Lecture - 18

Inverse Z-Transform

1.1 Introduction :

In this chapter we will discuss, the use of Z transform for analysis of LTI systems. If the signals are in Z domain then the stability of LTI system can be easily determined with the help of pole-zero plot. Similarly in the Z domain, causality of LTI system can be easily determined.

The solution of differential equations becomes simple in the Z domain. Using unilateral Z transform the transient response of zero state response of LTI system can be obtained. After completing analysis of systems in Z domain; it is required to convert the signals into discrete domain. For this inverse Z transform is used. Before studying analysis of system in Z domain; we will discuss different methods of inverse Z transform.

1.2 Inverse Z Transform :

In this section we will discuss how to obtain sequence x(n) from given X(Z). This procedure of obtaining x(n) from its Z-transform X(Z) is called as inverse Z-transform (IZT).

Methods for obtaining IZT :

The different methods for obtaining IZT from X(Z) are as follows :

(1) Inverse Z transform by inspection.

(2) Power series expansion.

(3) Partial fraction expansion.

(4) Residue method.

1.2.1 Inverse Z Transform by Inspection :

This is the simplest method of obtaining inverse Z transform. We have derived the equations for standard Z transform pairs. By using these standard pairs; inverse Z transform can be directly obtained.

For example if we want to obtain inverse Z transform of, $X(Z) = \frac{Z}{Z-3}, \; |Z| > 3$

Then we can use standard Z transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, \quad |Z| > |\alpha|$$

Here $\alpha = 3$. Thus inverse Z of $X(Z)$ is,

$$x(n) = 3^n u(n)$$

1.2.2 Power Series Expansion Method :

This method is also called as *long division method or direct division method.* Generally X(Z) is expressed as power series in 'Z'.

$$\therefore \; X(Z) = a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots\ldots + a_n Z^{-n} \qquad \ldots(1)$$

Equation (1) can be easily obtained by dividing numerator by the denominator of X(Z). Now according to the definition of 'Z' transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(2)$$

If the sequence is causal then the limits of 'n' will be from n = 0 to n = ∞.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad ...(3)$$

Expanding Equation (3) we get,

$$X(Z) = x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + ... + x(n)Z^{-n}$$

$$\text{But} \quad Z^{0} = 1.$$

$$\therefore \quad X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + ... + x(n)Z^{-n} \qquad ...(4)$$

Now by comparing Equations (1) and (4) we can write,

$$x(0) = a_0$$

$$x(1) = a_1$$

$$x(2) = a_2$$

$$\vdots$$

$$x(n) = a_n$$

Thus the general expression of discrete time causal sequence x (n) is,

$$\boxed{x(n) = a_n \quad n \geq 0} \qquad ...(5)$$

Solved Problems on IZT :

Prob. 1 : Determine the inverse Z-transform of

$$X(Z) = \frac{Z^2}{(0.5 - 1.5Z + Z^2)}$$ for ROC $|Z| < 0.5$ using long division method.

Soln. : The given ROC is $|Z| < 0.5$. That means ROC is interior part of circle having radius r = 0.5. Now while obtaining the long divisions remember the following instructions.

In case of anti-causal sequence :

(i) Carryout the long division by writing the polynomials in the reverse order that means starting with the most negative term on the left.

(ii) In case of causal sequence, carryout the long division without changing order of the polynomial. That means starting with the most positive term on the left.

The given expression is,

$$X(Z) = \frac{Z^2}{0.5 - 1.5Z + Z^2} \qquad ...(1)$$

While carrying out the long division; always convert the given expression in the simplest form. That means as far as possible it should be in the form of 1 divided by some polynomial. Thus dividing numerator and denominator by Z^{-2} we get,

$$X(Z) = \frac{1}{0.5Z^{-2} - 1.5Z^{-1} + 1} \qquad ...(2)$$

This is non-causal sequence. So numerator polynomial should start from maximum negative value of Z. Observe Equation (2). Here the polynomial is in the proper order. So we do not have to change the order of denominator polynomial. Now we will perform the long division as follows :

$$
\begin{array}{r|l}
 & 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 \\
\hline
0.5Z^{-2} - 1.5Z^{-1} + 1 & 1 \\
 & 1 - 3Z + 2Z^2 \\
 & - \quad + \quad - \\
\hline
 & 0 + 3Z - 2Z^2 \\
 & 3Z - 9Z^2 + 6Z^3 \\
 & - \quad + \quad - \\
\hline
 & 7Z^2 - 6Z^3 \\
 & 7Z^2 - 21Z^3 + 14Z^4 \\
 & - \quad + \quad - \\
\hline
 & 15Z^3 - 14Z^4 \\
 & 15Z^3 - 45Z^4 + 30Z^5 \\
 & - \quad + \quad - \\
\hline
 & 31Z^4 - 30Z^5
\end{array}
$$

Thus X (Z) can be approximately written as,

$$X(Z) = 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 + ... \qquad ...(3)$$

Now according to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

For anti-causal sequence; the limits of summation will be from n = − 1 to n = − ∞.

$$\therefore \quad X(Z) = \sum_{n=-1}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + x(-4)Z^4 + x(-5)Z^5 + ... \qquad ...(4)$$

Comparing Equations (3) and (4) we have,

$x(0) = 0 \qquad x(-3) = 6$

$x(-1) = 0 \qquad x(-4) = 14$

$x(-2) = 2 \qquad x(-5) = 30$ ....

