} \quad ...(15)$$

Now we will discuss the following conditions related to Equation (14).

**(i) When $r < 1$ then $\sigma < 0$**

Here $r < 1$, means interior part of circle having unit circle and $\sigma < 0$, means $\sigma$ is negative which is L.H.S. of s-plane. So this condition indicates that L.H.S. of s plane maps inside the unit circle.

**(ii) When $r = 1$ then $\sigma = 0$**

Now $r = 1$ means unit circle and $\sigma = 0$ means $j\Omega$ axis. Thus this condition indicates that the $j\Omega$ axis maps on the unit circle.

**(iii) When $r > 1$ then $\sigma > 0$**

Here $r > 1$, means exterior part of unit circle and $\sigma > 0$ indicates that $\sigma$ is positive means R.H.S. of s-plane. So this condition indicates that R.H.S. of s-plane maps outside the unit circle.

This mapping is similar to the mapping in impulse invariance method, as shown in Fig. K-2. But in impulse invariance method mapping is valid only for poles ; while in bilinear transformation, mapping is valid for poles as well as zeros.

## How stable analog filter is converted into stable digital filter ?

Analog filter is stable if the poles lie on the L.H.S. of s-plane. While the digital filter is stable if the poles are inside the unit circle in the Z-domain. Now condition (i) indicates that L.H.S of s-plane maps inside the unit circle. Thus stable analog filter is converted into stable digital filter.

## Frequency warping concept :

Here we will obtain the relationship of $j\Omega$ axis in s-plane to the unit circle in the Z-plane ($r = 1$).

Recall Equation (15).

$$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1}$$

For the unit circle, $r = 1$. Thus putting $r = 1$ in the equation of $\Omega$ we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1+2\cos\omega+1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2+2\cos\omega}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{\sin \omega}{1 + \cos \omega} \quad \text{...(16)}$$

We have the trignometric identities,

$$\sin \omega = 2 \sin \frac{\omega}{2} \cdot \cos \frac{\omega}{2} \text{ and } 2 \cos^2 \frac{\omega}{2} = 1 + \cos \omega.$$

Thus Equation (16) becomes,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2}}{2 \cos \omega / 2}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

$$\therefore \quad \omega = 2 \tan^{-1} \left( \frac{\Omega T_s}{2} \right) \quad \text{...(17)}$$

Now for different values of $\Omega T_s$; the graph of $\Omega T_s$ versus $\omega$ is as shown in Fig. K-3.

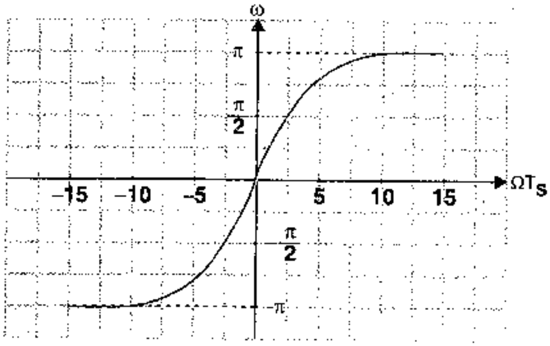


**Fig. K-3 : Mapping between ω and Ω**

## Observations :

1. The entire range in $\Omega$ is mapped only once into the range $-\pi \leq \omega \leq \pi$.
2. The mapping is one to one.
3. The mapping is highly non-linear.

Now because of the non-linearity of tangent function $2 \tan^{-1} \left( \frac{\Omega T_s}{2} \right)$; there exists frequency warping or frequency compression.

## What is frequency warping ?

Because of the non-linear mapping ; the amplitude response of digital IIR filter is expanded at lower frequencies and compressed at higher frequencies in comparison to the analog filter. This effect is called as frequency warping.

**Prewarping procedure :**

We have discussed the warping effect. Because of this effect, there is non-linear compression of $\Omega$ to $\omega$ values. To compensate this effect ; prewarping or prescaling procedure is used. This procedure is as follows :

(i) The '$\Omega$' scale is changed to prewarpped scale denoted by $\Omega^*$ and $\Omega^* = \frac{2}{T_s} \tan\left(\frac{\omega T_s}{2}\right)$

(ii) Then analog filter transfer function H ( s ) is obtained using values of $\Omega^*$.

