# Lecture - 2

**What is Discrete Fourier Transform (DFT)?**

According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{...(1)}$$

We know that $X(\omega)$ is fourier transform of discrete time signal $x(n)$. The range of '$\omega$' is from 0 to $2\pi$ or $-\pi$ to $\pi$. Thus it is not possible to compute $X(\omega)$ on digital computer. Because in Equation (1) the range of summation is from $-\infty$ to $+\infty$. But if we make this range finite then it is possible to do these calculations on digital computer.

When a fourier transform is calculated only at discrete points then it is called as discrete fourier transform (DFT).

## 1.1.1 Sampling the F.T. :

If we have aperiodic time domain signal then discrete time fourier transform (DTFT) is obtained. But DTFT is continuous in nature and its range is from $-\infty$ to $+\infty$. Then a finite range sequence is obtained by extracting a particular portion from such infinite sequence.

Now $X(\omega)$ is a continuous signal. A discrete signal is obtained by sampling $X(\omega)$. A particular sequence which is extracted from infinite sequence is called as windowed sequence. A windowed signal is considered as periodic signal. We can obtain periodic extension of this signal. This periodic extension in frequency domain is called as Discrete Fourier Transform (DFT). From this original sequence, x(n) is obtained by performing inverse process which is Inverse Discrete Fourier Transform (IDFT).

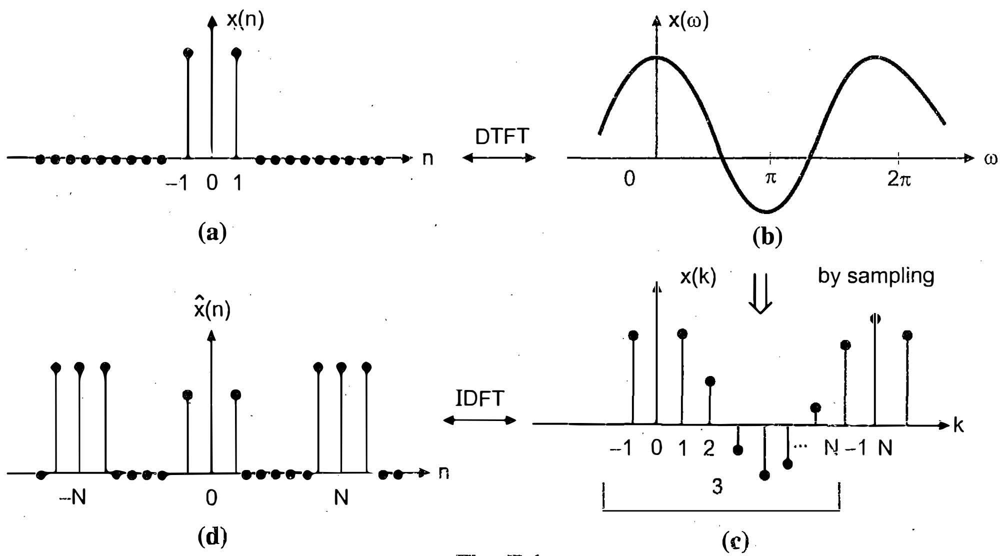



Fig. F-1

This process is explained graphically as shown in Fig. F-1. Fig. F-1(a) shows discrete time signal $x(n)$. By taking DTFT of $x(n)$; $X(\omega)$ is obtained as shown in Fig. F-1(b). The sampled version of $X(\omega)$ is denoted by $X(k)$ which is called as DFT. It is shown in Fig. F-1(c). By performing IDFT; original is obtained. It is denoted by $\hat{x}(n)$. It is shown in Fig. F-1(d). It is periodic extension of sequence $x(n)$.

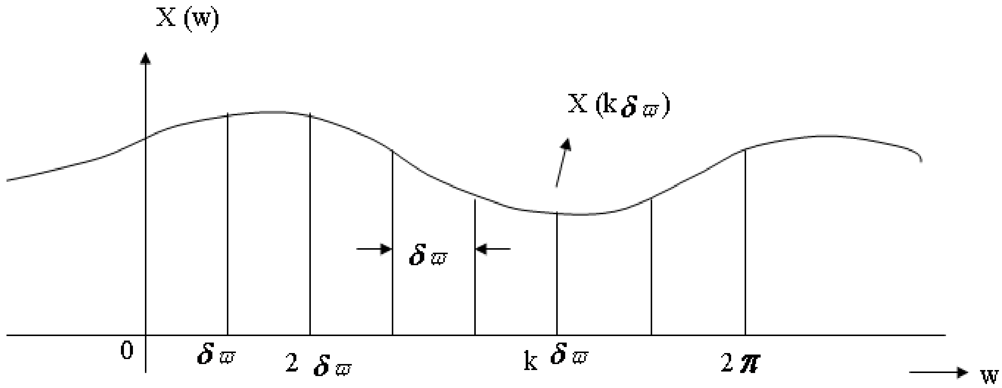
Here 'N' denotes the number of samples of input sequence and the number of frequency points in the DFT output.

