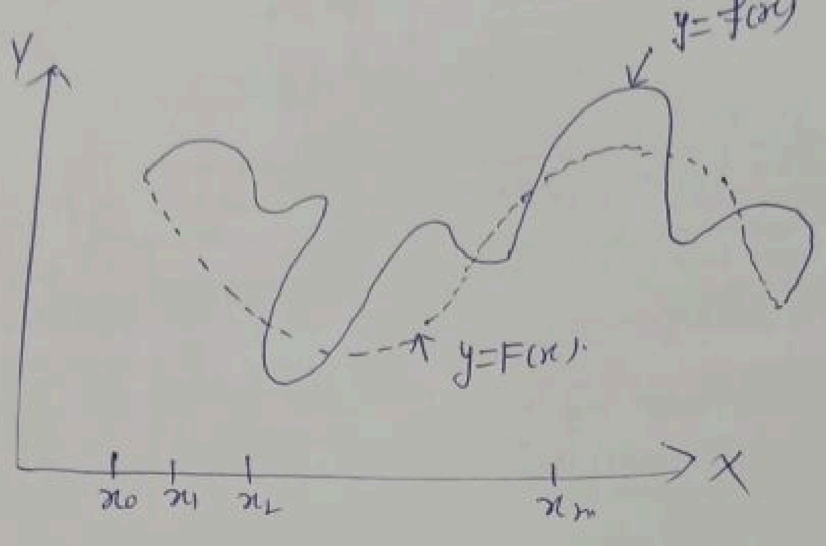
Interpolation. Page 1

To fix the height of dam across a river, an engineer utilize a set of data of the form (x_i, y_i) where x_i denotes the year and y_i, the peak flood level across the river; to estimate the heighest possible flood level in future. From the recorded data of (t_i, A_i), time and altitude of a rocket, To estimate what was the attitude at a particular time $t_0 (\neq t_i)$. Thus most of the exponential or obsered data is in the form a set of say $(n+1)$ orderd pairs $(x_0, y_0), --- (x_n, y_n)$ which is the tabular form of an unknown function $y = f(x)$. The process of determine the value of y for an $x \in [x_0, x_1]$ is known as interpolation.

Here $x_0, x_1, x_2, x_3, ---, x_n$ are called interpolation (or mesh) points.

"Interpolation is the "art of reading between the lines of a table."

The problem of interpolation is to contruct a new (interpolating) function $F(x)$ which ~~collocates~~ collocates (coincides) with the unknown function $f(x)$ at the tabulated $(n+1)$ interpolation point.



i.e $y_0 = f(x_0) = F(x_0)$

$y_1 = f(x_1) = F(x_1)$

$\vdots$

$y_n = f(x_n) = F(x_n)$.

Geometrically, this means that graphs of $y = f(x)$ and $y = F(x)$ coincide at $n+1$ points. Since a unique straight line passes through two given points, a unique parabola through three given points, a unique polynomial $F(x)$ of degree n can be determined (passing) satisfying the given set of $(n+1)$ points. In this case it is called a polynomial interpolation.

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Newton - Gregory Forward Interpolation Formula:

Suppose the value of $y_i = f(x_i)$ are given for equally spaced values of the independent variable (argument) $x_i = x_0 + ih$ for $i = 0, 1, 2, ---, n$ Here $\boxed{h}$ is known as the size of the interval or $\boxed{\text{spacing}}$, is constant.

Assuming the nth degree interpolating polynomial is given by

$$F(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + --- + --- + a_n (x - x_0)(x - x_1) --- (x - x_{n-1}) \quad \text{—} (1)$$

Using the $(n+1)$ conditions, $y_i = F(x_i)$ for $i = 0, 1, 2, ---n$, we determine the $n+1$ unknown coefficients $a_0, a_1, a_2, ---, a_n$ in (1).