(iii) By applying the bilinear transformation, the desired digital frequency response H ( Z ) is obtained.

**Advantages of bilinear transformation method :**

1. There is one to one transformation from the s-domain to the Z- domain.
2. The mapping is one to one.
3. There is no aliasing effect.
4. Stable analog filter is transformed into the stable digital filter.

**Disadvantage of bilinear transformation method :**

The mapping is non-linear and because of this; frequency warping effect takes place.

**Comparison between Impulse invariance and Bilinear transformation method :**

| Sr. No. | Impulse invariance method | Bilinear transformation method |
|---|---|---|
| 1. | Poles are transferred by using the equation, $\frac{1}{s - P_k} \rightarrow \frac{1}{1 - e^{P_k T_s} \cdot Z^{-1}}$ | Poles are transferred by using the equation, $s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$ |
| 2. | Mapping is many to one. | Mapping is one to one. |
| 3. | Aliasing effect is present. | Aliasing effect is not present. |
| 4. | It is not suitable to design high-pass filter and band reject filter. | High pass filter and band reject filter can be designed. |
| 5. | Only poles of the system can be mapped. | Poles as well as zeros can be mapped. |
| 6. | No frequency warping effect. | Frequency warping effect is present. |

**Solved Problems :**

**Prob. 1 :** An analog filter has the following transfer function $H(S) = \frac{1}{s+1}$. Using bilinear transformation technique, determine the transfer function of digital filter H ( Z ) and also write the difference equation of digital filter.

**Soln. :** The given transfer function is,

$$H(S) = \frac{1}{s+1} \quad ...(1)$$

In bilinear transformation H ( Z ) is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$$

Here $T_s$ is the sampling time; which is not given. So assume $T_s = 1$ sec.

$$\therefore \quad s = 2\left[\frac{Z-1}{Z+1}\right] \qquad ...(2)$$

Putting this value in Equation (1),

$$H(Z) = \frac{1}{1+2\left(\frac{Z-1}{Z+1}\right)} = \frac{1}{\frac{(Z+1)+2(Z-1)}{Z+1}}$$

$$\therefore \quad H(Z) = \frac{Z+1}{Z+1+2Z-2}$$

$$\therefore \quad H(Z) = \frac{Z+1}{3Z-1} \qquad ...(3)$$

This is the transfer function of digital filter. Now we have,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z+1}{3Z-1}$$

Converting into negative power of Z we get,

$$\frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}}{3-Z^{-1}}$$

$$\therefore \quad Y(Z)(3-Z^{-1}) = X(Z)(1+Z^{-1})$$

$$\therefore \quad 3Y(Z)-Z^{-1}\cdot Y(Z) = X(Z)+Z^{-1}X(Z)$$

Taking IZT of both sides,

$$3y(n)-y(n-1) = x(n)+x(n-1)$$

$$\therefore \quad 3y(n) = x(n)+x(n-1)+y(n-1)$$

$$\therefore \quad y(n) = \frac{1}{3}x(n)+\frac{1}{3}x(n-1)+\frac{1}{3}y(n-1)$$

This is the difference equation of digital filter.

---

**Prob. 2 :** The transfer function of analog filter is :

$$H(s) = \frac{3}{(s+2)(s+3)} \text{ with } T_s = 0.1 \text{ sec.}$$

Design the digital IIR filter using BLT.

**Soln. :** The given transfer function is,

$$H(s) = \frac{3}{(s+2)(s+3)} \qquad ...(1)$$

In bilinear transformation, H ( Z ) is obtained by putting :

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \qquad ...(2)$$

Here $T_s$ = sampling time = 0.1 sec.

$$s = \frac{2}{0.1}\left[\frac{Z-1}{Z+1}\right] = 20\left[\frac{Z-1}{Z+1}\right]$$

Putting this value in Equation (1) we get,

$$H(Z) = \frac{3}{\left[20\left(\frac{Z-1}{Z+1}\right)+2\right]\left[20\left(\frac{Z-1}{Z+1}\right)+3\right]}$$

$$\therefore \quad H(Z) = \frac{3}{\left[\frac{20Z-20}{Z+1}+2\right]\left[\frac{20Z-20}{Z+1}+3\right]}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)(Z+1)}{(20Z-20+2Z+2)(20Z-20+3Z+3)}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)^2}{(22Z-18)(23Z-17)}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-374Z-414Z+306}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-788Z+306} = \frac{Z^2+2Z+1}{168.67Z^2-262.67Z+102}$$

$$= \frac{1+2Z^{-1}+Z^{-2}}{168.67-262.67Z^{-1}+102Z^{-2}}$$

This is the required transfer function for digital IIR filter.