Thus sequence x (n) is,

$$x(n) = \{ \ldots x(-5), x(-4), x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)} \}$$

$$\therefore x(n) = \{ \ldots 30, 14, 6, 2, 0, \underset{\uparrow}{0} \}$$

Prob. 2 :

Find inverse Z of $X(Z) = \dfrac{1 - \frac{1}{2} Z^{-1}}{1 - \frac{1}{4} Z^{-2}}$ $\quad |Z| > 1/2$

Soln. :

The given ROC is $|Z| > 1/2$. That means it is causal sequence. Now for the causal sequence the denominator polynomial should have the maximum power of Z on its left. The given expression has the denominator in the proper form. So we don't have to rearrange the expression. Now we will perform the long division as follows :

$$\begin{array}{r|l}
 & 1 - \frac{1}{2} Z^{-1} + \frac{1}{4} Z^{-2} - \frac{1}{8} Z^{-3} \ldots\ldots \\
\hline
1 - \frac{1}{4} Z^{-2} & 1 - \frac{1}{2} Z^{-1} \\
 & 1 - \frac{1}{4} Z^{-2} \\
 & - \quad + \\
\hline
 & -\frac{1}{2} Z^{-1} + \frac{1}{4} Z^{-2} \\
 & -\frac{1}{2} Z^{-1} + \frac{1}{8} Z^{-3} \\
 & + \quad - \\
\hline
 & \frac{1}{4} Z^{-2} - \frac{1}{8} Z^{-3} \\
 & \frac{1}{4} Z^{-2} - \frac{1}{16} Z^{-4} \\
 & - \quad + \\
\hline
 & -\frac{1}{8} Z^{-3} + \frac{1}{16} Z^{-4} \\
 & -\frac{1}{8} Z^{-3} + \frac{1}{32} Z^{-5} \\
 & + \quad - \\
\hline
 & \frac{1}{16} Z^{-4} - \frac{1}{32} Z^{-5} \ldots\ldots
\end{array}$$

Thus the expression X (Z) can be approximately written as,

$$X(Z) = 1 - \frac{1}{2} Z^{-1} + \frac{1}{4} Z^{-2} - \frac{1}{8} Z^{-3} + \ldots\ldots \qquad \ldots(1)$$

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}.$$

The given sequence is a causal sequence. For causal sequence the limits of summation will be from n = 0 to n = ∞.

$$\therefore \ X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots\ldots$$

$$\therefore \quad X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots.. \qquad (2)$$

Comparing Equations (1) and (2) we get,

$$x(0) = 1 \qquad x(2) = \frac{1}{4}$$

$$x(1) = -\frac{1}{2} \qquad x(3) = -\frac{1}{8}$$

Thus the sequence x (n) is written as,

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots..\}$$

$$\therefore \ x(n) = \left\{ \underset{\uparrow}{1}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \ldots\ldots \right\} \qquad \ldots(3)$$

This is the required sequence x(n). Now Equation (3) can also be written as,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Prob. 3 : Find the inverse Z of :

$$X(Z) = \frac{Z}{Z-1} \qquad |Z| > 1$$

Soln. :

Since the given ROC is $|Z| > 1$; it is causal sequence. We know that for the causal sequence the denominator polynomial should have maximum power of Z on its left. Thus the given expression is in the proper form.

Note : We can also convert $X(Z) = \frac{Z}{Z-1}$ into negative powers of Z by multiplying and dividing by Z^{-1}. Thus $X(Z) = \frac{1}{1-Z^{-1}}$ and we can solve using the same method.

Now performing the long division we get,

$$
\begin{array}{r|l}
 & 1+Z^{-1}+Z^{-2}+Z^{-3}+\ldots\ldots \\
\hline
Z-1 & Z \\
 & Z-1 \\
 & -\quad + \\
\hline
 & 1 \\
 & 1-Z^{-1} \\
 & -\quad + \\
\hline
 & Z^{-1} \\
 & Z^{-1}-Z^{-2} \\
 & -\quad + \\
\hline
 & Z^{-2} \\
 & Z^{-2}-Z^{-3} \\
 & -\quad + \\
\hline
 & Z^{-3}
\end{array}
$$

Thus X (Z) can be approximately written as,

$$X(Z) = 1+Z^{-1}+Z^{-2}+Z^{-3}+\ldots\ldots \qquad \ldots(1)$$

Now the Z-transform of causal sequence is given by,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Expanding the summation,

$$X(Z) = x(0)+x(1)Z^{-1}+x(2)Z^{-2}+x(3)Z^{-3}+\ldots\ldots \qquad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots\ldots\}$$

$$\therefore \quad x(n) = \{ \underset{\uparrow}{1}, 1, 1, 1, \ldots\ldots\ldots \}$$

$$\therefore \quad x(n) = u(n)$$

$$\therefore \; Z^{-1}\left\{\frac{Z}{Z-1}\right\} = u(n)$$

This is standard Z-transform pair.

