Lecture – 3

Convolution

1.10 Linear Convolution :

Proof of linear convolution Or Proof of "LTI system is completely characterized by unit impulse response h (n)" :

Consider relaxed LSI (LTI) system. A relaxed system means if input $x(n)$ is zero then output $y(n)$ is zero. Let us say unit impulse $\delta(n)$ is applied to this system then its output is denoted by h (n). "h (n)" is called as impulse response of system.

Step I : $$\delta(n) \xrightarrow{T} y(n) = h(n)$$

Since the system is time invariant, if we delay input by 'k' samples then output should be delayed by same amount.

Step II : $$\delta(n-k) \xrightarrow{T} y(n) = h(n-k)$$

Now multiplying both sides by $x(k)$ we get,

Step III : $$x(k)\,\delta(n-k) \xrightarrow{T} y(n) = x(k)\,h(n-k)$$

Since the system is linear; we can apply superposition theorem. So taking summation at both sides we get,

Step IV :

$$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \xrightarrow{T} y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Thus we have,

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)} \quad ...(10)$$

This summation is called as convolution sum. In this case output $y(n)$ is obtained by taking convolution of $x(n)$ and $h(n)$. This is called as linear convolution and it is denoted by symbol '*'.

$$\therefore \quad x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

In this case we can say that the behaviour of LTI system is completely characterized by its impulse response.

Prob. 1 : Determine impulse response of the DT-LTI system in terms of unit impulse input $\delta(n)$ for $h(n) = 2^{-n} u(n) - 2^{-n} u(n-3)$.

Soln. :

Given :

$$h(n) = 2^{-n} u(n) - 2^{-n} u(n-3)$$

We have to express $h(n)$ in terms of unit impulse response $\delta(n)$.

$$\therefore \quad h(n) = 2^{-n} [u(n) - u(n-3)]$$

The sequence $u(n) - u(n-3)$ can be written as,

$$u(n) - u(n-3) = \{\underset{\uparrow}{1}, 1, 1\}$$

Since every sample is having unity magnitude,

we can write,

$$u(n) - u(n-3) = \{\underset{\uparrow}{1}, 1, 1\} = \delta(n) + \delta(n-1) + \delta(n-2)$$

$$\therefore \quad h(n) = 2^{-n} [\delta(n) + \delta(n-1) + \delta(n-2)]$$

$$\therefore \quad h(n) = 2^{-n} \delta(n) + 2^{-n} \delta(n-1) + 2^{-n} \delta(n-2)$$

This gives impulse response in terms of $\delta(n)$.

1.10.1 Computation of Linear Convolution :

The different methods used for the computation of linear convolution are as follows :

1) Graphical method
2) Using mathematical equation of convolution
3) Tabulation method
4) Multiplication method

1) Graphical method :

The linear convolution of two sequence is given by,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(1)$$

We will have to calculate output for different values of 'n'.

For n = 0 we get,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) \qquad ...(2)$$

Note that $h(-k)$ indicates folding of $h(k)$.

For n = 1 we get,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) \qquad ...(3)$$

Here the term $h(1-k)$ can be written as $h(-k+1)$. Thus Equation (3) becomes,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(-k+1) \qquad ...(4)$$

Here $h(-k+1)$ indicates shifting of folded signal $h(-k)$. It indicates that $h(-k)$ is delayed by '1' sample. Similarly for other values of 'n' output $y(n)$ is calculated.

Thus different operations involved in the calculation of linear convolution are as follows :

i) Folding operation : It indicates folding of sequence $h(k)$.

ii) Shifting operation : It indicates time shifting of $h(-k)$ eg. $h(-k+1)$.

iii) Multiplication : It indicates multiplication of $x(k)$ and $h(n-k)$.

iv) Summation : It indicates addition of all product terms obtained because of multiplication of $x(k)$ and $h(n-k)$

Solved Problems :

Prob. 1 : Obtain linear convolution of following sequences

$x(n) = \{1, 2, 1, 2\}$ and $h(n) = \{1, 1, 1\}$

Soln. : If the sequences are given in terms of 'n' then obtain $x(k)$ and $h(k)$ by replacing 'n' by 'k'.

$$\therefore \quad x(k) = \{1, 2, 1, 2\} \quad \text{and} \quad h(k) = \{1, 1, 1\}$$

Arrow is not mentioned in any sequences. So by default it is at first position.

$$\therefore \quad x(k) = \{\underset{\uparrow}{1}, 2, 1, 2\} \quad \text{and} \quad h(k) = \{\underset{\uparrow}{1}, 1, 1\}$$

Now we have equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad ....(1)$$

Now we have to decide range of 'n' and 'k'.