## 1.3 Butterworth Filter Approximation :

A typical characteristic of a butterworth low pass filter is as shown in Fig. K-5.

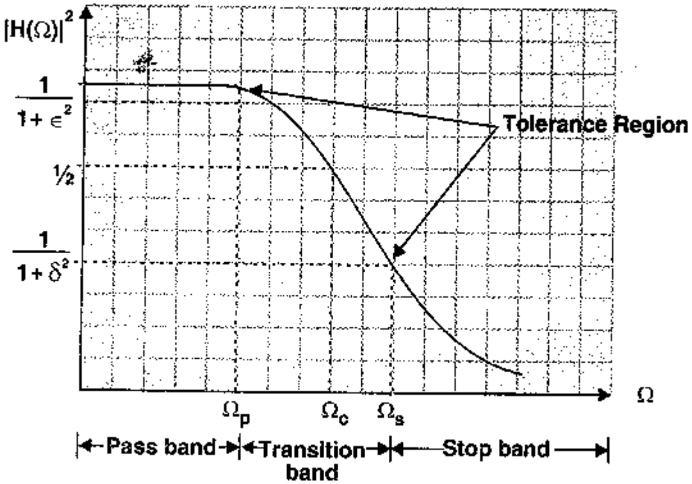


**Fig. K-5 : Typical characteristics of analog L.P.F.**

This type of response is called as butterworth response because its main characteristic is that the passband is maximally flat. That means there are no variations (ripples) in the passband.

Now the magnitude squared response of low pass butterworth filter is given by,

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad ...(1)$$

This equation is also expressed as,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad ..(2)$$

Here $|H(\Omega)|$ = Magnitude of analog low pass filter.

$\Omega_c$ = Cut-off frequency (– 3dB frequency).

$\Omega_p$ = Pass band edge frequency.

$1+\epsilon^2$ = Pass band edge value.

$1+\delta^2$ = Stop band edge value.

$\epsilon$ = Parameter related to ripples in pass band.

$\delta$ = Parameter related to ripples in stop band.

N = Order of the filter.

We know that, in case of low pass filter the frequencies will pass upto the value of cut-off frequency ($\Omega_c$). This is called as pass band. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in Fig. K-5. Ideally, at the value of cut-off frequency ($\Omega_c$) the frequencies should be stopped. But in practical cases this is not happening.

Now the order of filter is denoted by 'N'. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter 'N' increase's, the response of filter is more close to the ideal response as shown in Fig. K-6.

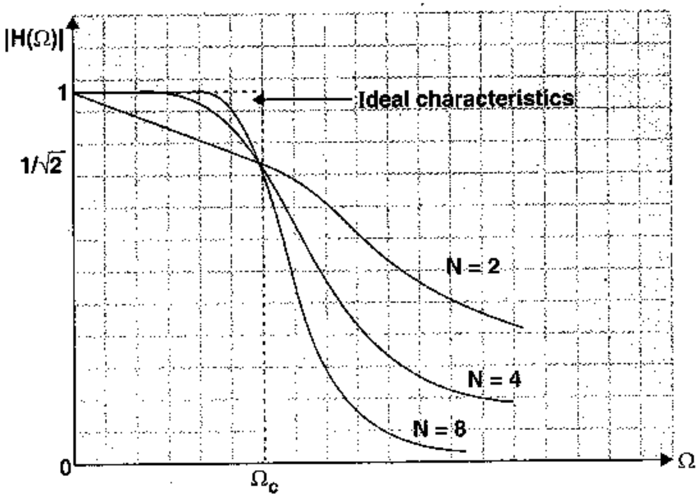


**Fig. K-6 : Effect of N on frequency response characteristics**

**Salient features of low pass butterworth filter :**

1. The magnitude response is nearly constant (equal to 1) at lower frequencies. That means pass band is maximally flat.
2. There are no ripples in the pass band and stop band.
3. The maximum gain occurs at $\Omega = 0$ and it is $| H(0) | = 1$.
4. The magnitude response is monotonically decreasing.

