Lecture - 17

Z-Transform Properties

1.3 Properties of Z-transform :

In this section we will discuss different properties of Z-transform. These properties are applicable to both double sided as well as single sided Z-transform.

1.3.1 Linearity :

Statement : It states that if $x(n) = a_1 x_1(n) + a_2 x_2(n)$

and if $x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{Z}{\longleftrightarrow} X_2(Z)$ then,

$$x(n) \overset{Z}{\longleftrightarrow} X(Z) = a_1 X_1(Z) + a_2 (X_2(Z)$$

Where a_1 and a_2 are constants.

Proof : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

$$\text{Here } x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$\therefore\ X(Z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] Z^{-n} \qquad ...(2)$$

Writing two terms separately we get,

$$X(Z) = \sum_{n=-\infty}^{\infty} a_1 x_1(n) Z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) Z^{-n}$$

Here a_1 and a_2 are constants. So we can take it outside the summation sign.

$$\therefore\ X(Z) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n} \qquad ...(3)$$

Comparing Equation (3) with Equation (1) we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

Comment on ROC :

The combined ROC is the overlap or intersection of the individual ROC's of $X_1(Z)$ and $X_2(Z)$.

Use of this property :

We can express the given signal in the form of sum of two or more elementary signals. Then find their individual Z- transforms and add them. This will simplify the solution.

Note : Linearity property states that Z- transform of linear combination of signals is same as linear combination of Z- transform. While the ROC of X(Z) is intersection or overlap of individual region of convergence of $X_1(Z)$ and $X_2(Z)$.

Solved Problems on Linearity Property :

Prob. 1 : Determine Z transform of

$x(n) = (n+1) u(n)$

Soln. : The given function is,

$$x(n) = (n+1) u(n)$$

$$\therefore\ x(n) = n u(n) + u(n) \qquad ...(1)$$

$$\text{Let } x_1(n) = n u(n) \text{ and } x_2(n) = u(n)$$

$$\therefore\ x(n) = x_1(n) + x_2(n) \qquad ...(2)$$

Here n u (n) is ramp sequence. We have standard Z transform pair,

$$n\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad X_1(Z) = Z\{n\,u(n)\} = \frac{Z}{(Z-1)^2}, |Z| > 1 \qquad ...(3)$$

Now $u(n)$ is a unit step and we have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

$$\therefore \quad X_2(Z) = Z\{u(n)\} = \frac{Z}{Z-1} \quad \text{ROC} : |Z| > 1 \qquad ...(4)$$

From Equation (2) we can write,

$$X(Z) = X_1(Z) + X_2(Z) \qquad ...(5)$$

Putting Equations (3) and (4) in Equation (5) we get,

$$\boxed{X(Z) = \frac{Z}{(Z-1)^2} + \frac{Z}{Z-1}, \text{ ROC} : |Z| > 1}$$

Thus ROC is exterior part of unit circle as shown in Fig. U-17.

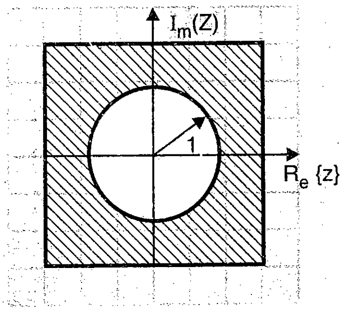


Fig. U-17

Prob. 2 : Determine Z-transform and ROC of signal :

$$x(n) = [3(4^n) - 5(3^n)]\,u(n)$$

Soln. : Let us bring it into the known form that is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \qquad ...(1)$$

$$\therefore \quad x(n) = 3(4^n)u(n) - 5(3^n)u(n) \qquad ...(2)$$

Comparing Equations (1) and (2) we can write,

$$x_1(n) = 4^n u(n) \quad \text{and} \quad x_2(n) = 3^n u(n)$$

Thus using linearity property,

$$X(Z) = 3X_1(Z) - 5X_2(Z) \qquad ...(3)$$

Now recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{and ROC is } |Z| > |\alpha|$$

Here $x_1(n) = 4^n u(n) \quad \therefore \alpha = 4$

$$\therefore \quad X_1(Z) = \frac{Z}{Z-4}, \qquad \text{ROC } |Z| > 4 \qquad \text{...(4)}$$

And $x_2(n) = 3^n u(n) \quad \therefore \alpha = 3$

$$\therefore \quad X_2(Z) = \frac{Z}{Z-3}, \qquad \text{ROC } |Z| > 3 \qquad \text{...(5)}$$

Putting Equations (4) and (5) in Equation (3) we get,

$$\boxed{X(Z) = 3 \times \frac{Z}{Z-4} - 5 \times \frac{Z}{Z-3}}$$

ROC :

For $X_1(Z)$, ROC is $|Z| > 4$ and for $X_2(Z)$, ROC is $|Z| > 3$. Now we have to decide the common ROC such that both $X_1(Z)$ and $X_2(Z)$ will converge. We have discussed that the combined ROC is overlap or intersection of two ROC's. The common ROC is $|Z| > 4$. It is as shown in Fig. U-18.

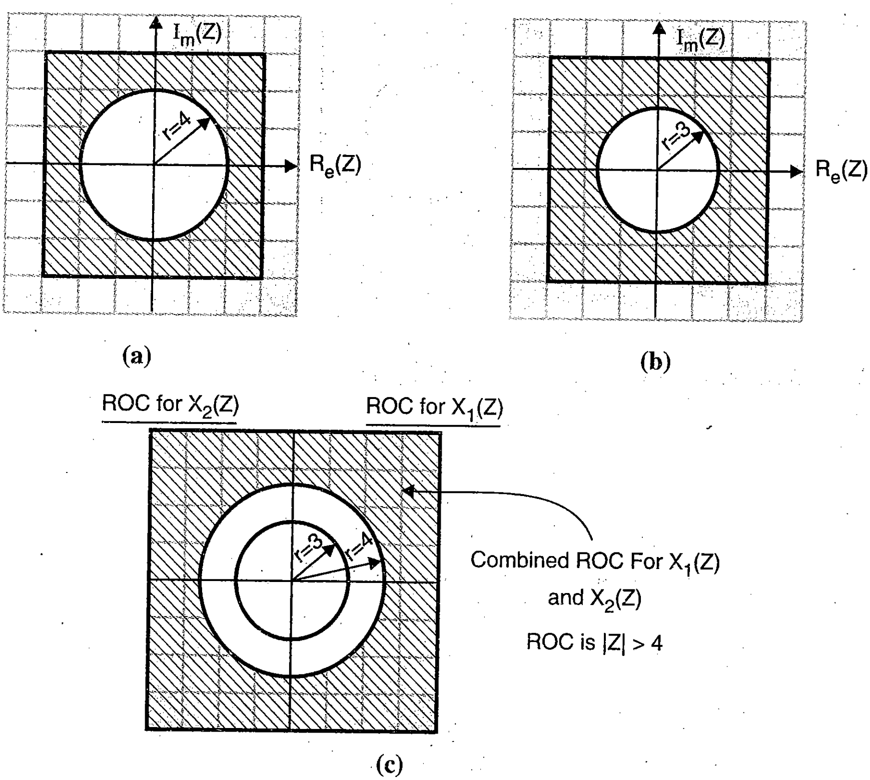


Fig. U-18 : ROC of given signal

Note : We cannot take combined ROC as $|Z| > 3$. Because for $X_2(Z)$ ROC is $|Z| > 4$. So in this case $X_2(Z)$ will not converge. But if we take ROC $|Z| > 4$ then by default it is > 3, so both sequences will converge. That means in this case $|Z| > 4$ is the common ROC for both sequences.

Prob. 3 : Determine the Z-transform and sketch the ROC of :

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n, & n > 0, \\ \left(\frac{1}{2}\right)^{-n}, & n < 0, \end{cases}$$

Soln. : For the range $n > 0$, we will get a causal sequence and for the range $n < 0$, we will get anticausal sequence. Thus the given expression can be written as,

$$x(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1) \quad \text{...(1)}$$

The known form of sequence $x(n)$ is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \quad \text{...(2)}$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \quad \text{and} \quad x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

Using linearity property,

$$X(Z) = X_1(Z) + X_2(Z) \quad \text{...(3)}$$

We have standard Z-transform pair for causal sequence.

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha|$$

$$\text{Here } x_1(n) = \left(\frac{1}{3}\right)^n u(n) \qquad \therefore \alpha = \frac{1}{3}$$

$$\therefore X_1(Z) = \frac{Z}{Z - \frac{1}{3}} \qquad \text{ROC is } |Z| > \left|\frac{1}{3}\right| \quad \text{...(4)}$$

$$\text{Now } x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1) \quad \text{...(5)}$$

Recall the standard Z-transform pair for anticausal sequence,

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| < |\alpha|$$

From this we can write,

$$+\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} -\frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| < |\alpha|$$

Now Equation (5) can be written as,

$$x_2(n) = \left[\left(\frac{1}{2}\right)^{-1}\right]^n u(-n-1) = \left[\left(\frac{1}{1/2}\right)^n\right] u(-n-1)$$

$$\therefore \quad x_2(n) = 2^n u(-n-1)$$

Thus, here $\alpha = 2$,

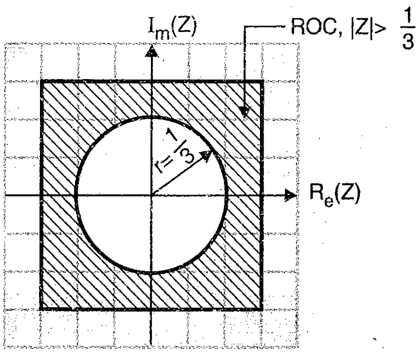
$$\therefore \quad X_2(Z) = \frac{Z}{Z-2} \qquad \text{ROC is } |Z| < |2| \qquad \text{...(6)}$$

Putting Equations (4) and (5) in Equation (3) we get,

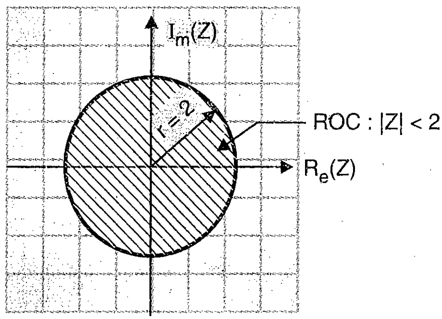
$$\boxed{X(Z) = \frac{Z}{Z-\frac{1}{3}} + \frac{Z}{Z-2}}$$

ROC :

For $X_1(Z)$, ROC is $|Z| > \frac{1}{3}$ and for $X_2(Z)$ ROC is $|Z| < 2$. We know that the combined ROC is overlap or intersection of two ROC's. This ROC is shown in Fig. U-19. Thus combined ROC is $\left|\frac{1}{3}\right| < |Z| < |2|$.