Range of 'n' :

Basically to calculate $y(n)$ we should know from which value of 'n'. We should start (lower range of n) and what is ending value of 'n' (higher range of n). We will use following notations to decide range of $y(n)$.

$$y_l = \text{Lowest range of } y(n)$$

$$y_h = \text{Highest range of } y(n)$$

$$x_l = \text{Lowest range of } x(k)$$

$$x_h = \text{Highest range of } x(k)$$

$$h_l = \text{Lowest range of } h(k)$$

$$h_h = \text{Highest range of } h(k)$$

Now use following formulae to calculate range of $y(n)$

$$y_l = x_l + h_l \text{ and } y_h = x_h + h_h$$

From the given sequences we have,

$$x_l = 0, \quad x_h = 3, \quad h_l = 0 \text{ and } \quad h_h = 2$$

$$\therefore \quad y_l = x_l + h_l = 0 + 0 = 0$$

$$\text{and} \quad y_h = x_h + h_h = 3 + 2 = 5$$

Thus range of $y(n)$ is from $y(0)$ to $y(5)$.

Range of 'k' : The value of k in the summation sign will be always same as the sequence $x(k)$. Thus in this case the range of k is from $k = 0$ to 3. So Equation (1) becomes,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(n-k) \qquad \text{...(2)}$$

Now we will calculate output $y(n)$ by putting different values of n from 0 to 5.

Calculation of y (0) :

Putting $n = 0$ in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)(0-k) = \sum_{k=0}^{3} x(k)\,h(-k) \qquad \text{...(3)}$$

Now we have to perform calculations using graphical method. Here $h(-k)$ indicates folded version of $h(k)$.

Step I : Sketch $x(k)$.

Sketch of $x(k)$ is as shown in Fig. A-36(a)

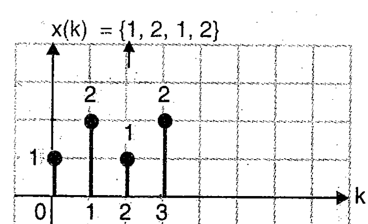


Fig. A-36(a) : Sequence x (k)

Step II : Sketch h (k)

Sketch of h (k) is shown in Fig. A-36(b).

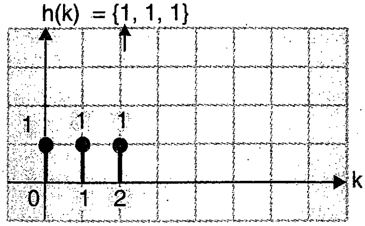


Fig. A-36(b) : Sequence h (k)

Step III : Fold h (k) to obtain h (– k).

Sketch of h (– k) is shown in Fig. A-36(c)

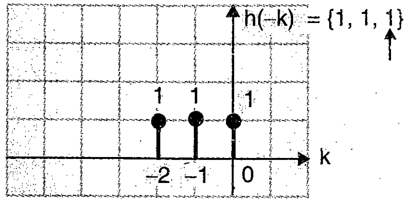


Fig. A-36(c) : Folded sequence h (– k)

Step IV : Obtain multiplication of x (k) and h (– k). This multiplication takes place on sample to sample basis as shown in Fig. A-36(d).