## DFT (Frequency Domain Sampling)

The Fourier series describes periodic signals by discrete spectra, where as the DTFT describes discrete signals by periodic spectra. These results are a consequence of the fact that sampling on domain induces periodic extension in the other. As a result, signals that are both discrete and periodic in one domain are also periodic and discrete in the other. This is the basis for the formulation of the DFT.

Consider aperiodic discrete time signal x (n) with FT X(w) = $\sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$

Since X (w) is periodic with period $2\pi$, sample X(w) periodically with N equidistance samples with spacing $\delta w = \frac{2\pi}{N}$.




K = 0, 1, 2…..N-1

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi}{N}Kn}$$

The summation can be subdivided into an infinite no. of summations, where each sum contains

$$X\left(\frac{2\pi k}{N}\right) = \ldots\ldots\ldots\ldots + \sum_{n=-N}^{-1} x(n)e^{-j\frac{2\pi}{N}Kn} + \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}Kn} + \sum_{n=N}^{2N-1} x(n)e^{-j\frac{2\pi}{N}Kn} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi}{N}Kn}$$

Put n = n-lN

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-j\frac{2\pi}{N}K(n-lN)}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN) e^{-j\frac{2\pi}{N}Kn}$$

$$X(k) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}Kn} \qquad \text{Where} \quad x_p(n) = \sum_{=-\infty}^{\infty} x(n-lN)e$$

We know that $x_p(n) = \sum_{k=0}^{N-1} C_k\, e^{j\frac{2\pi}{N}Kn}$ n= 0 to N-1

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}Kn} \qquad k=0 \text{ to } N-1$$

Therefore $C_k = \frac{1}{N} X(k)$ k=0 to N-1

IDFT ------------ $x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}Kn}$ n = 0 to N-1

This provides the reconstruction of periodic signal $x_p(n)$ from the samples of spectrum X(w).

The spectrum of aperiodic discrete time signal with finite duration L<N, can be exactly recovered from its samples at frequency $W_k = \frac{2\pi k}{N}$.

## 1.2 Definition of DFT :

### The Discrete Fourier Transform :

**Definition of DFT :**

It is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

**Mathematical Equations :**

The DFT of discrete sequence x ( n ) is denoted by X ( k ). It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \qquad ...(1)$$

Here k = 0, 1, 2 ... N − 1

Since this summation is taken for 'N' points; it is called as 'N' point DFT.

We can obtain discrete sequence x ( n ) from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \qquad ...(2)$$

Here n = 0, 1, 2, .... N − 1

This is called as 'N' point IDFT.

Now we will define the new term 'W' as,

$$W_N = e^{-j2\pi/N} \qquad ...(3)$$

This is called as twiddle factor. Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor we can write equations of DFT and IDFT as follows :

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad ...(4)$$

Here n = 0, 1, 2 ... N − 1

$$\text{and} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad ...(5)$$

Here $n = 0, 1, 2, \ldots N-1$

### 1.2.1 Relationship between DTFT and DFT :

The DTFT is discrete time fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad ...(6)$$

The range of $\omega$ is from $-\pi$ to $\pi$ or 0 to $2\pi$.

Now we know that discrete fourier transform (DFT) is obtained by sampling one cycle of fourier transform. And DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\,2\pi kn/N} \qquad ...(7)$$

Comparing Equations (6) and (7), we can say that DFT is obtained from DTFT by putting $\omega = \frac{2\pi k}{N}$

$$\therefore \; X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

By comparing DFT with DTFT we can write,

1. The continuous frequency spectrum $X(\omega)$ is replaced by discrete fourier spectrum $X(k)$.
2. Infinite summation in DTFT is replaced by finite summation in DFT.
3. The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$; where $T_s$ is sampling time.

### 1.2.2 DFT of Standard Signals :

In this sub-section, we will obtain DFT of some standard signals with the help of solved examples as follows :

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**Prob. 1 :** Obtain DFT of unit impulse $\delta(n)$.

**Soln. :**

$$\text{Here} \quad x(n) = \delta(n) \qquad ...(1)$$

According to the definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(2)$$

But $\delta(n) = 1$ only at $n = 0$,

It is shown in Fig. F-2

Thus Equation (2) becomes,

$$X(k) = \delta(0) e^{0} = 1$$

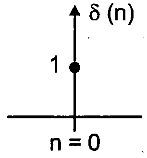


**Fig. F-2**

$$\therefore \delta(n) \overset{DFT}{\longleftrightarrow} 1$$

This is the standard DFT pair.