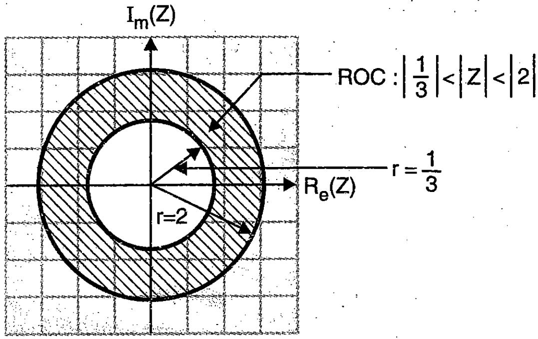


(a)



(b)

Fig. U-19



(c)

Fig. U-19 : ROC of given sequence

Prob. 4 : Find Z transform of following function along with ROC, $x(n) = a^n u(n) + \delta(n-5)$.

Soln. : The given function is,

$$x(n) = a^n u(n) + \delta(n-5) \qquad ...(1)$$

$$\text{Let} \quad x_1(n) = a^n u(n) \qquad ...(2)$$

$$\text{and} \quad x_2(n) = \delta(n-5) \qquad ...(3)$$

$$\therefore \quad x(n) = x_1(n) + x_2(n) \qquad ...(4)$$

First we will calculate Z transform of $x_1(n)$.

We have,

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}; \qquad \text{ROC} : |Z| > |a|$$

$$\therefore \quad Z\{a^n u(n)\} = X_1(Z) = \frac{Z}{Z-a}; \qquad \text{ROC } |Z| > |a|$$

Now we will calculate Z transform of $x_2(n)$.

We have standard Z transform pair,

$$\delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}; \qquad \text{ROC : Entire Z plane except } Z = 0.$$

$$\therefore \quad Z\{\delta(n-5)\} = X_2(Z) = Z^{-5}; \qquad \text{ROC : Entire Z plane except } Z = 0.$$

From Equation (4) we can write,

$$X(Z) = X_1(Z) + X_2(Z)$$

$$\boxed{\therefore \quad X(Z) = \frac{Z}{Z-a} + Z^{-5}}$$

The combined ROC is $|Z| > |a|$.

Prob. 5 : Determine the Z-transform of :

$$x(n) = (\cos \omega_0 n)\, u(n)$$

Soln. : According to Euler's identity we have,

$$\cos \theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$$

$$\therefore \quad \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\therefore \quad x(n) = \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n) \qquad \text{...(1)}$$

Now we have the known form,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \qquad \text{...(2)}$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = e^{j\omega_0 n}, \qquad \text{here } a_1 = \frac{1}{2}$$

$$\text{and } x_2(n) = e^{-j\omega_0 n}, \qquad \text{here } a_2 = \frac{1}{2}$$

Using Linearity property we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \qquad \text{...(3)}$$

$$\text{Here } x_1(n) = e^{j\omega_0 n} u(n) = \left(e^{j\omega_0}\right)^n u(n)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha|$$

Here $\alpha = e^{j\omega_0}$ Thus we get,

$$X_1(Z) = \frac{Z}{Z - e^{j\omega_0}} \qquad \text{ROC is } |Z| > |e^{j\omega_0}| \qquad \text{...(4)}$$

$$\text{Now } x_2(n) = e^{-j\omega_0 n} u(n) = \left(e^{-j\omega_0}\right)^n u(n)$$

Here $\alpha = e^{-j\omega_0}$

$$\therefore \quad X_2(Z) = \frac{Z}{Z - e^{-j\omega_0}} \qquad \text{ROC is } |Z| > |e^{-j\omega_0}| . \qquad \text{...(5)}$$

Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = a_1 \frac{Z}{Z - e^{j\omega_0}} + a_2 \frac{Z}{Z - e^{-j\omega_0}}$$

But $a_1 = a_2 = \frac{1}{2}$

$$\boxed{\therefore X(Z) = \frac{1}{2}\left[\frac{Z}{Z-e^{j\omega_0}} + \frac{Z}{Z-e^{-j\omega_0}}\right]} \quad ...(6)$$

ROC :

For $X_1(Z)$, ROC is $|Z| > |e^{j\omega_0}|$

$$\text{We have} \quad e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore \quad e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$$

$$\text{Now} \quad |e^{j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

Similarly for $X_2(Z)$, ROC is $|Z| > |e^{-j\omega_0 n}|$

$$\text{We have,} \quad e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$$

$$\text{Thus} \quad \left|e^{-j\omega_0}\right| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

So the combined ROC is $|Z| > 1$.

This is shown in Fig. U-20.

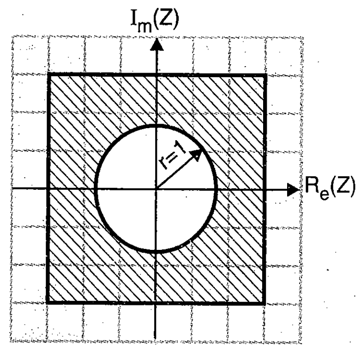


Fig. U-20 : ROC of $x(n) = (\cos\omega_0 n)\, u(n)$

This is a standard Z-transform pair.

We can further simplify Equation (6) as follows.

We have $e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$ and $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Putting these values in Equation (6) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} + \frac{Z}{Z-(\cos\omega_0 - j\sin\omega_0)}\right] \quad ...(7)$$

Consider the first term inside the bracket,

$$\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z-\cos\omega_0 - j\sin\omega_0}$$

Rationalizing the equation,

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{(Z - \cos\omega_0 - j\sin\omega_0)(Z - \cos\omega_0 + j\sin\omega_0)}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 - Z\cos\omega_0 + \cos^2\omega_0 - j\sin\omega_0\cos\omega_0 - jZ\sin\omega_0 + j\sin\omega_0\cos\omega_0 - j^2\sin^2\omega_0}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(8)$$

.... [Here $j^2 = -1$ and $\sin^2\omega_0 + \cos^2\omega_0 = 1$]

Now consider second term,

$$\frac{Z}{Z - (\cos\omega_0 - j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 + j\sin\omega_0}$$

Rationalizing the equation.

$$= \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{(Z - \cos\omega_0 + j\sin\omega_0)(Z - \cos\omega_0 - j\sin\omega_0)}$$

$$= \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(9)$$

Putting Equations (8) and (9) in Equation (7) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} + \frac{Z(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{1}{2}\left[\frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 + Z^2 - Z\cos\omega_0 - jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{1}{2}\left[\frac{2Z^2 - 2Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore \quad X(Z) = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Therefore the standard Z-transform identity is,

$$(\cos \omega_0 n)\, u(n) \overset{Z}{\longleftrightarrow} \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}, \quad \text{ROC is } |Z| > 1$$

Prob. 6 : Determine Z-transform of :

$$x(n) = \sin \omega_0 n\, u(n)$$

Soln. : We can obtain Z-transform of $\sin \omega_0 n\, u(n)$ similar to the last problem using Euler's identify. This problem can also be solved using another method, which is more simple.

We have standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha| \qquad \text{...(1)}$$

Let $\alpha = e^{j\omega n}$. Thus we can write,

$$Z\left\{e^{j\omega n} u(n)\right\} = \frac{Z}{Z - e^{j\omega}} \qquad \text{ROC : } |Z| > |e^{j\omega_0}| \qquad \text{...(2)}$$

But we have, $$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

Putting this value in Equation (3) we get,

$$Z\left\{\cos\omega_0 n + j\sin\omega_0 n\right\} = \frac{Z}{Z - (\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 - j\sin\omega_0}$$

$$= \frac{Z(Z - \cos\omega_0 + j\sin\omega_0)}{(Z - \cos\omega_0 - j\sin\omega_0)(Z - \cos\omega_0 + j\sin\omega_0)}$$

$$\therefore \quad Z\left\{\cos\omega_0 n + j\sin\omega_0 n\right\} = \frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore \quad Z\left\{\cos\omega_0 n\right\} + Z\left\{j\sin\omega_0 n\right\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} + j\frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{...(3)}$$

Comparing imaginary part we get,

$$Z\left\{j\sin\omega_0 n\right\} = \frac{jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore \quad Z\left\{\sin\omega_0 n\right\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC is } |Z| > |e^{j\omega_0}|$$

Similarly comparing real part we can obtain Z-transform of $\cos\omega_0 n$. Now ROC is $|Z| > |e^{j\omega_0}|$. But we know that $|e^{j\omega_0}| = 1$. Thus ROC is $|Z| > 1$. This is a standard Z-transform pair.

$$\therefore \sin\omega_0 n\, u(n) \overset{Z}{\longleftrightarrow} \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC} : |Z| > 1$$

1.3.2 Time Shifting :

Statement :

$$\text{If} \quad x(n) \overset{Z}{\longleftrightarrow} X(Z)$$

$$\text{then} \quad x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

The ROC of $Z^{-k}X(Z)$ is same as that of $X(Z)$ except for $Z = 0$ if $k > 0$ and $Z = \infty$ if $k < \infty$

Proof : According to definition of Z-transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \text{...(1)}$$

Then $Z\{x(n-k)\}$ can be written as,

$$\therefore \quad Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) Z^{-n} \qquad \text{...(2)}$$

Now Z^{-n} can be written as $Z^{-n} = Z^{-(n-k)} \cdot Z^{-k}$. Thus Equation (2) becomes,

$$Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-k)} \cdot Z^{-k}$$

Since the limits of summation are in terms of 'n' we can take Z^{-k} outside the summation sign.