Prob. 4 : Obtain inverse Z of :

$$X(Z) = \frac{Z}{Z-a} \quad \text{if} \quad |Z| < |a|$$

Soln. : Given ROC is $|Z| < |a|$. So the sequence is anti-causal. That means the numerator polynomial should have minimum power of Z on its left. Thus we can write $X(Z)$ as :

$$X(Z) = \frac{Z}{-a+Z} \quad \text{if} \quad |Z| < |a| \qquad \text{...(1)}$$

Now we will perform the long division as follows :

$$-a^{-1}Z - a^{-2}Z^2 - a^{-3}Z^3 \ldots\ldots$$

$$-a+Z \;\big)\; Z$$

$$Z - a^{-1}Z^2$$

$$- \quad +$$

$$a^{-1}Z^2$$

$$a^{-1}Z^2 - a^{-2}Z^3$$

$$- \quad +$$

$$a^{-2}Z^3$$

$$a^{-2}Z^3 - a^{-3}Z^4$$

$$- \quad +$$

$$a^{-3}Z^4$$

$$\vdots$$

Thus $X(Z)$ can be approximately written as,

$$X(Z) = -a^{-1}Z^1 - a^{-2}Z^2 - a^{-3}Z^3 - \ldots\ldots \qquad \text{...(2)}$$

According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

For anti-causal sequence, the range of n is from $n = -1$ to $n = -\infty$

$$\therefore \; X(Z) = \sum_{n=-1}^{-\infty} x(n)Z^{-n}$$

Expanding the summation,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + \ldots\ldots \qquad \text{...(3)}$$

Comparing Equations (2) and (3) we get,

$$x(-1) = -a^{-1}$$

$$x(-2) = -a^{-2}$$

$$x(-3) = -a^{-3} \ldots\ldots$$

Thus sequence x(n) can be written as,

$$x(n) = \{ ..... -a^{-3}, -a^{-2}, -a^{-1}, \underset{\uparrow}{0} \} \quad ...(4)$$

Equation (4) can be written as,

$$x(n) = -a^n u(-n-1)$$

This is the standard Z-transform pair.

Prob. 5 : Find inverse Z-transform :

$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a|$

Soln. : The given expression is,

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a| \quad ...(1)$$

Now the power series expansion for $\log(1 + x)$ with $|x| < 1$ is given by,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - .... = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad ...(2)$$

Let $x = aZ^{-1}$ Thus Equation (2) becomes,

$$X(Z) = \log(1 + aZ^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} [aZ^{-1}]^n$$

$$\therefore \quad X(Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n \cdot Z^{-n} \quad ...(3)$$

Now according to the definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(4)$$

But the given sequence is causal, since $|Z| > |a|$. And we will assume that the sequence is present from $n = 1$ to $n = \infty$. Thus changing the limits of summation in Equation (4) we get,

$$X(Z) = \sum_{n=1}^{\infty} x(n) Z^{-n} \quad ...(5)$$

Now comparing Equations (3) and (5), we get,

$$x(n) = \frac{(-1)^{n+1}}{n} a^n \quad \text{if } n \geq 1$$

This is the required IZT.

Limitations of power series expansion method :

If 'n' is large then the long division will become tedious. The method is used only when the resulting pattern is simple enough or if we want the values of only first few samples.

1.2.3 Partial Fraction Expansion (P.F.E.) Method :

This is possible for the Z transforms which are rational in nature, that means they are expressed as the ratio of two polynomials.

That means

$$X(Z) = \frac{N(Z)}{D(Z)} = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}}{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + ... a_N Z^{-N}} \quad ...(1)$$

Here $N(Z)$ = Numerator polynomial

$D(Z)$ = Denominator polynomial

$b_0, b_1 .... b_M$ = Coefficients of numerator

$a_0, a_1 ... a_N$ = Coefficients of denominator

M = Degree of numerator

N = Degree of denominator

Steps to follow in partial fraction expansion :

(1) Check whether the given function is in proper form or not. The function is said to be in proper form when following conditions are satisfied.

(i) The coefficient a_0 in Equation (1) should be equal to 1 ($a_0 = 1$). If $a_0 \neq 1$ then the polynomials are adjusted accordingly.

(ii) In Equation (1) $a_n \neq 1$ and the degree of numerator should be less than the degree of denominator ($M < N$). If this condition is not satisfied then the long division is carried out to make $M < N$.

Thus the first step is to obtain the function in the proper form.

(2) Multiply the numerator and denominator by Z^N. That means convert the function in terms of positive powers of Z.

(3) Obtain the equation $\frac{X(Z)}{Z}$.

(4) Factorise the denominator and obtain the roots. Then the denominator will be in the form $(Z - P_1)(Z - P_2) .... (Z - P_N)$. Here $P_1, P_2 .... P_N$ are called as poles.