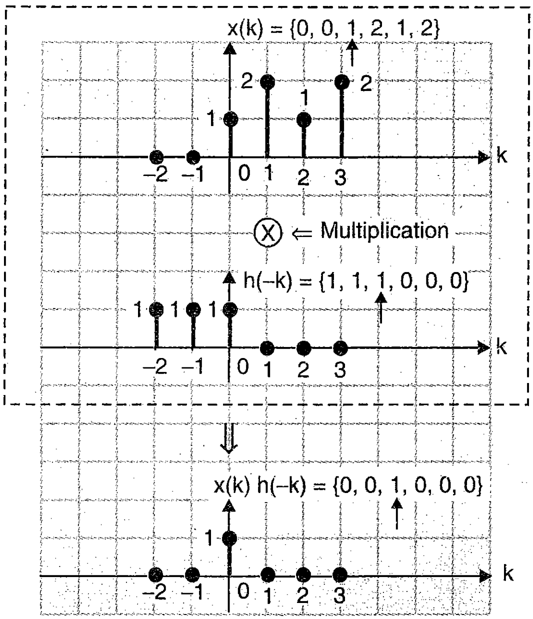


Fig. A-36(d) : Product of x (k) and h (– k)

Step V : Take the summation of all product terms.

In this case as result of multiplication we have only one sample at n = 0.

$$\boxed{\therefore \quad y(0) = 1}$$

Calculation of y (1) : Putting n = 1 in Equation (2) we get,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(1-k) \qquad \text{...(4)}$$

$h(1-k)$ can be written as $h(-k+1)$. Thus Equation (4) becomes,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(-k+1) \qquad \text{...(5)}$$

Step I : Now $h(-k+1)$ indicates delay of $h(-k)$ by '1' sample. So shift the sequence in Fig. A-36(c) towards right by '1' sample. This sequence is shown in Fig. A-37(a).

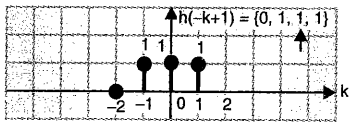


Fig. A-37(a) : Sequence h (– k + 1)

Step II : Obtain multiplication of $x(k)$ and $h(-k+1)$. It is shown in Fig. A-37(b).

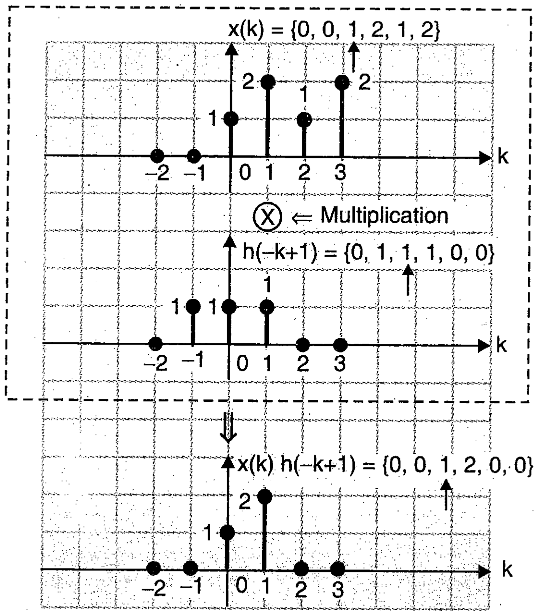


Fig. A-37(b) : Product of x (k) and h (– k + 1)

Step III : Now according to Equation (5) we have to add all product terms shown in Fig. A-37(b) to obtain $y(n)$.

$$\therefore \quad y(1) = (1\times 1)+(2\times 1) = 1 + 2$$

$$\boxed{\therefore \quad y(1) = 3}$$

Calculation of y (2) :

Putting n = 2 in Equation (2) we get,

$$y(2) = \sum_{k=0}^{3} x(k)\,h(2-k) \qquad \text{...(6)}$$

The term $h(2-k)$ is same as $h(-k+2)$.

$$\therefore \quad y(2) = \sum_{k=0}^{3} x(k)\,h(-k+2) \qquad \text{...(7)}$$

Step I : Here $h(-k+2)$ indicates delay of $h(-k)$ by '2' samples. This is obtained by shifting sequence in Fig. A-36(c) towards right by '2' samples. This sequence is shown in Fig. A-38(a).