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-k)} \qquad \text{...(3)}$$

Now put $n - k = m$ on R.H.S. The limits will change as follows.

at $n = -\infty$, $-\infty - k = m \Rightarrow m = -\infty$

at $n = +\infty$, $\infty - k = m \Rightarrow m = \infty$

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} \sum_{m=-\infty}^{\infty} x(m) Z^{-m} \qquad \text{...(4)}$$

Compare Equation (4) with Equation (1),

$$\therefore \quad Z\{x(n-k)\} = Z^{-k} X(Z)$$

$$\boxed{\therefore\ x(n-k) \stackrel{Z}{\longleftrightarrow} Z^{-k} X(Z)}$$

Similarly it can be shown that,

$$\boxed{x(n+k) \stackrel{Z}{\longleftrightarrow} Z^{+k} X(Z)}$$

Note : Here $x(n-k)$ indicates that the sequence is shifted in time domain (delayed). Thus shifting the sequence in time domain by $(-k)$ samples corresponds to multiplication by Z^{-k} in the frequency domain.

Solved Problems on Time Shifting Property :

Prob. 1 : Find the Z-transform of :

$$x(n) = \delta(n-k)$$

Soln. :

We know that $\delta(n)$ is unit impulse and $Z\{\delta(n)\} = 1$ that means $\delta(n) \stackrel{Z}{\longleftrightarrow} 1$

According to time shifting property we have,

$$x(n-k) \stackrel{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

$$\boxed{\therefore\ \delta(n-k) \stackrel{Z}{\longleftrightarrow} Z^{-k} \cdot 1}$$

ROC : Entire Z-plane except $Z = 0$.

Prob. 2 : Find the Z-transform of :

$$x(n) = \delta(n+2)$$

Soln. : We have,

$$\delta(n) \overset{Z}{\longleftrightarrow} 1$$

Using time shifting property we can write,

$$x(n+k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

$$\boxed{\therefore\ \delta(n+2) \overset{Z}{\longleftrightarrow} Z^2}$$

ROC : Entire Z-plane except $Z = \infty$

Prob. 3 : It is given that :

$$x_1(n) = \{1, 2, 3, 4, 0, 1\}$$
$$\uparrow$$

Using time shifting property find Z-transform of $x_2(n)$ where :

$$x_2(n) = \{1, 2, 3, 4, 0, 1\}$$
$$\uparrow$$

Soln. : Comparing the given sequences, we can conclude that $x_2(n)$ is advanced version of $x_1(n)$ and the advance is by two units.

$$\therefore \quad x_2(n) = x_1(n+2) \qquad \text{...(1)}$$

According to time shifting property

$$x(n+k) \overset{Z}{\longleftrightarrow} Z^k X(Z)$$

$$\therefore \quad x(n+2) \overset{Z}{\longleftrightarrow} Z^2 X(Z) \qquad \text{...(2)}$$

Now we will obtain $X_1(Z)$

Here $x_1(n) = \{1, 2, 3, 4, 0, 1\}$
$$\uparrow$$

According to definition of Z-transform.

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Here range of sequence x (n) is from n = 0 to 5

$$\therefore \quad X_1(Z) = \sum_{n=0}^{5} x(n) Z^{-n}$$

Expanding the summation and putting the values of x (n) we get,

$$X_1(Z) = 1Z^0 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + 0 + 1Z^{-5} \quad ...(3)$$

Using Equations (1) and (2) we can write,

$$Z\left\{x_2(n)\right\} = Z^2\left[1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + Z^{-5}\right]$$

$$\boxed{\therefore \quad X_2(Z) = Z^2 + 2Z + 3 + 4Z^{-1} + Z^{-3}}$$

ROC : This is two sided finite duration sequence. Thus ROC is entire Z-plane except $Z = 0$ and $Z = \infty$.

Prob. 4 : Determine the Z-transform of the following finite duration sequence

$$x(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Soln. : Since the magnitude of x (n) is 1, it is an unit step but having a finite duration. This signal is generated from unit step as shown in Fig. U-21.

Thus the given signal can be expressed as,

$$x(n) = u(n) - u(n-N) \quad ...(1)$$

Here u (n) is unit step and u (n − N) is delayed unit step.

According to linearity property the Z-transform of Equation (1) can be written as,

$$X(Z) = Z\left\{u(n)\right\} - Z\left\{u(n-N)\right\} \quad ...(2)$$

We know that Z-transform of unit step is,

$$Z\left\{u(n)\right\} = \frac{Z}{Z-1} \qquad \textbf{ROC : } |Z| > 1 \quad ...(3)$$

Now consider second term u (n − N). It is unit step delayed by 'N' samples. According to time shifting property we can write,

$$Z\left\{u(n-N)\right\} = Z^{-N} \cdot \frac{Z}{Z-1} : \text{ ROC } |Z| > 1 \quad ...(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$X(Z) = \frac{Z}{Z-1} - \frac{Z^{-N} \cdot Z}{Z-1} = \frac{Z(1-Z^{-N})}{Z-1}$$

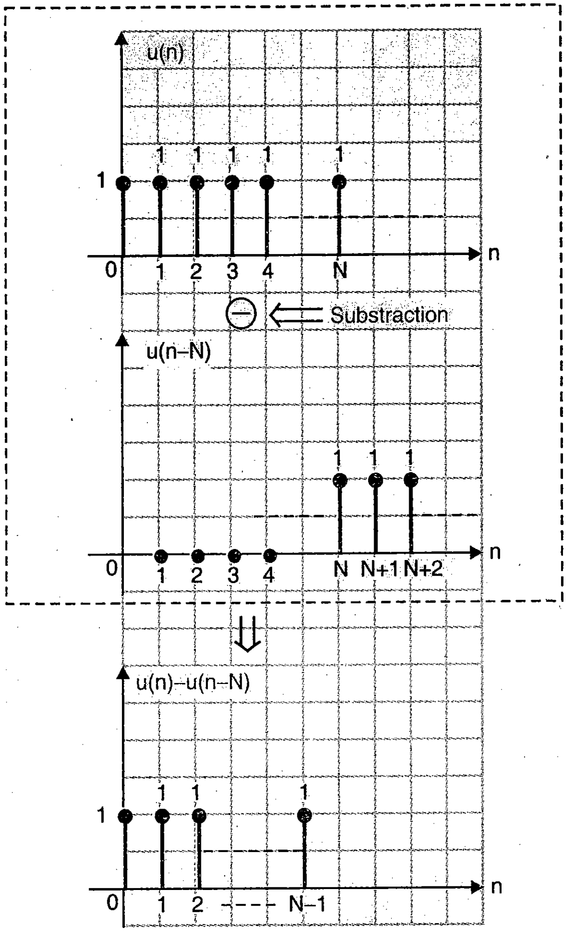


Fig. U-21 : Generation of x (n)

This is standard Z-transform pair

$$\therefore u(n) - u(n-N) \overset{Z}{\longleftrightarrow} \frac{Z(1-Z^{-N})}{Z-1}$$

Note : If combination of several signals has finite duration then ROC is exclusively dictated by the finite duration of the signal and not by the ROC of individual transform. This is finite duration unit step and it is causal. So its ROC is entire Z plane except Z = 0.

Prob. 5 : Express the Z transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k) \text{ in terms of } X(Z).$$

Soln. : The given expression is,

$$y(n) = \sum_{k=-\infty}^{n} x(k) \quad ...(1)$$

Let us expand the summation

$$\therefore \quad y(n) = x(-\infty) + \ldots + x(0) + x(1) + \ldots + x(n-1) + x(n) \quad ...(2)$$

Replace 'n' by 'n – 1' in Equation (1)

$$\therefore \quad y(n-1) = \sum_{k=-\infty}^{n-1} x(k) \quad ...(3)$$

Expanding Equation (3) we get,

$$y(n-1) = x(-\infty) + \ldots + x(0) + x(1) + \ldots + x(n-1) \quad ...(4)$$

Subtracting Equation (4) from Equation (2) we get,

$$y(n) - y(n-1) = x(n) \quad ...(5)$$

Taking Z transform of both sides,

$$Y(Z) - Z^{-1} Y(Z) = X(Z)$$

$$\therefore \quad Y(Z)[1 - Z^{-1}] = X(Z)$$

$$\boxed{\therefore \quad X(Z) = Y(Z)[1 - Z^{-1}]}$$

1.3.3 Scaling in the Z-domain (Multiplication by Exponential Sequence) :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ ROC : $r_1 < |Z| < r_2$

then $a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$ ROC : $|a| r_1 < |Z| < |a| r_2$

For any constant 'a' real or complex.

Proof : According to definition of Z-transform.