(5) Write down the equation in partial fraction expansion form as follows.

$$\frac{X(Z)}{Z} = \frac{A_1}{Z - P_1} + \frac{A_2}{Z - P_2} + ... + \frac{A_N}{Z - P_N} \quad ...(2)$$

Here $A_1, A_2, ... A_N$ are coefficients. The coefficient A_K is calculated as,

$$A_K = (Z - P_K) \cdot \frac{X(Z)}{Z} \bigg|_{Z = P_K} \quad ...(3)$$

Equation (3) indicates that the calculation $(Z - P_K) \cdot \frac{X(Z)}{Z}$ is performed by putting $Z = P_K$.

Here $P_K = P_1, P_2, ...$

(6) By calculating values of $A_1, A_2, ... A_N$, transfer 'Z' to the R.H.S. of Equation (2). Now we standard Z transform pairs to obtain inverse Z transform.

$$a^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \textbf{ROC } |Z| > |a| \qquad ...(4)$$

and $$-a^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a}, \qquad \text{ROC } |Z| < |a| \qquad ...(5)$$

Thus depending on given ROC and by using Equations (4) and (5), the sequence x (n) is obtained as follows :

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \qquad \text{if ROC : } |Z| > |P_K|$$

That means causal sequence ...(6)

$$\text{and } x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1) \qquad \text{if ROC : } |Z| < |P_K|$$

That means anticausal sequence. ...(7)

Solved Problems :

While solving the numericals we will consider different cases depending on the nature of poles.

Case I : When there are simple poles.

Prob. 1 : Find inverse Z of the following :

$$X(Z) = \frac{1 - 1/2\,Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}}, \; |Z| > \frac{1}{2}$$

Soln. :

Step I : We will check whether the given function is in proper form or not.

Here $a_0 = 1$, M = Degree of numerator = 1, N = Degree of denominator = 2

$\therefore$ M < N and $a_0 = 1$

Hence the given function is in proper form.

Note : While checking the degree of numerator and denominator; do not consider the sign.

Step II : We will convert the given function in terms of positive powers of Z. The given function is,

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}} \qquad ...(1)$$

To obtain positive powers of Z; multiply numerators and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2\left[1 - 1/2\, Z^{-1}\right]}{Z^2\left[1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right]}$$

$$\therefore \quad X(Z) = \frac{Z^2 - \frac{1}{2}Z}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \qquad ...(2)$$

Step III : We will convert Equation (2) in the form of $\frac{X(Z)}{Z}$. To do this, take Z common from the numerator.

$$\therefore \quad X(Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{Z^2 + \frac{3}{4}Z + \frac{1}{8}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z - 1/2}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \qquad ...(3)$$

Step IV : We will obtain the roots (poles) of the denominator. To obtain poles P_1 and P_2 use following equation,

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad ...(4)$$

The equation of denominator is, $Z^2 + \frac{3}{4}Z + \frac{1}{8}$

Thus here $a = 1$, $b = \frac{3}{4}$ and $c = \frac{1}{8}$

Putting these values in Equation (4) we get,

$$P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - 4\left(1 \times \frac{1}{8}\right)}}{(2 \times 1)} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}}{2}$$

$$\therefore \quad P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9-8}{16}}}{2} = \frac{-\frac{3}{4} \pm \frac{1}{4}}{2} = -\frac{3}{8} \pm \frac{1}{8}$$

$$\therefore \quad P_1 = -\frac{3}{8} - \frac{1}{8} = -\frac{4}{8} = -\frac{1}{2}$$

$$\text{and} \quad P_2 = -\frac{3}{8} + \frac{1}{8} = \frac{-3+1}{8} = -\frac{1}{4}$$

Step V : We will write down Equation (3) in the partial fraction expansion form.

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z-\frac{1}{2}}{(Z-P_1)(Z-P_2)} = \frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)} \qquad \text{...(5)}$$

Thus in terms of coefficients A_1 and A_2 we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z+\frac{1}{2}\right)} + \frac{A_2}{\left(Z+\frac{1}{4}\right)} \qquad \text{...(6)}$$

Step VI : We will obtain values of A_1 and A_2. The general expression to calculate A_K is,

$$A_K = (Z-P_K)\frac{X(Z)}{Z}\Bigg|_{Z=P_K}$$

$$\therefore \quad A_1 = (Z-P_1)\frac{X(Z)}{Z}\Bigg|_{Z=P_1}$$

Here $P_1 = -1/2$

$$\therefore \quad A_1 = \left(Z+\frac{1}{2}\right)\cdot\frac{X(Z)}{Z}\Bigg|_{Z=-\frac{1}{2}}$$

Putting the value of $\frac{X(Z)}{Z}$ from Equation (5) we get,

$$A_1 = \left(Z+\frac{1}{2}\right)\cdot\frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)}\Bigg|_{Z=-\frac{1}{2}}$$

$$\therefore \quad A_1 = \frac{Z-\frac{1}{2}}{Z+\frac{1}{4}}\Bigg|_{Z=-\frac{1}{2}} = \frac{-\frac{1}{2}-\frac{1}{2}}{-\frac{1}{2}+\frac{1}{4}} = \frac{-1}{\frac{-2+1}{4}} = \frac{-1}{-\frac{1}{4}} = 4$$