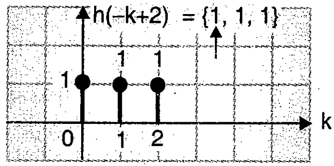


Fig. A-38(a) : Sequence h (− k + 2)

Step II : Obtain multiplication of $x(k)$ and $h(-k+2)$. It is shown in Fig. A-38(b).

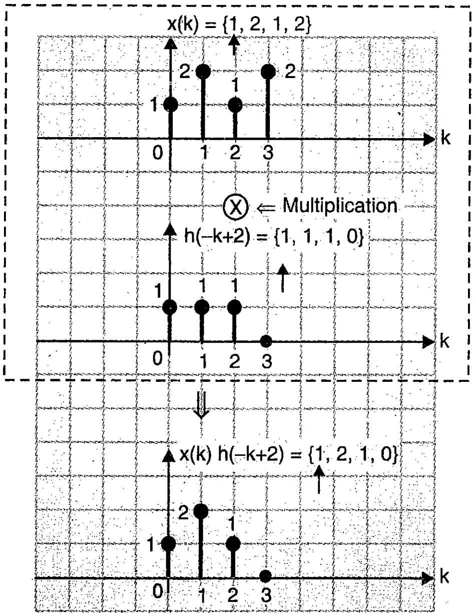


Fig. A-38(b) : Product of x (k) and h (− k + 2)

Step III : Add all product terms to obtain y (2).

$$\therefore \quad y(2) = (1\times 1)+(1\times 2)+(1\times 1) = 1+2+1$$

$$\boxed{\therefore \; y(2) = 4}$$

Calculation of y (3) : Putting n = 3 in Equation (2) we get,

$$y(3) = \sum_{k=0}^{3} x(k)\,h(3-k) \qquad \text{...(8)}$$

The term h (3 − k) is same as h (− k + 3)

$$\therefore \quad y(3) = \sum_{k=0}^{3} x(k)\,h(-k+3) \qquad \text{...(9)}$$

Step I : Here h (− k + 3) indicates delay of h (− k) by 3 samples. It is shown in Fig. A-39(a).

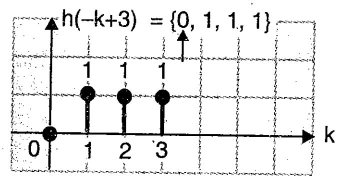


Fig. A-39(a) : Sequence h (− k + 3)

Step II : Obtain multiplication of x (k) and h (− k + 3). This sequence is shown in Fig. A-39(b).

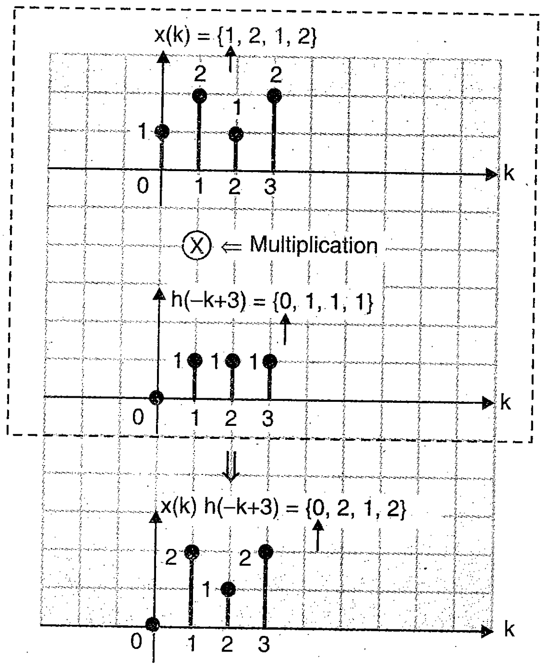


Fig. A-39(b) : Product of x (k) and h (− k + 3)

Step III : Add all product terms to obtain y (3).

$$\therefore \quad y(3) = (0\times 1)+(1\times 2)+(1\times 1)+(2\times 1) = 0+2+1+2$$

$$\therefore \quad y(3) = 5$$

Calculation of y (4) : Putting n = 4 in Equation (2) we get,

$$y(4) = \sum_{k=0}^{4} x(k)\,h(4-k) \qquad \ldots(10)$$

The term $h(4-k)$ is same as $h(-k+4)$

$$\therefore \quad y(4) = \sum_{k=0}^{4} x(k)\,h(-k+4) \qquad \ldots(11)$$

Step I : Here $h(-k+4)$ indicates delay of $h(-k)$ by '4' samples. It is shown in Fig. A-40(a).