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(1)$$

$$\therefore \quad Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, a^n Z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\,(a^{-1} Z)^{-n}$$

$$\therefore \quad Z\left\{a^n x(n)\right\} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{Z}{a}\right)^{-n} \qquad \text{...(2)}$$

Comparing Equations (1) and (2),

$$Z\left\{a^n x(n)\right\} = X\left(\frac{Z}{a}\right)$$

$$\boxed{\therefore \quad a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)}$$

ROC : ROC of $X(Z)$ is $r_1 < |Z| < r_2$

To obtain ROC of $X\left(\frac{Z}{a}\right)$, replace 'Z' by $\frac{Z}{a}$.

$$\therefore \text{ ROC of } X\left(\frac{Z}{a}\right) : \qquad r_1 < \left|\frac{Z}{a}\right| < r_2$$

$$\boxed{\therefore \quad \text{ROC of } X\left(\frac{Z}{a}\right) : \quad |a|\, r_1 < |Z| < |a|\, r_2}$$

Note : (1) Scaling in the time domain corresponds to shrinking or expanding the Z-plane.
(2) By taking 'a' as complex variable one can change scale of radius and ω in the Z-plane.

Solved Problems on Scaling Property :

Prob. 1 : Obtain Z-transform of $x(n) = a^n u(n)$ using scaling property.

Soln. : Here $u(n)$ is a unit step. Its Z- transform is,

$$Z\left\{u(n)\right\} = \frac{Z}{Z-1} \qquad \textbf{ROC : } |Z| > 1 \qquad \text{...(1)}$$

According to linearity property we have,

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right) \qquad \text{...(2)}$$

Let $x(n) = u(n)$. Thus $Z\{a^n x(n)\} = Z\{a^n u(n)\}$ is obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore \quad Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} \qquad \text{ROC } \left|\frac{Z}{a}\right| > 1$$

$$\boxed{\therefore \; Z\{a^n u(n)\} = \frac{Z}{Z-a} \qquad \text{ROC : } |Z| > |a|}$$

We have already obtained Z-transform of this sequence. Thus using scaling property, same result is obtained.

Prob. 2 : Obtain Z-transform of $x(n) = a^n \cos \omega_0 n \, u(n)$.

Soln. : We have already obtained the Z-transform of $\cos \omega_0 n \, u(n)$.

Recall that result,

$$Z\{\cos \omega_0 n \;\; u(n)\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC : } |Z| > 1 \qquad \text{...(1)}$$

According to scaling property we have,

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right) \qquad \text{...(2)}$$

Remember that whenever we use scaling property; we will have to replace Z by $\frac{Z}{a}$ in the equation of $X(Z)$. Thus replacing Z by $\frac{Z}{a}$ in Equation (1) we get,

$$Z\{a^n \cos\omega_0 n\} = \frac{\left(\frac{Z}{a}\right)^2 - \left(\frac{Z}{a}\right)\cos\omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos\omega_0 + 1} \qquad \text{ROC : } \left|\frac{Z}{a}\right| > 1$$

Multiplying and dividing numerator and denominator by a^2 we get,

$$\boxed{Z\{a^n \cos\omega_0 n\} = \frac{Z^2 - aZ\cos\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2} \qquad \text{ROC : } |Z| > |a|}$$

Here ROC is $|Z| > |a|$. That means it is exterior part of circle having radius $r = a$. This is shown in Fig. U-22.

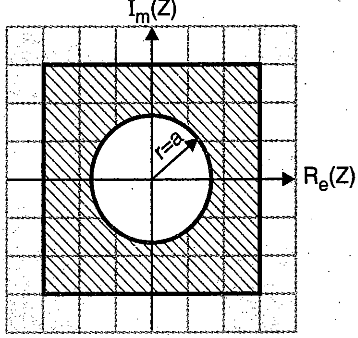


Fig. U-22 : ROC of $(a^n \cos \omega_0 n) u (n)$

Prob. 3 : Find Z transform of x (n) and draw its ROC,

$$x(n) = \left[(0.5)^n \sin \frac{\pi n}{4} \right] u(n)$$

Soln. : The given function is,

$$x(n) = \left[(0.5)^n \sin \frac{\pi n}{4} \right] u(n)$$

$$\therefore \quad x(n) = (0.5)^n \sin \frac{\pi n}{4} u(n) \qquad \text{...(1)}$$

We have standard Z transform pair,

$$\sin \omega_0 n \, u(n) \overset{Z}{\longleftrightarrow} \frac{Z \sin \omega_0}{Z^2 - 2 Z \cos \omega_0 + 1} \qquad \text{ROC : } |Z| > 1 \qquad \text{...(2)}$$

Consider the term $\sin \frac{\pi n}{4} u(n)$. Here, let $\omega_0 = \frac{\pi}{4}$. Thus Equation (2) becomes,

$$\sin \frac{\pi n}{4} u(n) \overset{Z}{\longleftrightarrow} \frac{Z \sin \frac{\pi}{4}}{Z^2 - 2 Z \cos \frac{\pi}{4} + 1}, \qquad \text{ROC : } |Z| > 1$$

$$\therefore \quad Z \left\{ \sin \frac{\pi}{4} n \, u(n) \right\} = \frac{0.707 Z}{Z^2 - 1.414 Z + 1}, \qquad \text{ROC : } |Z| > 1 \qquad \text{...(3)}$$

According to the scaling property we have,

$$a^n u(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a} \right)$$

Here $a = 0.5$. Thus Z transform of given function, $x(n)$ is obtained by replacing Z by $\frac{Z}{0.5}$ that means by 2Z in Equation (3) we get,

$$\therefore \quad Z\left\{\left[(0.5)^n \sin\frac{\pi n}{4}\right]u(n)\right\} = \frac{0.707\,(2Z)}{(2Z)^2 - 1.414\,(2Z) + 1}, \qquad \text{ROC}: |2Z| > 1$$

$$\boxed{\therefore \quad X(Z) = \frac{1.414\,Z}{4Z^2 - 2.828 + 1}, \qquad \text{ROC}: |Z| > \frac{1}{2}}$$

Thus ROC is exterior part of circle having radius $\frac{1}{2}$ as shown in Fig. U-23.

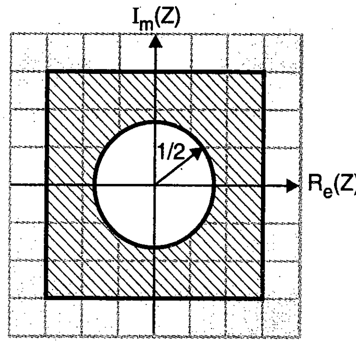


Fig. U-23

Prob. 4 : Obtain Z-transform of $x(n) = a^n \sin\omega_0 n\, u(n)$.

Soln. : We have already obtained Z-transform of $\sin\omega_0 n$. Recall that result,

$$Z\{\sin\omega_0 n\, u(n)\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC}: |Z| > 1 \qquad \text{...(1)}$$

Here we have,

$x(n) = a^n \sin\omega_0 n\, u(n)$. Thus according to scaling property, $X(Z)$ can be obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore \quad Z\{a^n \sin\omega_0 n\, u(n)\} = \frac{\left(\frac{Z}{a}\right)\sin\omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos\omega_0 + 1} \qquad \text{ROC}: \left|\frac{Z}{a}\right| > 1.$$

Dividing numerator and denominator by a^2 we get,

$$\boxed{Z\left\{a^n \sin \omega_0 n\, u(n)\right\} = \frac{aZ \sin \omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2} \qquad \text{ROC} : |Z| > |a|}$$

Prob. 5 : Find Z-transform and sketch the ROC

$$x(n) = (-1)^n 2^{-n} u(n)$$

Soln. :

$$\text{Given} \quad x(n) = (-1)^n 2^{-n} u(n)$$

$$\therefore \quad x(n) = (-1)^n \frac{1}{(2)^n} u(n)$$

$$\therefore \quad x(n) = \left(-\frac{1}{2}\right)^n u(n) \qquad \text{...(1)}$$

Here u (n) is unit step and its Z-transform is given by,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1 \qquad \text{...(2)}$$

According to scaling property, we can write,

$$Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} \quad : \quad \text{ROC} \left|\frac{Z}{a}\right| > 1 \qquad \text{...(3)}$$

Observe Equation (1). Here a = – 1/2. Putting this value in Equation (3) we get,

$$Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{Z/(-1/2)}{\left(\frac{Z}{-1/2}\right) - 1} \quad : \quad \text{ROC } |Z| > \left|\left(-\frac{1}{2}\right)\right|$$

$$= \frac{-2Z}{-2Z - 1} \quad : \quad \text{ROC } |Z| > \left|-\frac{1}{2}\right|$$

$$\therefore \quad Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{2Z}{2Z+1} \quad : \text{ROC } |Z| > 1/2$$

Thus ROC is exterior part of circle having radius r = 1/2.

Prob. 6 : Determine Z-transform including ROC of the following,

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$$

Soln. : The given expression can be written as,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-10) \qquad \text{...(1)}$$

According to linearity property we have,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \qquad \text{...(2)}$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \textbf{ROC : } |Z| > |\alpha| \qquad ...(3)$$

Consider first term of Equation (1),

$$\text{Let,} \quad x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

Using Equation (3) we can write,

$$X_1(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{Z}{Z-1/2} = \frac{2Z}{2Z-1} \qquad \text{ROC : } |Z| > \frac{1}{2} \qquad ...(4)$$

Now consider second term of Equation (1),

$$\textbf{Let} \quad x_2(n) = \left(\frac{1}{2}\right)^n u(n-10) \qquad ...(5)$$

First we will obtain Z-transform of $u(n-10)$

Recall the time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-K} \cdot X(Z)$$

Now we have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}; \qquad \text{ROC : } |Z| > 1$$

$$\therefore \; Z\{u(n-10)\} = Z^{-10}\left(\frac{Z}{Z-1}\right); \qquad \text{ROC } |Z| > 1$$

$$\therefore \; Z\{u(n-10)\} = \frac{Z^{-9}}{Z-1} \qquad ...(6)$$

According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

That means we have to replace Z by $\frac{Z}{a}$.