**Design equations and design steps :**

Let $A_p$ = Attenuation in pass band.

$A_s$ = Attenuation in stop band.

$\Omega_p$ = Pass band edge frequency.

$\Omega_c$ = Cut-off frequency.

$\Omega_s$ = Stop band edge frequen

In the problem the specifications of required digital filter will be given and it will be asked to design a particular discrete time butterworth filter. Then the following steps should be used :

**Step I :** From the given specifications of digital filter ; obtain equivalent analog filter as follows :

**(a) For impulse invariance method :**

$$\Omega = \frac{\omega}{T_s}$$

**(b) For bilinear transformation method :**

$$\Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

Here $\Omega$ = Frequency of analog filter $\omega$ = Frequency of digital filter

$T_s$ = Sampling time

**Step II :** Calculate the order 'N' of filter using the equation,

$$N = \frac{1}{2} \times \frac{\log \left[ \frac{\left( \frac{1}{A_s^2} - 1 \right)}{\left( \frac{1}{A_p^2} - 1 \right)} \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

or

$$N = \frac{\log \left[ \left( 1/\delta_s^2 \right) - 1 \right]}{2 \log \left( \Omega_s / \Omega_c \right)}$$

Here $\delta_s$ = Attenuation in stop band.

If the specifications are given in decibels (dB) then use the equation,

$$N = \frac{1}{2} \frac{\log \left[ \frac{10^{0.1 A_s (dB)} - 1}{10^{0.1 A_p (dB)} - 1} \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

**Step III :** Calculation of cut-off frequency ( $\Omega_c$ ) :

The cut-off frequency ( $\Omega_c$ ) of analog filter is calculated as :

**(a) For impulse invariance method :**

$$\Omega_c = \frac{\omega_c}{T_s}$$

**(b) For bilinear transformation method :**

$$\Omega_c = \frac{2}{T_s} \tan \frac{\omega_c}{2}$$

When $\omega_c$ is not given then use the equation,

(i) $$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}}$$ and if specifications are in dB then.

(ii) $$\Omega_c = \frac{\Omega_p}{10^{0.1 A_p} - 1}$$

**Step IV :** Calculate the poles using,

$$P_k = \pm \Omega_c e^{j(N+2k+1)\pi/2N}, \qquad k = 0, 1, 2, \ldots N-1.$$

If the poles are complex conjugate then organize the poles ($P_k$) as complex conjugate pairs that means,

$s_1$ and $s_1^*$, $s_2$ and $s_2^*$ etc.

**Step V :** Calculate the system transfer function of analog filter using,

$$H(s) = \frac{\Omega_c^N}{(s-P_1)(s-P_2)\ldots}$$

and if poles are complex conjugate then,

$$H(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

**Step VI :** Design the digital filter using impulse invariance method or bilinear transformation method.

**Solved Problems :**

---

**Prob. 1 :** A digital filter has frequency specification as :

Passband frequency = $\omega_p = 0.2\pi$

Stopband frequency = $\omega_s = 0.3\pi$

What are the corresponding specifications for pass band and stop band frequencies in analog domain if,

(I) Impulse invariance technique is used for designing.

(II) Bilinear transformation is used for designing.

**Soln. :** Assume sampling time $T_s = 1$.

**(I) For impulse invariance method we have,**

$$\Omega_p = \frac{\omega_p}{T_s}$$

$$= 0.2\pi = 0.63 \text{ rad/sec}$$

$$\text{and} \quad \Omega_s = \frac{\omega_s}{T_s} \qquad \therefore \quad \Omega_s = \frac{0.3\pi}{T_s} = 0.3\pi$$

$$\Omega_s = 0.94 \text{ rad/sec}$$

**(II) For bilinear transformation we have,**

$$\Omega_p = \frac{2}{T_s} \tan\left(\frac{\omega_p}{2}\right)$$

$$\therefore \quad \Omega_p = 2\tan\left(\frac{0.2\pi}{2}\right) = 2\tan\left(\frac{0.2 \times 180}{2}\right)$$

$$\Omega_p = 0.65 \text{ rad/sec}$$

$$\text{and} \quad \Omega_s = \frac{2}{T_s} \tan\left(\frac{\omega_s}{2}\right)$$

$$\therefore \quad \omega_s = 2\tan\left(\frac{0.3\pi}{2}\right) = 2\tan\left(\frac{0.3 \times 180}{2}\right)$$

$$\Omega_s = 1.019 \text{ rad/sec}$$

---

**Prob. 2 :** Design a second order DT butterworth filter with cut-off frequency of 1 KHz and sampling frequency of $10^4$ samples/sec. by bilinear transformation.