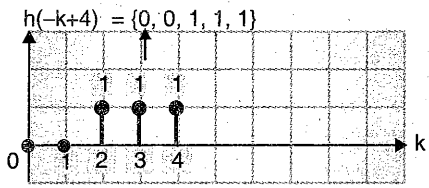


Fig. A-40(a) : Sequence h (− k + 4)

Step II : Obtain multiplication of $x(k)$ and $h(-k+4)$. This sequence is shown in Fig. A-40(b).

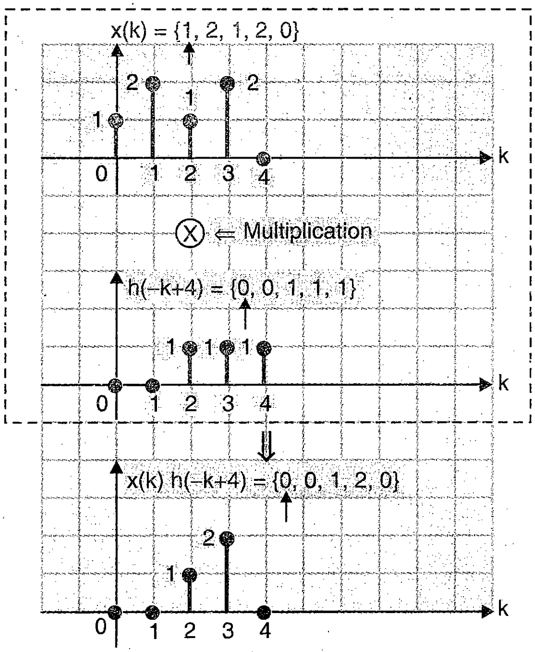


Fig. A-40(b) : Product of x (k) and h (− k + 4)

Step III : Add all product terms to obtain $y(4)$.

$$\therefore \quad y(4) = (1\times 0)+(2\times 0)+(1\times 1)+(2\times 1)+(0\times 1) = 0+0+1+2+0$$

$$\therefore \boxed{y(4) = 3}$$

Calculation of y (5) : Putting n = 5 in Equation (2) we get,

$$y(5) = \sum_{k=0}^{3} x(k)\,h(5-k) \qquad \text{...(12)}$$

The term h (5 − k) is same as h (− k + 5)

$$\therefore \quad y(5) = \sum_{k=0}^{3} x(k)\,h(-k+5) \qquad \text{...(13)}$$

Step I : Here h (− k + 5) indicates delay of h (− k) by '5' samples. It is shown in Fig. A-41(a).

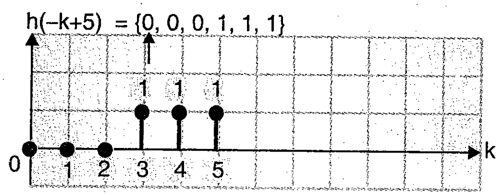


Fig. A-41(a) : Sequence h (− k + 5)

Step II : Obtain multiplication of x (k) and h (− k + 5). This sequence is shown in Fig. A-41(b).