In this case, $a = \frac{1}{2}$. Thus applying scaling property to Equation (6) we can write,

$$Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left[\frac{Z}{(1/2)}\right]^{-9}}{\frac{Z}{1/2}-1} = \frac{(Z^{-9})\left(\frac{1}{1/2}\right)^{-9}}{\frac{Z}{1/2}-1}; \qquad \text{ROC : } \left|\frac{Z}{1/2}\right| > 1$$

$$= \frac{Z^{-9} \cdot \left(\frac{1}{2}\right)^{+9}}{2Z-1}; \quad \text{ROC} : |Z| > \frac{1}{2}$$

$$X_2(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} \quad \text{ROC} : |Z| > \frac{1}{2} \qquad ...(7)$$

Putting Equations (4) and (7) in Equation (2) we get,

$$X(Z) = \frac{2Z}{2Z-1} - \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC} : |Z| > \frac{1}{2}$$

$$\therefore \quad X(Z) = \frac{2Z \times Z^9 - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC} : |Z| > \frac{1}{2}$$

$$\boxed{\therefore \quad X(Z) = \frac{2Z^{10} - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC} : |Z| > \frac{1}{2}}$$

Thus ROC is exterior part of circle having radius r = 1/2.

Prob. 7 : Determine Z transform and draw ROC of the following signal $x(n) = (2)^{n+2} u(n-1)$. Is the signal causal ?

Soln. : The given signal is,

$$x(n) = (2)^{n+2} u(n-1)$$

$$\therefore \quad x(n) = 2^n \cdot 2^2 u(n-1)$$

$$\therefore \quad x(n) = 4 \cdot 2^n u(n-1) \qquad ...(1)$$

First we will obtain Z transform of $u(n-1)$. We have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \qquad \text{ROC} : |Z| > 1$$

According to shifting property we have,

$$u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\left[\frac{Z}{Z-1}\right], \quad \text{ROC} : |Z| > 1$$

$$\therefore \quad Z\{u(n-1)\} = \frac{1}{Z-1}, \qquad \text{ROC} : |Z| > 1 \qquad ...(2)$$

Now according to the scaling property, we have,

$$Z\left\{a^n u(n)\right\} = X\left(\frac{Z}{a}\right)$$

Here a = 2. Thus applying scaling property to Equation (2).

$$Z\left\{2^n u(n-1)\right\} = \frac{1}{\frac{Z}{2}-1}, \qquad \text{ROC : } \left|\frac{Z}{2}\right| > 1$$

$$\therefore \quad Z\left\{2^n u(n-1)\right\} = \frac{2}{Z-2}, \qquad \text{ROC : } |Z| > 2$$

Thus for Equation (1) we can write,

$$Z\left\{4 \cdot 2^n u(n-1)\right\} = 4 \cdot \frac{2}{Z-2}, \qquad \text{ROC : } |Z| > 2$$

$$\therefore \quad X(Z) = \frac{8}{Z-2}, \qquad \text{ROC : } |Z| > 2$$

Here ROC is exterior part of circle having radius 2, as shown in Fig. U-24. Since ROC is exterior part of circle; the given sequence is causal.

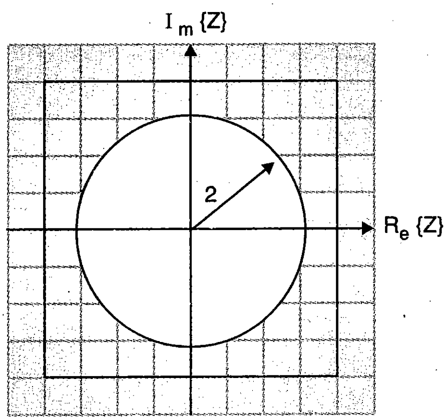


Fig. U-24

1.3.4 Time Reversal Property :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ ROC : $r_1 < |Z| < r_2$

Then $x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$ ROC : $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof : According to definition of Z-transform we have,

$$Z\left\{x(n)\right\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore \ Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)Z^{-n} \quad ...(1)$$

Put $l = -n$; the limits will change as follows :
when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l = -\infty$.

$$\therefore \ Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l)Z^{l}$$

$$\therefore \ Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)(Z^{-1})^{-l} \quad ...(2)$$

Comparing this equation with the definition of Z-transform we get,

$$\boxed{Z\{x(-n)\} = X(Z^{-1})} \quad ...(3)$$

Note : Here $x(-n)$ is the folded version of $x(n)$. So $x(-n)$ is the time reversed signal. Thus the folding of signal in time domain is equivalent to replacing Z by Z^{-1} in the Z-domain. Replacing Z by Z^{-1} is called as inversion. Hence folding in time domain is equivalent to inversion in the Z-domain.

Comment on ROC : ROC of $x(n)$ is inverse of that of $x(-n)$. This means that, if Z_0 belongs to the ROC of $x(n)$ then $1/Z_0$ is in the ROC for $x(-n)$.

Given ROC is $r_1 < |Z| < r_2$. Thus ROC of $X(Z^{-1})$ is obtained by replacing Z by Z^{-1}.

Thus ROC of Z^{-1} : $r_1 < |Z^{-1}| < r_2$

$$\therefore \ r_1 < \left|\frac{1}{Z}\right| < r_2$$

$$\therefore \ |Z| < \frac{1}{r_1} \text{ and } |Z| > \frac{1}{r_2}$$

$$\boxed{\therefore \ \frac{1}{r_2} < |Z| < \frac{1}{r_1}}$$

Solved Problems on Time Reversal Property :

Prob. 1 : Obtain the Z-transform of signal $x(n) = u(-n)$.

Soln. : We have the Z-transform of unit step,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1 \quad ..(1)$$

According to the time reversal property,

$$Z\{x(-n)\} = X(Z^{-1})$$

Thus Z-transform of $x(-n)$ is obtained by replacing Z by Z^{-1} in Equation (1).

$$\therefore\ Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1} \qquad \text{ROC} : |Z^{-1}| > 1$$

Multiplying numerator and denominator by Z we get,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad \text{ROC} : |Z^{-1}| > 1$$

Here ROC is $|Z^{-1}| > 1$

$\therefore \left|\frac{1}{Z}\right| > 1$ that means $|Z| < 1$.

This is the standard Z-transform pair,

$$\boxed{\therefore\ u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1}$$

Thus ROC is interior part of circle having radius r = 1. ***The circle or radius = 1 is also called as unit circle***. Thus ROC is interior part of unit circle as shown in Fig. U-25.

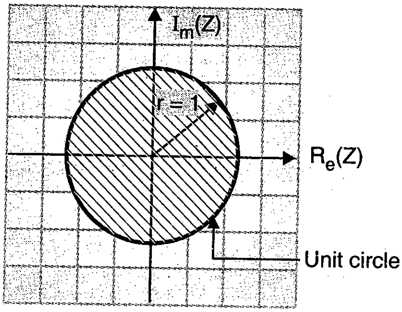


Fig. U-25 : ROC of u (– n)

Prob. 2 : Determine Z-transform and ROC of :

$$x(n) = \left(\frac{1}{2}\right)^n u(-n)$$

Soln. :

We have $Z\{u(-n)\} = \dfrac{1}{1-Z}$

ROC : $|Z| < 1$

Here the given signal is $\left(\frac{1}{2}\right)^n u(-n)$. According to scaling property we have,

$$Z\left\{a^n x(n)\right\} = X\left(\frac{Z}{a}\right)$$

That means we will have to replace Z by $\frac{Z}{a}$. In this example $a = \frac{1}{2}$.

$$\text{Thus , } Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-\frac{Z}{1/2}}$$

$$\text{ROC : } \left|\frac{Z}{1/2}\right| < 1$$

$$\therefore Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-2Z} \qquad \text{ROC : } |Z| < \frac{1}{2}$$

Thus ROC is interior part of circle having radius r = 1/2. This ROC is similar to Fig. U-22. Only difference is here, r = 1/2.

Prob. 3 : The Z-transform of DT signal x (n) is given by $x(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z^2+4}$ ROC $|Z| > 2$.

Determine Z transform and ROC of the following signals using properties of Z-transform.

(i) $2^n x(n)$ (ii) $nx(n)$ (iii) $x(-n)$ (iv) $x(n-4)$.

Soln. :

We have,

$$Z\left\{u(n)\right\} = \frac{Z}{Z-1}$$

$$\text{ROC } |Z| > 1$$

$$\text{and } Z\left\{u(-n)\right\} = \frac{Z^{-1}}{Z^{-1}-1}$$

$$\text{ROC } |Z^{-1}| > 1$$

$$= \frac{1}{1-Z}$$

$$\text{ROC} \left| \frac{1}{Z} \right| < 1$$

$$\text{i.e. } 1 < |Z|$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} + \frac{1}{1-Z}$$

$$\text{ROC} : |Z| > 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} - \frac{(-1)}{1-Z} = \frac{Z}{Z-1} + \frac{(-1)}{Z-1} = \frac{Z-1}{Z-1}$$

$$\therefore \quad X(Z) = 1$$

Thus ROC is entire Z plane.

Prob. 4 : Find Z-transform of following signals and comment on the results.