**Soln. :**

In this problem, the specifications of digital filter are not given directly. So first we have to obtain required design specifications for digital filter. Then for butterworth approximation we have to convert these specifications into specifications of equivalent analog filter. Finally using bilinear transformation we have to obtain H ( Z ).

### Given specifications of analog filter :

Order of filter, N = 2

Cut-off frequency of analog filter, $F_c = 1$ KHz = 1000 Hz.

Sampling frequency, $F_s = 10^4$ samples/sec. = 10,000 Hz.

### Part A : Calculation of the required design specification of digital filter :

We have the equation to convert continuous frequency (F) into discrete frequency (f). It is,

$$f = \frac{F}{F_s} \qquad \therefore \quad f_c = \frac{F_c}{F_s}$$

**Soln. :** Given,

$$\text{Cut-off frequency } = F_c = 1 \text{ kHz}$$

$$\text{and sampling frequency } F_s = 10 \text{ kHz}$$

$$\therefore \text{ Sampling time } T_s = \frac{1}{10 \text{ kHz}}$$

Given transfer function is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \quad ...(1)$$

We know that prewarping frequency is,

$$\Omega_p^* = \frac{2}{T_s} \tan\left(\frac{\omega_c T_s}{2}\right) \quad ...(2)$$

Here we will assume $\frac{2}{T_s} = 1$ because in BLT we have to put $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$; so in the final stage $\frac{2}{T_s}$ will cancel.

$$\therefore \Omega_p^* = \tan\left(\frac{\omega_c T_s}{2}\right)$$

$$= \tan\left(\frac{2\pi \times 1 \times 10^3}{2 \times 10 \times 10^3}\right)$$

$$\therefore \Omega_p^* = 0.324 \quad ...(3)$$

Now we will use frequency transformation as follows :

$$H^*(s) = H(s)\Big|_{s = \frac{s}{\Omega_p^*}}$$

$$\therefore H^*(s) = \frac{1}{\left(\frac{s}{0.324}\right)^2 + \sqrt{2}\left(\frac{s}{0.324}\right) + 1}$$

$$\therefore H^*(s) = \frac{(0.324)^2}{s^2 + \sqrt{2}\, s \times 0.324 + (0.324)^2}$$

$$\therefore H^*(s) = \frac{(0.324)^2}{s^2 + 0.458 + (0.324)^2} \quad ...(4)$$

Using BLT we can obtain H ( Z ) as follows :

$$H(Z) = H^*(s)\Big|_{s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)}$$

But we have assumed $\frac{2}{T_s} = 1$

$$\therefore \quad H(Z) = H^*(s) \Big|_{s=\left(\frac{Z-1}{Z+1}\right)}$$

Thus Equation (4) becomes,

$$H(Z) = \frac{(0.324)^2}{\left(\frac{Z-1}{Z+1}\right)^2 + 0.458\left(\frac{Z-1}{Z+1}\right) + (0.324)^2}$$

$$\therefore \quad H(Z) = \frac{(0.324)^2 (Z+1)^2}{(Z-1)^2 + 0.458(Z-1)(Z+1) + (0.324)^2 (Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{0.1049 Z^2 + 0.2099 Z + 0.1049}{Z^2 - 2Z + 1 + 0.458 Z^2 - 0.458 + 0.1049 Z^2 + 0.2099 Z + 0.1049}$$

$$\therefore \quad H(Z) = \frac{0.1049 Z^2 + 0.2099 Z + 0.1049}{1.563 Z^2 - 1.79 Z + 0.647}$$

To convert it into negative powers of Z, multiply numerator and denominator by $Z^{-2}$,

$$\boxed{\therefore \quad H(Z) = \frac{0.1049 Z^{-2} + 0.2099 Z^{-1} + 0.1049}{0.647 Z^{-2} - 1.79 Z^{-1} + 1.563}}$$