Similarly, $$A_2 = (Z-P_2)\frac{X(Z)}{Z}\Bigg|_{Z=P_2}$$

Here $P_2 = -\frac{1}{4}$

$$\therefore \quad A_2 = \left(Z+\frac{1}{4}\right)\cdot\frac{Z-\frac{1}{2}}{\left(Z+\frac{1}{2}\right)\left(Z+\frac{1}{4}\right)}\Bigg|_{Z=-\frac{1}{4}}$$

$$\therefore \quad A_2 = \left.\frac{Z-\frac{1}{2}}{Z+\frac{1}{2}}\right|_{Z=-\frac{1}{4}} = \frac{-\frac{1}{4}-\frac{1}{2}}{-\frac{1}{4}+\frac{1}{2}} = \frac{-\frac{3}{4}}{1/4} = -3$$

Thus $A_1 = 4$ and $A_2 = -3$.

Step VII : Putting values of A_1 and A_2 in Equation (6) we get,

$$\frac{X(Z)}{Z} = \frac{4}{Z+\frac{1}{2}} - \frac{3}{Z+\frac{1}{4}}$$

Now we will obtain equation of X (Z) by transferring Z to the R.H.S.

$$\therefore \quad X(Z) = \frac{4Z}{Z+\frac{1}{2}} - \frac{3Z}{Z+\frac{1}{4}} \qquad ...(7)$$

Step VIII : The first term of Equation (7) is $4 \cdot \frac{Z}{Z+\frac{1}{2}}$. This term is causal when $|Z| > \frac{1}{2}$.

Note : The standard form is $\frac{Z}{Z+a}$. It is causal when $|Z| > |a|$.

Now the given ROC is $|Z| > \frac{1}{2}$. So first term is causal.

Now recall the standard equation,

$$IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \text{ when } |Z| > |P_K| \qquad ..(8)$$

Here $P_K = P_1 = -\frac{1}{2}$

$$\therefore \quad IZT\left\{\frac{Z}{Z+\frac{1}{2}}\right\} = \left(-\frac{1}{2}\right)^n u(n) \qquad ...(9)$$

Now consider second term. It is $\frac{Z}{Z+\frac{1}{4}}$. This term is causal when $|Z| > \frac{1}{4}$. Given ROC is $|Z| > \frac{1}{2}$. That means by default it is $> \frac{1}{4}$. Thus the second term is also causal. Thus using Equation (8).

$$IZT\left\{\frac{Z}{Z+\frac{1}{4}}\right\} = \left(-\frac{1}{4}\right)^n u(n) \qquad ...(10)$$

... since $P_2 = -\frac{1}{4}$

Thus from Equations (9) and (10) we can write IZT of given function as,

$$x(n) = 4\left(-\frac{1}{2}\right)^n u(n) - 3\left(-\frac{1}{4}\right)^n u(n)$$

Prob. 2 : Obtain inverse Z of the following :

$$X(Z) = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{1}{4}Z^{-2}}, \qquad |Z| > \frac{1}{2}$$

Soln. :

Step I : First we will check whether given function is in the proper form or not.

Here $a_0 = 1$, $M = 1$ and $N = 2$.

Since $M < N$; the function is in the proper form.

Step II : The given function is,

$$X(Z) = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{1}{4}Z^{-2}} \qquad \text{...(1)}$$

Multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2-\frac{1}{2}Z}{Z^2-\frac{1}{4}} \qquad \text{...(2)}$$

Step III : We will obtain the equation of $\frac{X(Z)}{Z}$ as follows :

$$X(Z) = \frac{Z\left(Z-\frac{1}{2}\right)}{Z^2-\frac{1}{4}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{\left(Z-\frac{1}{2}\right)}{Z^2-\frac{1}{4}} \qquad \text{...(3)}$$

Step IV : We will obtain the roots of denominator $Z^2-\frac{1}{4}$.

$$\text{Here } Z^2-\frac{1}{4} = Z^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \text{ Roots are } \left(Z-\frac{1}{2}\right) \text{ and } \left(Z+\frac{1}{2}\right)$$

Thus the poles are,

$$\left(Z-\frac{1}{2}\right) = (Z-P_1) \Rightarrow P_1 = \frac{1}{2}$$

$$\text{and} \quad \left(Z+\frac{1}{2}\right) = (Z-P_2) \Rightarrow P_2 = -\frac{1}{2}$$

Step V : Thus Equation (3) can be written as,

$$\frac{X(Z)}{Z} = \frac{Z-\frac{1}{2}}{\left(Z-\frac{1}{2}\right)\left(Z+\frac{1}{2}\right)} = \frac{1}{Z+\frac{1}{2}} \qquad ...(4)$$

Step VI : The partial fraction expansion form is,

$\frac{X(Z)}{Z} = \frac{A_1}{Z-P_1}$. Thus $A_1 = 1$. We do not have to calculate this value.