(i) $x(n) = a^n u(n-1)$ (ii) $x(n) = a^n u(-n-1)$

(iii) $x(n) = -(a^n) u(n-1)$ (iv) $x(n) = -(a^n) u(-n-1)$

Soln. :

(i) $\mathbf{x(n) = a^n u(n-1)}$

Here $u(n-1)$ is delayed unit step. First we will obtain $Z\{u(n-1)\}$.

Now we have,

$$Z\{u(n)\} = \frac{Z}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Applying time shifting property we can write,

$$Z\{u(n-1)\} = Z^{-1} \times \frac{Z}{Z-1} = \frac{1}{Z-1} \qquad \text{ROC} : |Z| > 1$$

Now we have to obtain $Z\{a^n u(n-1)\}$.

Applying scaling property we get,

$$Z\{a^n u(n-1)\} = \frac{1}{\frac{Z}{a} - 1} \qquad \text{ROC} : \left| \frac{Z}{a} \right| > 1$$

$$\boxed{\therefore Z\{a^n u(n-1)\} = \frac{a}{Z-a} \qquad \text{ROC} : |Z| > a} \qquad \text{...(1)}$$

Thus ROC is exterior part of circle having radius, r = a.

(ii) $\mathbf{x(n) = a^n u(-n-1)}$

First we will obtain $Z\{u(-n-1)\}$. Here $u(-n)$ is folded unit step. The Z- transform of folded unit step is given by,

$$Z\{u(-n)\} = \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

Now recall time shifting property, it is

$$\text{If} \quad x(n) \overset{Z}{\longleftrightarrow} X(Z) \quad \text{then} \quad x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z).$$

Here we have folded sequence. For folded sequence this property can be written as,

$$x(-n-k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

This is because folded sequence is mirror image of original sequence.

Thus we can write,

$$Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\}$$

$$= Z^{1} \cdot \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

$$\boxed{\therefore \; Z\{u(-n-1)\} = \frac{Z}{1-Z} \qquad \text{ROC} : |Z| < 1}$$

Now we have to obtain $Z\{a^n u(-n-1)\}$. Using scaling property we get,

$$Z\{a^n u(-n-1)\} = \frac{Z/a}{1-Z/a} \qquad \text{ROC} : \left|\frac{Z}{a}\right| < 1$$

$$\boxed{\therefore \; Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \qquad \text{ROC} : |Z| < |a|} \qquad ...(2)$$

Thus ROC is interior part of circle having radius $r = a$.

(iii) $\mathbf{x(n) = -a^n u(n-1)}$

From Equation (1) we have,

$$Z\{a^n u(n-1)\} = \frac{a}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

$$\text{Thus} \quad Z\{-a^n u(n-1)\} = -\frac{a}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

$$\boxed{\therefore \; Z\{-a^n u(n-1)\} = \frac{a}{a-Z} \qquad \text{ROC} : |Z| > |a|} \qquad ..(3)$$

(iv) $\mathbf{x(n) = -(a)^n u(-n-1)}$

From Equation (2) we have,

$$Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \qquad \text{ROC} : |Z| < |a|$$

Thus $Z\left\{-a^n u(-n-1)\right\} = -\dfrac{Z}{a-Z}$ ROC : $|Z| < |a|$

$$\therefore \; Z\left\{-a^n u(-n-1)\right\} = \frac{Z}{Z-a} \quad \text{ROC} : |Z| < |a| \qquad ...(4)$$

Note : Compare given Equations (1) and (3) as well as (2) and (4). Here magnitude of sequence is changed in time domain. From the result of Z-transform, Equations (1) and (3), and Equations (2) and (4), we can say that the magnitude in Z-domain is also reversed. Thus changing the magnitude in time domain, changes the magnitude in Z-domain also; while the ROC remains same.

1.3.5 Differentiation in Z domain :

Statement : **If** $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

then $n\,x(n) \overset{Z}{\longleftrightarrow} -Z\dfrac{dX(Z)}{dZ}$

Proof : According to the definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)\,Z^{-n} \qquad ...(1)$$

Differentiate both sides with respect to Z,

$$\frac{dX(Z)}{dZ} = \frac{d}{dZ}\left[\sum_{n=-\infty}^{\infty} x(n)\,Z^{-n}\right]$$

We can transfer d/dZ inside the summation sign,

$$\therefore \; \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} \frac{d}{dZ}\left[x(n)\,Z^{-n}\right] = \sum_{n=-\infty}^{\infty} (-n)\cdot x(n)\,Z^{-n-1}$$

But Z^{-n-1} can be written as $Z^{-n}\cdot Z^{-1}$

$$\therefore \; \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} (-n)\,x(n)\,Z^{-n}\cdot Z^{-1}$$

Since the limits of summation are in terms of n ; we can take Z^{-1} outside the summation.

$$\therefore \; \frac{dX(Z)}{dZ} = Z^{-1}\sum_{n=-\infty}^{\infty} (-n)\,x(n)\,Z^{-n}$$

Taking negative sign outside the summation,

$$\frac{dX(Z)}{dZ} = -Z^{-1} \sum_{n=-\infty}^{\infty} [n\,x(n)]Z^{-n}$$

$$\therefore \quad \frac{dX(Z)}{dZ} = -\frac{1}{Z} \sum_{n=-\infty}^{\infty} [n\,x(n)]Z^{-n}$$

$$\therefore \quad -Z\frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} [n\,x(n)]Z^{-n} \qquad ...(2)$$

Comparing R.H.S. with Equation (1), we can say that $\sum_{n=-\infty}^{\infty} [n\,x(n)]Z^{-n}$ is the Z- transform of $n\,x(n)$. Thus Equation (2) becomes,

$$-Z\frac{dX(Z)}{dZ} = Z\{n\,x(n)\}$$

$$\boxed{\therefore \quad n\,x(n) \overset{Z}{\longleftrightarrow} -Z\frac{dX(Z)}{dZ}}$$

Hence proved. Note that both transforms have the same ROC.

Note : Multiplying the sequence in time domain by 'n' is equivalent to multiplying the derivative of its Z-transform, by $-Z$ in the 'Z' domain.

Solved Problems on Differentiation Property :

Prob. 1 : Find Z transform and ROC of following sequence

$$x(n) = \frac{1}{2}\delta(n+1) + 5\left(\frac{1}{2}\right)^{-n} u(-n) + u(-n-1)$$

Soln. : The given expression is,

$$x(n) = \frac{1}{2}\delta(n+1) + 5\left(\frac{1}{2}\right)^{-n} u(n) + u(-n-1) \qquad ...(1)$$

It can be written as,

$$x(n) = x_1(n) + x_2(n) + x_3(n) \qquad ...(2)$$

Here $x_1(n) = \frac{1}{2}\delta(n+1)$ ..

We have, $\delta(n) \overset{Z}{\longleftrightarrow} 1$ ROC : Entire Z plane except Z = 0

According to time shifting property,

$$\delta(n+1) \overset{Z}{\longleftrightarrow} Z^1 \qquad \text{ROC : Entire Z plane except } Z = 0$$

$$\therefore \quad Z\left\{\frac{1}{2}\delta(n+1)\right\} = X_1(Z) = \frac{1}{2}Z \qquad \text{ROC : Entire Z plane except } Z = 0. \qquad ...(4)$$

$$\text{Now} \quad x_2(n) = 5\left(\frac{1}{2}\right)^{-n} u(n)$$

$$\therefore \quad x_2(n) = 5 \cdot \frac{1}{\left(\frac{1}{2}\right)^n} u(n) = 5 \cdot \frac{2^n}{(1)^n} u(n)$$

$$\therefore \quad x_2(n) = 5 \cdot 2^n u(n) \qquad ...(5)$$

We have standard Z transform pair,

$$u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-Z}, \qquad \text{ROC : } |Z| < 1$$

$$\therefore \quad a^n u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{Z}{a}} \qquad \text{ROC : } |Z| < a$$

Here a = 2

$$\therefore \quad 2^n u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{Z}{2}} \qquad \text{ROC : } |Z| < 2$$

$$\therefore \quad Z\left\{5 \cdot 2^n u(n)\right\} = X_2(Z) = \frac{5}{1-\frac{Z}{2}} \qquad \text{ROC : } |Z| < 2 \qquad ...(6)$$

$$\text{Now} \quad x_3(n) = u(-n-1)$$

First we will obtain $Z\{u(-n)\}$. Here $u(-n)$ is folded unit step. The Z-transform of folded unit step is,

$$Z\{u(-n)\} = \frac{1}{1-Z}, \qquad \text{ROC : } |Z| < 1$$

According to the time shifting property we have,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$$

For folded sequence this property can be written as,

$$x(-n-k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

$$\therefore \quad Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\}$$

$$\therefore \quad Z\{u(-n-1)\} = Z^{+1} \cdot \frac{1}{1-Z} \qquad \text{ROC} : |Z| < 1$$

$$\therefore \quad Z\{u(-n-1)\} = X_3(Z) = \frac{Z}{1-Z}, \qquad \text{ROC} : |Z| < 1 \qquad \text{...(7)}$$

The total Z-transform is obtained by applying linearity property to Equation (2).

$$\therefore \quad X(Z) = X_1(Z) + X_2(Z) + X_3(Z)$$

Thus from Equations (4), (6) and (7) we can write,

$$\boxed{X(Z) = \frac{1}{2} Z + \frac{5}{1-\frac{Z}{2}} + \frac{Z}{1-Z}}$$

We have ROCs, $|Z| < 1$ and $|Z| < 2$. Thus combined ROC is $|Z| < 1$ and it is shown in Fig. U-26.