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**Prob. 2 :** Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

**Soln. :** We know that $\delta(n - n_0)$ indicates unit impulse delayed by '$n_0$' samples.

Here $x(n) = \delta(n - n_0)$ ...(1)

Now we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(2)$$

But $\delta(n - n_0) = 1$ only at $n = n_0$.

It is shown in Fig. F-3

Thus Equation (2) becomes,

$$X(k) = 1 \cdot e^{-j2\pi kn_0/N}$$

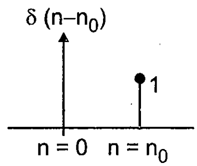


**Fig. F-3**

$$\therefore \delta(n - n_0) \overset{DFT}{\longleftrightarrow} e^{-j2\pi kn_0/N}$$

Similarly we can write,

$$\delta(n + n_0) \overset{DFT}{\longleftrightarrow} e^{j2\pi kn_0/N}$$

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**Prob. 3 :** Obtain N-point DFT of exponential sequence :

$$x(n) = a^n u(n) \text{ for } 0 \le n \le N-1$$

**Soln. :** According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(1)$$

Here $x(n) = a^n u(n)$

The multiplication of $a^n$ with $u(n)$ indicates sequence is positive. Putting $x(n) = a^n$ in Equation (1) we get,

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} (ae^{-j2\pi k/N})^n \qquad ...(2)$$

Now use the standard summation formula,

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here $N_1 = 0$, $N_2 = N-1$ and $A = ae^{-j2\pi k/N}$

$$\therefore \quad X(k) = \frac{(ae^{-j2\pi k/N})^0 - (ae^{-j2\pi k/N})^{N-1+1}}{1 - ae^{-j2\pi k/N}}$$

$$\therefore \quad X(k) = \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j2\pi k/N}} \qquad ...(3)$$

Using Euler's identity to the numerator term, we get,

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But k is an integer

$$\therefore \quad \cos 2\pi k = 1 \text{ and } \sin 2\pi k = 0$$

$$\therefore \quad e^{-j2\pi k} = 1 - j0 = 1$$

$$\therefore \quad X(k) = \frac{1 - a^N}{1 - a e^{-j2\pi k/N}}$$

$$\therefore \; a^n u(n) \overset{DFT}{\longleftrightarrow} \frac{1 - a^N}{1 - a e^{-j2\pi k/N}}$$

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**Prob. 4 :** Find the DFT of following window function,

$$w(n) = u(n) - u(n-N)$$

**Soln. :** According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad ...(1)$$

The given equation is $x(n) = w(n) = 1$ for $0 \le n \le N-1$. We will assume some value of N. Let N = 4; so we will get 4-point DFT.

$$\therefore \quad X(k) = \sum_{n=0}^{3} 1 \cdot e^{-j2\pi kn/4} \qquad \text{...(2)}$$

The range of k is from '0' to N − 1. So in this case 'k' will vary from 0 to 3.

For $k = 0 \Rightarrow X(0) = \sum_{n=0}^{3} 1 \cdot e^{0} = \sum_{n=0}^{3} 1 = 1 + 1 + 1 + 1 = 4$

For $k = 1 \Rightarrow X(1) = \sum_{n=0}^{3} e^{-j2\pi n/4}$

$$\therefore \quad X(1) = e^{0} + e^{-j2\pi/4} + e^{-j4\pi/4} + e^{-j6\pi/4}$$

$$\therefore \quad X(1) = 1 + \left(\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4}\right) + \left(\cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4}\right) + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right)$$

$$\therefore \quad X(1) = 1 + (0 - j) + (-1 - 0) + (0 + j)$$

$$\therefore \quad X(1) = 1 - j - 1 + j = 0$$

For $k = 2 \Rightarrow X(2) = \sum_{n=0}^{3} e^{-j2\pi \times 2n/4} = \sum_{n=0}^{3} e^{-j\pi n}$

$$\therefore \quad X(2) = e^{0} + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$$

$$\therefore \quad X(2) = 1 + (\cos\pi - j\sin\pi) + (\cos 2\pi - j\sin 2\pi) + (\cos 3\pi - j\sin 3\pi)$$

$$\therefore \quad X(2) = 1 + (-1 - 0) + (1 - 0) + (-1 - 0) = 1 - 1 + 1 - 1 = 0$$

For $k = 3 \Rightarrow X(3) = \sum_{n=0}^{3} e^{-j2\pi \times 3n/4} = \sum_{n=0}^{3} e^{-j6\pi n/4}$

$$\therefore \quad X(3) = e^{0} + e^{-j6\pi/4} + e^{-j3\pi} + e^{-j9\pi/2}$$

$$\therefore \quad X(3) = 1 + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right) + (\cos 3\pi - j\sin 3\pi) + \left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$\therefore \quad X(3) = 1 + (0 + j) + (-1 - 0) + (0 - j) = 1 + j - 1 - j = 0$$

$$\therefore \quad X(k) = \{4, 0, 0, 0\}$$