Step VII : Now from Equation (4);

$$X(Z) = \frac{Z}{Z+\frac{1}{2}} \qquad ...(5)$$

This function is causal if $|Z| > \frac{1}{2}$. The given ROC is $|Z| > \frac{1}{2}$.

Thus given function is causal. Now we will use the standard formula of causal sequence.

$$IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Prob. 3 : Determine IZT of the following :

$$X(Z) = \frac{3}{Z-\frac{1}{4}-\frac{1}{8}Z^{-1}}\ ;\ x(n) \text{ is causal.}$$

Soln. :

Step I : Here $a_0 = 1$, $M = 0$ and $N = 1$.

Since $M < N$ the function is in the proper form.

Step II : The given equation is,

$$X(Z) = \frac{3}{Z-\frac{1}{4}-\frac{1}{8}Z^{-1}} \qquad ...(1)$$

To convert it into positive terms of Z, multiply and divide by Z.

$$\therefore \quad X(Z) = \frac{3Z}{Z\left(Z-\frac{1}{4}-\frac{1}{8}Z^{-1}\right)}$$

$$\therefore \quad X(Z) = \frac{3Z}{Z^2-\frac{1}{4}Z-\frac{1}{8}} \qquad ...(2)$$

Step III : The equation of $\frac{X(Z)}{Z}$ is,

$$\frac{X(Z)}{Z} = \frac{3}{Z^2 - \frac{1}{4}Z - \frac{1}{8}} \quad \text{...(3)}$$

Step IV : The denominator equation is, $Z^2 - \frac{1}{4}Z - \frac{1}{8}$.

Here $a = 1$, $b = -\frac{1}{4}$ and $c = -\frac{1}{8}$

Thus poles $P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore \quad P_1, P_2 = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}}}{2} = \frac{\frac{1}{4} \pm \sqrt{\frac{9}{16}}}{2} = \frac{\frac{1}{4} \pm \frac{3}{4}}{2}$$

$$\therefore \quad P_1 = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2} \text{ and } P_2 = \frac{1}{8} - \frac{3}{8} = \frac{-2}{8} = -\frac{1}{4}.$$

Thus $P_1 = \frac{1}{2}$ and $P_2 = -\frac{1}{4}$

Step V : Thus Equation (3) can be written as,

$$\frac{X(Z)}{Z} = \frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)} \quad \text{...(4)}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{A_1}{\left(Z - \frac{1}{2}\right)} + \frac{A_2}{\left(Z + \frac{1}{4}\right)} \quad \text{...(5)}$$

Step VI : Now we will calculate values of A_1 and A_2.

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\bigg|_{Z = P_1}$$

$$\therefore \quad A_1 = \left(Z - \frac{1}{2}\right)\frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\Bigg|_{Z = \frac{1}{2}}$$

$$\therefore \quad A_1 = \frac{3}{\frac{1}{2} + \frac{1}{4}} = \frac{3}{3/4} = 4$$

$$\text{and } A_2 = (Z - P_2)\frac{X(Z)}{Z}\bigg|_{Z = P_2}$$

$$\therefore \quad A_2 = \left(Z + \frac{1}{4}\right)\frac{3}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\Bigg|_{Z = -\frac{1}{4}}$$

$$\therefore \quad A_2 = \frac{3}{-\frac{1}{4}-\frac{1}{2}} = \frac{3}{-3/4} = -4$$

Step VII : Putting values of A_1 and A_2 in Equation (5),

$$\frac{X(Z)}{Z} = \frac{4}{Z-\frac{1}{2}} - \frac{4}{Z+\frac{1}{4}}$$

$$\therefore \quad X(Z) = 4\frac{Z}{Z-\frac{1}{2}} - 4\frac{Z}{Z+\frac{1}{4}} \quad \text{...(6)}$$

Step VIII : The given function is stable ; thus we will use the formula

$$\text{IZT}\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore \quad x(n) = 4(P_1)^n u(n) - 4(P_2)^n u(n)$$

$$\therefore \quad x(n) = 4\left(\frac{1}{2}\right)^n u(n) - 4\left(-\frac{1}{4}\right)^n u(n)$$

$$\therefore \quad x(n) = 4\left[\left(\frac{1}{2}\right)^n - \left(-\frac{1}{4}\right)^n\right] u(n)$$

Prob. 4 : Determine all possible sequences $x(n)$ associated with Z transform

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})}$$

Soln. : The given function is,

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})} = \frac{5Z^{-1}}{3-Z^{-1}-6Z^{-1}-2Z^{-2}}$$

$$\therefore \quad X(Z) = \frac{5Z^{-1}}{3-7Z^{-1}-2Z^{-2}} \quad \text{...(1)}$$

Step I : Here M = 1 and N = 2 ; M < N but $a_0 = 3$. We want $a_0 = 1$.