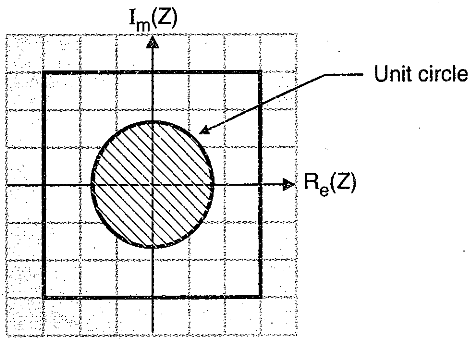


Fig. U-26

Prob. 2 : Using differentiation property obtain the Z-transform of unit ramp sequence.

Soln. : We know that unit ramp sequence is given by,

$$x(n) = n\,u(n) \qquad \text{...(1)}$$

According to differentiation property we have,

$$Z\{n\,x(n)\} = -Z \frac{dX(Z)}{dZ} \qquad \text{...(2)}$$

In this case we can write,

$$Z\{n\,u(n)\} = -Z \frac{d}{dZ} [Z(u(n))]$$

We have $Z\{u(n)\} = \dfrac{Z}{Z-1}$ ROC : $|Z| > 1$

$$\therefore \quad Z\{n\,u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-1}\right]$$

$$\therefore \quad Z\{n\,u(n)\} = -Z\frac{d}{dZ}\left[(Z)\cdot(Z-1)^{-1}\right]$$

$$= -Z\left[Z\cdot\frac{d}{dZ}\left[(Z-1)^{-1}\right]+(Z-1)^{-1}\cdot\frac{d}{dZ}Z\right]$$

$$= -Z\left[Z(-1)(Z-1)^{-2}+(Z-1)^{-1}\cdot 1\right]$$

$$= -Z\left[\frac{-Z}{(Z-1)^2}+\frac{1}{(Z-1)}\right] = -Z\left[\frac{-Z+Z-1}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{-1}{(Z-1)^2}\right]$$

$$\therefore \quad Z\{n\,u(n)\} = \frac{Z}{(Z-1)^2} \qquad \text{ROC : } |Z| > 1$$

This is the standard Z-transform pair. Earlier we had obtained the Z-transform of $n\,u(n)$ without using differentiation property.

Prob. 3 : Obtain the Z-transform of signal,

$x(n) = n\,a^n\,u(n)$

Soln. :

Let $x_1(n) = a^n\,u(n)$ $\therefore$ $x(n) = n\,x_1(n)$

According to differentiation property,

$$n\,x_1(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dZ}X_1(Z) \qquad \text{...(1)}$$

In this case we can write,

$$Z\{n\,a^n\,u(n)\} = -Z\frac{d}{dZ}\left[Z\{a^n\,u(n)\}\right] \qquad \text{...(2)}$$

Here the Z-transform of $a^n\,u(n)$ is given by,

$$Z\{(a^n\,u(n)\} = \frac{Z}{Z-a} \qquad \text{ROC : } |Z| > |a|$$

Putting this value in Equation (2) we get,

$$Z\{n\,a^n\,u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-a}\right]$$

$$\therefore \quad Z\{n\,a^n\,u(n)\} = -Z\frac{d}{dZ}\left[Z\cdot(Z-a)^{-1}\right]$$

$$= -Z\left\{Z \cdot \frac{d}{dZ}\left[(Z-a)^{-1}\right] + (Z-a)^{-1} \cdot \frac{d}{dZ} Z\right\}$$

$$= -Z\left[Z \cdot (Z-a)^{-2} \cdot (-1) + (Z-a)^{-1} \cdot 1\right]$$

$$= -Z\left[\frac{-Z}{(Z-a)^2} + \frac{1}{(Z-a)}\right] = -Z\left[\frac{-Z+Z-a}{(Z-a)^2}\right]$$

$$= -Z\left[\frac{-a}{(Z-a)^2}\right]$$

$$\therefore \quad Z\left\{n\,a^n u(n)\right\} = \frac{aZ}{(Z-a)^2} \qquad \text{ROC : } |Z| > |a| \qquad \text{...(3)}$$

This is also standard Z-transform pair.

$$\therefore \quad n\,a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2} \qquad \text{ROC : } |Z| > |a|$$

Note : From this equation we can obtain Z- transform of unit ramp sequence, $n\,u(n)$. By putting a = 1 in Equation (3) we get,

$$Z\left\{n\,u(n)\right\} = \frac{Z}{(Z-1)^2} \qquad \text{ROC : } |Z| > |a|$$

Prob. 4 : The z transform of DT signal $x(n)$ is given by $x(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z^2+4}$ ROC $|Z| > 2$.

Determine Z transform and ROC of the following signals using properties of Z transform.

(i) $2^n x(n)$ (ii) $nx(n)$ (iii) $x(-n)$ (iv) $x(n-4)$.

Soln. :

(i) $2^n x(n)$

According to the scaling property.

$$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$$

Here $\quad X(Z) = \frac{Z}{Z^2+4}$, ROC $|Z| > 2$

$$\therefore \quad 2^n x(n) \overset{Z}{\longleftrightarrow} \frac{Z_{12}}{\left(\frac{Z}{2}\right)^2 + 4} \text{ ROC } \left|\frac{Z}{2}\right| > 2$$

$$\therefore \quad Z\left\{2^n x(n)\right\} = \frac{Z_{12}}{\frac{Z^2}{4} + 4}, \text{ ROC } |Z| > 4$$

(ii) $n\,x(n)$

According to the differentiation property,

$$n\,x(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dz}X(Z)$$

$$\therefore \quad Z\{n\,x(n)\} = -Z\frac{d}{dz}\left\{\frac{Z}{Z^2+4}\right\} = -Z\left\{Z(-1)(Z^2+4)^{-2}+(Z^2+4)^{-1}\right\}$$

$$\therefore \quad Z\{n\,x(n)\} = \frac{Z^2}{Z^2+4}-\frac{Z}{Z^2+4} = \frac{Z^2-Z}{Z^2+4}$$

The ROC remains same as $X(Z)$.

(iii) $x(-n)$

According to time reversal property,

$$x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$$

$$\therefore \quad Z\{x(-n)\} = \frac{Z^{-1}}{(Z^{-1})^2+4} \quad \text{ROC } |Z^{-1}| > 2$$

$$= \frac{Z^{-1}}{Z^{-2}+4} \quad \text{ROC } \left|\frac{1}{Z}\right| > 2 \text{ i.e. } 1 > 2|Z| \text{ i.e. } |Z| > \frac{1}{2}$$

$$\therefore \quad Z\{x(-n)\} = \frac{Z}{1+4Z^2} \quad \text{ROC } |Z| > \frac{1}{2}$$

(iv) $x(n-4)$

According to time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X(Z)$$

$$\therefore \quad Z\{x(n-4)\} = Z^{-4}\left[\frac{Z}{Z^2+4}\right] = \frac{Z^{-3}}{Z^2+4}$$

Since $k = 4$ which is greater than zero, ROC is same as $X(Z)$ except $Z = 0$.

1.3.6 Convolution of Two Sequences :

Statement : If $x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{Z}{\longleftrightarrow} X_2(Z)$

Then $x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \qquad ...(1)$$

$$\therefore \; x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad \text{...(2)}$$

Taking Z-transform of x (n),

$$X(Z) = Z\left\{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right\} \qquad \text{...(3)}$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \qquad \text{...(4)}$$

In Equation (3) take the bracket term as $x(n)$. Thus Equation (4) becomes,

$$\therefore \; Z\{x(n)\} = \sum_{n=-\infty}^{\infty}\left[\underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)}\right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n}\right] \qquad \text{...(5)}$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^{k} \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (5) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)} \cdot Z^{-k}\right]$$

In the second summation; the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore \; Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k}\right]\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)}\right]$$

In the second summation put $n-k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m \qquad \therefore \; m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m \quad \therefore \quad m = \infty$

$$\therefore \quad Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k}\right]\left[\sum_{m=-\infty}^{\infty} x_2(m) Z^{-m}\right] \qquad ...(6)$$

Compare each bracket of R.H.S. with the definition of Z- transform.

$$\therefore \quad Z\{x(n)\} = X_1(Z) \cdot X_2(Z) \qquad ...(7)$$

But here $\quad x(n) = x_1(n) * x_2(n)$

$$\therefore \quad Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$\boxed{\therefore \quad x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)}$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.

Note : Convolution of two sequences in time domain is equivalent to multiplication of its Z- transforms in Z domain.

Solved Problems :

Prob. 1 : Find the linear convolution of $x_1(n)$ and $x_2(n)$ using Z-transform

$$x_1(n) = \{1, 2, \underset{\uparrow}{3}, 4\} \text{ and } x_2(n) = \{1, 2, \underset{\uparrow}{0}, 2, 1\}.$$

Soln. : According to the property of linear convolution in Z- domain we have,

$$Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z) \qquad ...(1)$$

Step I : We have $x_1(n) = \{1, 2, \underset{\uparrow}{3}, 4\}$

According to the definition of Z-transform,

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n}$$

Here limits of summation are from $n = -2$ to $n = 1$.

$$\therefore \quad X_1(Z) = \sum_{n=-2}^{1} x_1(n) Z^{-n}.$$

$$= x_1(-2) Z^2 + x_1(-1) Z + x_1(0) Z^0 + x_1(1) Z^{-1}$$

$$\therefore \quad X_1(Z) = 1Z^2 + 2Z + 3 + 4Z^{-1} \qquad ...(2)$$

Let us determine the ROC of $X_1(Z)$.

$$\text{We have } X_1(Z) = Z^2 + 2Z + 3 + \frac{4}{Z}$$

ROC : (i) Putting Z = 0 we get,

$$X_1(Z) = 0 + 0 + 3 + \frac{4}{0} = 0 + 0 + 3 + \infty = \infty$$

Thus $Z = 0$ is not allowed.