Putting $x = x_0$ in (1), we get

$$y_0 = F(x_0) = a_0 + 0 + 0 + --- + 0 \quad \therefore a_0 = y_0$$

Putting $x = x_1$ in (1), we have

$$y_1 = F(x_1) = a_0 + a_1 (x_1 - x_0).$$

But $a_0 = y_0$ and $x_1 - x_0 = h$.

$$a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \frac{1}{h} \Delta y_0 .$$

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Now with $x = x_2$ in (1), we get

$$y_2 = F(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= a_0 + a_1 \cdot 2h + a_2 \cdot 2h \cdot h$$

$\boxed{\begin{matrix} x_i = x_0 + ih \\ x_2 = x_0 + 2h \end{matrix}}$

$$a_2 = \frac{y_2 - a_0 - 2h \cdot a_1}{2h^2}$$

$$= \frac{y_2 - y_0 - 2h \frac{1}{h} \Delta y_0}{2h^2}$$

$$a_2 = \frac{y_2 - y_0 - 2(y_1 - y_0)}{2h}$$

$$= \frac{y_2 - 2y_1 + y_0}{2h}$$

$$= \frac{1}{2h} \Delta^2 y_0$$

Similarly, at $x = x_3$ we have.

$$y_3 = F(x_3) = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

$$= a_0 + a_1 \cdot 3h + a_2 \cdot 3h \cdot 2h + a_3 \cdot 3h \cdot 2h \cdot h$$

Solving $a_3 = \dfrac{y_3 - 3y_2 + 3y_1 - y_0}{3!\,h^2}$

$$= \frac{1}{3!\,h^2} \Delta^3 y_0.$$

This way, we get

$$a_4 = \frac{1}{4!h^4}\Delta^4 y_0,$$

$$a_5 = \frac{1}{5!h^5}\Delta^5 y_0 \text{ etc. and}$$

$$\boxed{a_n = \frac{1}{n!h^n}\Delta^n y_0} \quad -(2)$$

Substituting these values of $a_0, a_1, ---, a_n$ in (1) we get the Newton - Gregory forward interpolation formula (also known as Newton first interpolation formula) as.

$$y = F(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + --- + \frac{\Delta^n y_0}{n!h^n}(x-x_0)(x-x_1) --- (x-x_{n-1}) \quad -(3)$$

Introducing $q = \frac{x-x_0}{h}$, the above formula (3) can be written in more conveient way

Now $\frac{x-x_1}{h} = \frac{x-(x_0+h)}{h} = \frac{x-x_0}{h} - 1 = q-1.$

$$\frac{x-x_2}{h} = \frac{x-(x_0+2h)}{h} = \frac{x-x_0}{h} - 2 = q-2 \text{ etc.}$$

$$\frac{x-x_{n-1}}{h} = \frac{x-(x_0+(n-1)h)}{h} = q-(n-1) = q-n+1$$

Substituting these values, we have

$$F(x) = F(x_0 + hq) = g(q) = y_0 + \Delta y_0 \cdot q + \frac{\Delta^2 y_0}{2!} q(q-1) + \frac{\Delta^3 y_0}{3!} q(q-1)(q-2) + \cdots$$

$$+ \frac{q(q-1)\cdots(q-n+1)}{n!} \Delta^n y_0 \quad \text{—} \ (4)$$

For $n=1$, in (4), we get linear interpolation

$$P_1(x) = y_0 + q\Delta y_0$$

For $n-2$, in (4), we have parabolic interpolation

$$\boxed{P_2(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2} \Delta^2 y_0.}$$

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Difference of a Generalized Power

The first difference of the factorial is

$$\Delta[x]^n = [x+h]^n - [x]^n$$

$$= \{(x+h)(x)(x-h) --- (x-(n-2)h)\}$$
$$- \{x(x-h) --- (x-(n-1)h)\}.$$

$$= \{x(x-h) ---- (x-(n-2)h)\} \times$$
$$\times \{x+h - (x-(n-1)h)\}$$

$$= \{x(x-h) --- (x-(n-2)h)\}$$

$$= nh$$
$$= nh[x]^{n-1}.$$

Thus $\boxed{\Delta[x]^n = nh[x]^{n-1}.}$

Now the second diff.

$$\Delta^2[x]^n = \Delta(\Delta[x]^n)$$

$$= \Delta(nh[x]^{n-1})$$

$$= nh\,\Delta[x]^{n-1}$$

$$= nh \cdot (n-1) \cdot h \cdot [x]^{n-2}$$

$$\Delta^2[x]^n = n(n-1)h^2[x]^{n-2}.$$

By Mathematical induction

$$\Delta^k[x]^n = n(n-1) ---- (n-(k-1))h^k[x]^{n-k}.$$

where $k = 1, 2, 3, ---, n$.