Thus to convert the given function into proper form; multiply numerator and denominator by $\frac{1}{3}$.

$$\therefore \quad X(Z) = \frac{\frac{5}{3}Z^{-1}}{1-\frac{7}{3}Z^{-1}-\frac{2}{3}Z^{-2}}$$

Step II : Multiply numerator and denominator by Z^2

$$\therefore \quad X(Z) = \frac{\frac{5}{3}Z}{Z^2 - \frac{7}{3}Z - \frac{2}{3}}$$

Step III : The equation of $\frac{X(Z)}{Z}$ is,

$$\frac{X(Z)}{Z} = \frac{\frac{5}{3}}{Z^2 - \frac{7}{3}Z - \frac{2}{3}} \qquad ...(2)$$

Step IV : We will obtain roots of denominator

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{7}{3} \pm \sqrt{\frac{49}{9} - \frac{8}{3}}}{2} = \frac{7}{6} \pm \frac{5}{6}$$

$$\therefore \quad P_1 = \frac{7}{6} + \frac{5}{6} = 2 \text{ and } P_2 = \frac{7}{6} - \frac{5}{6} = \frac{1}{3}.$$

Step V : Thus Equation (2) can be written as,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{\frac{5}{3}}{(Z-2)\left(Z - \frac{1}{3}\right)} \qquad ...(3)$$

In the form of partial fraction expansion we can write,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{A_1}{Z-2} + \frac{A_2}{Z - \frac{1}{3}} \qquad ...(4)$$

Step VI : Now,

$$A_1 = (Z - P_1) \cdot \frac{X(Z)}{Z} \Bigg|_{Z = P_1}$$

$$\therefore \quad A_1 = (Z-2) \frac{5/3}{(Z-2)(Z-\frac{1}{3})} \Bigg|_{Z=2}$$

$$= \frac{5/3}{2 - 1/3} = \frac{5/3}{5/3} = \mathbf{1}$$

and

$$A_2 = (Z - \frac{1}{3}) \cdot \frac{5/3}{(Z-2)(Z-\frac{1}{3})} \Bigg|_{Z = \frac{1}{3}}$$

$$= \frac{5/3}{1/3 - 2} = \frac{5/3}{-5/3} = -1$$

Step VII : Putting values of A_1 and A_2 in Equation (4) we get,

$$\frac{X(Z)}{Z} = \frac{1}{Z-2} - \frac{1}{Z - \frac{1}{3}}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-2} - \frac{Z}{Z-1/3} \quad ...(5)$$

Step VIII : To obtain all possible sequences we will consider the following ROC.

(i) When ROC is $|Z| > 2$:

The first term of Equation (5) will be causal if $|Z| > 2$ and the second term will be causal if $|Z| > 1/3$. Now if ROC is $|Z| > 2$ then by default it is greater than 1/3. Thus for this ROC both the terms are causal.

$$\therefore \quad x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n) \quad ...(6)$$

$$\therefore \quad x(n) = (2)^n u(n) - (1/3)^n u(n)$$

(ii) When ROC is $2 > |Z| > 1/3$:

For the first term to be causal, the required ROC is $|Z| > 2$. But the given ROC is $2 > |Z|$ that means $|Z| < 2$. Thus first term is anticausal. We will use the standard formula of IZT for anticausal sequence

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1) \quad ...(7)$$

Thus IZT of first term is,

$$x_1(n) = -(2)^n u(-n-1)$$

Now for the second term to be causal the required ROC is $|Z| > \frac{1}{3}$ and the given ROC is $|Z| > \frac{1}{3}$. So second term is causal. Thus using Equation (6) IZT of second term is,

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

Thus IZT of total function is,

$$x(n) = x_1(n) + x_2(n)$$

$$\therefore \quad x(n) = -(2)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$$

(iii) When ROC is $|Z| < \frac{1}{3}$:

We know that the required ROC for first and second term to be causal are $|Z| > 2$ and $|Z| > \frac{1}{3}$ respectively. But the given ROC is $|Z| < \frac{1}{3}$. Thus both the terms will be anticausal. Now using Equation (7),

$$x(n) = -(2)^n u(-n-1) - \left(\frac{1}{3}\right)^n u(-n-1)$$

Prob. 5 : Using partial fraction expansion method find inverse Z transform of following system given by,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)}$$

for ROC being $|Z| > 3$, $2 < |Z| < 3$, $|Z| < 2$.

Soln. : The given function is,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)} \quad ...(1)$$

Step I :

$$\frac{X(Z)}{Z} = \frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)} \quad ...(2)$$

Equation (2) is in the proper form. The roots of denominator are already given; so the poles are $P_1 = 1$, $P_2 = 2$ and $P_3 = 3$.