(ii) Putting $Z = \infty$ we get,

$$X_1(Z) = \infty + \infty + 3 + \frac{4}{\infty} = \infty + \infty + 3 + 0 = \infty$$

Thus $Z = \infty$ is not allowed.

Therefore ROC is entire Z plane except $Z = 0$ and $Z = \infty$

Step II : Now we will obtain Z-transform of $x_2(n)$.

$$\text{We have } x_2(n) = \{1, 2, \underset{\uparrow}{0}, 2, 1\}$$

Using definition of Z-transform,

$$X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

Here the range of n is $n = -2$ to $n = +2$

$$\therefore \quad X_2(Z) = \sum_{n=-2}^{2} x_2(n) Z^{-n}$$

$$\therefore \quad X_2(Z) = x_2(-2) Z^2 + x_2(-1) Z^1 + x_2(0) Z^0 + x_2(1) Z^{-1} + x_2(2) Z^{-2}$$

$$\therefore \quad X_2(Z) = 1 Z^2 + 2Z + 0Z^0 + 2 Z^{-1} + 1 Z^{-2}$$

$$\therefore \quad X_2(Z) = Z^2 + 2Z + 2Z^{-1} + Z^{-2} \qquad ...(3)$$

Let us determine the ROC of $X_2(Z)$

$$\text{We have } X_2(Z) = Z^2 + 2Z + \frac{2}{Z} + \frac{1}{Z^2}$$

ROC :

(i) Putting Z = 0 we get,

$$X_2(Z) = 0 + 0 + \frac{2}{0} + \frac{1}{0} = 0 + 0 + \infty + \infty = \infty.$$

Thus $Z = 0$ is not allowed.

(ii) Putting $Z = \infty$ we get,

$$X_2(Z) = \infty + \infty + \frac{2}{\infty} + \frac{1}{\infty} = \infty + \infty + 0 + 0 = \infty.$$

Thus $Z = \infty$ is not allowed.

Therefore the ROC is entire Z plane except $Z = 0$ and $Z = \infty$.

Step III : Now we have,

$$X(Z) = X_1(Z) \cdot X_2(Z)$$

$$\therefore \quad X(Z) = (Z^2 + 2Z + 3 + 4Z^{-1}) \cdot (Z^2 + 2Z + 2Z^{-1} + Z^{-2})$$

$$\therefore \quad X(Z) = Z^4 + 2Z^3 + 2Z + 1 + 2Z^3 + 4Z^2 + 4 + 2Z^{-1} + 3Z^2 + 6Z + 6Z^{-1} + 3Z^{-2}$$
$$+ 4Z + 8 + 8Z^{-2} + 4Z^{-3}$$

$$\therefore \quad X(Z) = Z^4 + 4Z^3 + 7Z^2 + 12Z + 13 + 8Z^{-1} + 11Z^{-2} + 4Z^{-3} \qquad ...(4)$$

Step IV : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

From Equation (4), we get the range of n. It is from $n = -4$ to $n = 3$.

$$\therefore \quad X(Z) = \sum_{n=-4}^{3} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-4)Z^4 + x(-3)Z^3 + x(-2)Z^2 + x(-1)Z + x(0)$$
$$+ x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} \qquad ...(5)$$

Comparing Equations (4) and (5),

$x(-4) = 1$, $x(-3) = 4$, $x(-2) = 7$, $x(-1) = 12$,
$x(0) = 13$, $x(1) = 8$, $x(2) = 11$, $x(3) = 4$.

Thus we can write,

$$x(n) = \{x(-4), x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}, x(1), x(2), x(3)\}$$

$$\boxed{\therefore \quad x(n) = \{1, 4, 7, 12, \underset{\uparrow}{13}, 8, 11, 4\}}$$

1.3.7 Initial Value Theorem :

Statement : If x (n) is a causal sequence then its initial value is given by,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

Proof : According to the definition of Z-transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(1)}$$

But if x (n) is a causal sequence then limits of summation will be from n = 0 to n = ∞.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{...(2)}$$

Expanding the summation we get,

$$X(Z) = x(0) Z^0 + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + \ldots$$

Applying limits as $Z \to \infty$ we get,

$$\lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) Z^{-1} + \lim_{Z \to \infty} x(2) Z^{-2} + \ldots \text{ (as } Z^0 = 1)$$

$$\therefore \quad \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) \cdot \frac{1}{Z} + \lim_{Z \to \infty} x(2) \frac{1}{Z^2} + \ldots \quad \text{...(3)}$$

Now as $Z \to \infty$ the terms $\frac{1}{Z} = \frac{1}{Z^2}$ etc. will be $\frac{1}{\infty}$ which is zero. Thus Equation (3) becomes,

$$\lim_{Z \to \infty} X(Z) = x(0) \quad \text{...(4)}$$

But x (0) is called as initial value of x (n).

$$\boxed{\therefore \text{ Initial value} = x(0) = \lim_{Z \to \infty} X(Z)}$$

Hence proved.

1.3.8 Final Value Theorem :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ then,

$$x(\infty) = \lim_{Z \to 1} [(Z-1)X(Z)]$$

Proof : According to the definition of Z-transform for causal sequence we have,

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{...(1)}$$

Consider the term $x(n) - x(n-1)$, using Equation (1), Z-transform of this term can be expressed as,

$$Z\{x(n) - x(n-1)\} = \sum_{n=0}^{\infty} [x(n) - x(n-1)] Z^{-n}$$

$$\therefore \quad Z\{x(n)\} - Z\{x(n-1)\} = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n} \quad \text{...(2)}$$

Now we have $Z\{x(n)\} = X(Z)$ and

According to shifting property, $Z\{x(n-1)\} = Z^{-1} X(Z)$

$$\therefore \quad X(Z) - Z^{-1} X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n}$$

$$\therefore \quad X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n} \quad \text{...(3)}$$

Now consider the second summation. Here Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \cdot Z^{1} \cdot Z^{-1} = Z^{-(n-1)} \cdot Z^{-1}$$

$$\therefore \quad X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)} Z^{-1}$$

$$\therefore \quad X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - Z^{-1} \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)}$$

Taking limit as $Z \to 1$ on both sides,

$$\lim_{Z \to 1} X(Z)(1-Z^{-1}) = \lim_{Z \to 1} \sum_{n=0}^{\infty} x(n) Z^{-n} - \lim_{Z \to 1} Z^{-1} \cdot \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)}$$

Remember that the first summation is $Z\{x(n)\}$ and second summation is $Z\{x(n-1)\}$. Now expanding the summations we can write,

$$\therefore \quad \lim_{Z \to 1} X(Z)(1-Z^{-1}) = \lim_{Z \to 1} \Big[(X(0)-Z^{-1}X(-1)) + (X(1)-Z^{-1}X(0)) + (X(2)-Z^{-1}X(1)) + \ldots + [X(\infty)-Z^{-1}X(\infty-1)) \Big] \quad \ldots(4)$$

Here as $\lim_{Z \to 1}$ we get $\lim_{Z \to 1} Z^{-1} = \lim_{Z \to 1} \frac{1}{Z} = 1$ and $X(-1) = 0$

because $x(n) = 0$ for $n = -1$ since it is a causal sequence.

$$\therefore \quad \lim_{Z \to 1} X(Z)(1-Z^{-1}) = [X(0)-0] + [X(1)-X(0)] + \ldots + [X(\infty)-X(\infty-1)]$$

$= X(\infty)$ ... (Here we are adding the difference between two terms and this addition is upto ∞).

$$\boxed{\therefore \; X(\infty) = \lim_{Z \to 1} X(Z)(1-Z^{-1})}$$

Hence proved.

Solved Problems :

Prob. 1 : Find initial and final values of $x(n)$ if Z transform is,

$$X(Z) = \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$$

$$\boxed{\therefore \; x(0) = 0}$$

According final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z-1) \cdot X(Z) = \lim_{Z \to 1} (Z-1) \cdot \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$$

We will obtain roots of denominator term.

$$Z^2 + \frac{1}{6}Z - \frac{1}{6} \Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\frac{1}{6} \pm \sqrt{\frac{1}{36} + \frac{4}{6}}}{2}$$

$$\therefore \quad \text{roots} = \frac{-\frac{1}{6} \pm \sqrt{\frac{1+24}{36}}}{2} = \frac{-\frac{1}{6} \pm \frac{5}{6}}{2}$$

$$\therefore \quad \text{roots} = \frac{1}{3}, -\frac{1}{2}$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2}{\left(Z-\frac{1}{3}\right)\left(Z+\frac{1}{2}\right)}$$

$$\boxed{\therefore \quad x(\infty) = 0}$$

Prob. 2 : Determine the value of signal $x(n)$ at $n = 0$ and $n = \infty$ if $X(Z) = \dfrac{2Z^2 + 0.25}{(Z+0.25)(Z-1)}$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

$$x(0) = \lim_{Z \to \infty} \frac{2Z^2 + 0.25}{(Z+0.25)(Z-1)}$$

Multiplying numerator and denominator by Z^{-2};

$$x(0) = \lim_{Z \to \infty} \frac{2 + 0.25 Z^{-2}}{Z^{-1}(Z+0.25) \cdot Z^{-1}(Z-1)} = \lim_{Z \to \infty} \frac{2 + \frac{0.25}{Z^2}}{\left(1+\frac{0.25}{Z}\right)\left(1-\frac{1}{Z}\right)}$$

$$\boxed{\therefore \quad x(0) = \frac{2}{1} = 2}$$

According to final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z-1) X(Z)$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2Z^2 + 0.25}{(Z+0.25)(Z-1)} = \lim_{Z \to 1} \frac{2Z^2 + 0.25}{Z+0.25} = \frac{2+0.25}{1+0.25}$$

$$\boxed{\therefore \quad x(\infty) = 1.8}$$