Step II : In the P.F.E. form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-2} + \frac{A_3}{Z-3} \quad ...(3)$$

We will obtain values of A_1, A_2 and A_3 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\Big|_{Z = P_1}$$

$$\therefore \quad A_1 = (Z-1)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\Big|_{Z = 1}$$

$$\therefore \quad A_1 = \frac{1 - 4 + 5}{(1-2)(1-3)} = \frac{2}{(-1)(-2)}$$

$$\therefore A_1 = +1$$

$$A_2 = (Z - P_2)\cdot\frac{X(Z)}{Z}\Big|_{Z = P_2}$$

$$\therefore \quad A_2 = (Z-2)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\Big|_{Z = 2}$$

$$\therefore \quad A_2 = \frac{4 - 8 + 5}{(2-1)(2-3)} = \frac{1}{(1)(-1)}$$

$$\therefore A_2 = -1$$

$$\text{And} \quad A_3 = (Z-P_3)\frac{X(Z)}{Z}\Bigg|_{Z=P_3}$$

$$\therefore \quad A_3 = (Z-3)\cdot\frac{Z^2-4Z+5}{(Z-1)(Z-2)(Z-3)}\Bigg|_{Z=3}$$

$$\therefore \quad A_3 = \frac{9-12+5}{(3-1)(3-2)} = \frac{2}{(2)(1)}$$

$$\therefore \quad A_3 = 1$$

Step III : Putting values of A_1, A_2 and A_3 in Equation (3) we get,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{Z-1} - \frac{1}{Z-2} + \frac{1}{Z-3}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} - \frac{Z}{Z-2} + \frac{Z}{Z-3} \qquad \text{...(4)}$$

Step IV : We will obtain x (n) for different ROC conditions as follows :

i) **For ROC : | Z | > 3 :**

Since ROC is | Z | > 3 then all three terms in Equation (4) are causal.

$$\therefore \quad x(n) = (1)^n u(n) - (2)^n u(n) + (3)^n u(n)$$

ii) **For ROC : 2 < | Z | < 3 :**

That means | Z | > 2 and | Z | < 3

For this ROC; first term is causal, second term is causal and the third term is anticausal.

$$\therefore \quad x(n) = (1)^n u(n) - (2)^n u(n) - (3)^n u(-n-1)$$

iii) **For ROC : | Z | < 2 :**

In this case all terms of Equation (4), will be anti-causal.

$$\therefore \quad x(n) = -(1)^n u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1)$$

Prob. 6 : Determine inverse Z-transform of

$$X(Z) = \frac{1}{1-1.5\,Z^{-1}+0.5\,Z^{-2}}$$

if (i) ROC : | Z | > 1

(ii) ROC : | Z | < 0.5

(iii) ROC : 0.5 < | Z | < 1.0

Soln. :

Step I : Here $a_0 = 1$, M = 0, N = 2

Thus the given function is in the proper form.

Step II : To convert X(Z) into positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore \; X(Z) = \frac{Z^2}{Z^2 - 1.5\,Z + 0.5} \quad \text{...(1)}$$

Step III : We will obtain the expression $\frac{X(Z)}{Z}$

$$\therefore \; \frac{X(Z)}{Z} = \frac{Z}{Z^2 - 1.5\,Z + 0.5} \quad \text{...(2)}$$

Step IV : We will obtain the roots of denominator $Z^2 - 1.5\,Z + 0.5$

Here a = 1, b = – 1.5 and c = 0.5

$$\therefore \; P_{1,}\,P_2 = \frac{1.5 \pm \sqrt{(1.5)^2 - 4 \times 0.5}}{2}$$

$$\therefore \; P_1 = 1 \text{ and } P_2 = 0.5$$

Step V : In PFE form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-0.5} \quad \text{...(3)}$$

Step VI : We will calculate the values of A_1 and A_2 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\cdot\Big|_{Z = P_1}$$

$$\therefore \; A_1 = (Z-1)\cdot\frac{Z}{(Z-1)(Z-0.5)}\cdot\Big|_{Z=1}$$

$$\therefore \; A_1 = \frac{1}{1-0.5} = \frac{1}{0.5}$$

$$\boxed{\therefore \; A_1 = 2}$$

and

$$A_2 = (Z - P_2)\frac{X(Z)}{Z}\cdot\Big|_{Z = P_2}$$

$$\therefore \; A_2 = (Z-0.5)\frac{Z}{(Z-1)(Z-0.5)}\cdot\Big|_{Z=0.5}$$

$$\therefore \; A_2 = \frac{0.5}{0.5-1} = \frac{0.5}{-0.5}$$

$$\boxed{\therefore \; A_2 = -1}$$

Step VII : Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{2}{Z-1} - \frac{1}{Z-0.5}$$

$$\therefore \quad X(Z) = \frac{2Z}{Z-1} - \frac{Z}{Z-0.5} \qquad \text{...(4)}$$

Step VIII : We will obtain x (n) according to given ROC conditions.

(i) ROC : | Z | > 1 :

It means that | Z | > 0.5 also. So both the terms of Equation (4) are causal. For the causal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z-P_k}\right\} = (P_k)^n u(n)$$

$$\therefore \; x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

(ii) ROC : | Z | < 0.5 :

For this ROC both the terms of Equation (4) become anticausal. And for anticausal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z-P_k}\right\} = -(P_k)^n u(-n-1)$$

$$\therefore \; x(n) = 2[-(1)^n u(-n-1)] - [-(0.5)^n u(-n-1)]$$

(iii) ROC : 0.5 < | Z | < 1 :

For this ROC ; the first term is anticausal and second term is causal.

$$\therefore \; x(n) = 2[-(1)^n u(-n-1)] - (0.5)^n u(n)$$