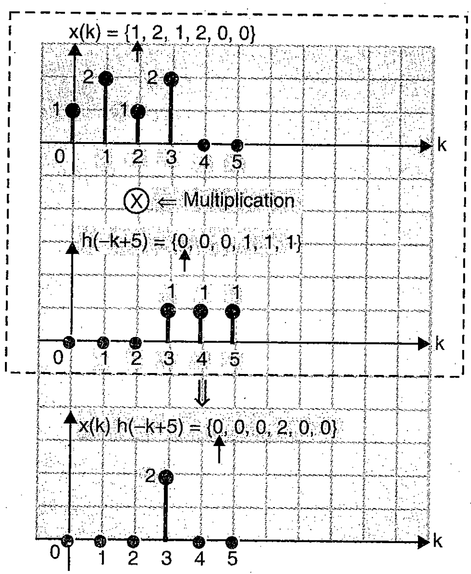


Fig. A-41(b) : Product of x (k) and h (− k + 5)

Step III : Add all product terms to obtain y (5).

$$\therefore \quad y(5) = (1 \times 0) + (2 \times 0) + (1 \times 0) + (2 \times 1) + (0 \times 1) + (0 \times 1)$$

The result of convolution, y (n) is represented graphically as shown in Fig. A-42.

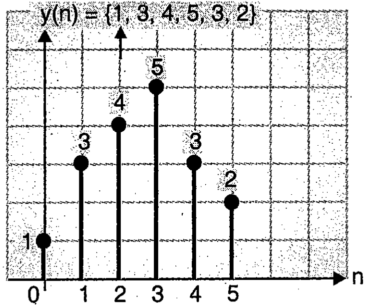


Fig. A-42 : Sketch of y (n) = { 1, 3, 4, 5, 3, 2 }
$\uparrow$

3) Tabulation method of linear convolution :

This is the simplest method of performing linear convolution.

$$\text{Let} \quad x(n) = \{\underset{\uparrow}{x(0)}, x(1), x(2)\} \text{ and } h(n) = \{\underset{\uparrow}{h(0)}, h(1), h(2)\}$$

Step I : Form the matrix as shown in Fig. A-57.

$$\begin{array}{c|ccc} & \multicolumn{3}{c}{\overbrace{\quad x(0) \qquad x(1) \qquad x(2) \quad}^{x(n)}} \\ \hline h(n)\left\{\begin{array}{c} h(0) \\ \\ h(1) \\ \\ h(2) \end{array}\right. & & & \end{array}$$

Fig. A.57 : Matrix of x (n) and h (n)

Note that we can interchange the positions of x (n) and h (n).

Step II : Multiply the corresponding elements of x (n) and h (n) as shown in Fig. A-58.

	x (0)	x (1)	x (2)
h (0)	h (0) x (0)	h (0) x (1)	h (0) x (2)
h (1)	h (1) x (0)	h (1) x (1)	h (1) x (2)
h (2)	h (2) x (0)	h (2) x (1)	h (2) x (2)

Fig. A-58 : Multiplication of elements x (n) and h (n)

Step III : Separate out the elements diagonally as shown in Fig. A-59.

	x(0)	x(1)	x(2)
h(0)	h(0)x(0)	h(0)x(1)	h(0)x(2)
h(1)	h(1)x(0)	h(1)x(1)	h(1)x(2)
h(2)	h(2)x(0)	h(2)x(1)	h(2)x(2)

Fig. A-59 : Diagonally separation of elements

Step IV : Simply add the elements in that particular block. This addition gives corresponding values of y (n).

Here range of 'n' is :

$$y_l = x_l + h_l = 0 + 0 = 0$$

and $$y_h = x_h + h_h = 2 + 2 = 4$$

Thus we get,

$$y(0) = h(0)x(0)$$
$$y(1) = h(1)x(1) + h(0)x(1)$$
$$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$$
$$y(3) = h(2)x(1) + h(1)x(2)$$

and $$y(4) = h(2)x(2)$$

After calculating all these values; write down the result of convolution as,

$$y(n) = \{\underset{\uparrow}{y(0)}, y(1), y(2), y(3), y(4)\}$$

Prob. 9 : Compute the convolution $y(n) = x(n) * h(n)$

where $x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$ and $h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

Soln. : Here x (n) contains '5' samples and h (n) has '4' samples. To make the length of x (n) and h (n) same; rewrite the sequence h (n) as follows :

$$h(n) = \{1, -2, -3, \underset{\uparrow}{4}, 0\}$$

We can add zeros at the end or at the beginning of sequence. This will not affect the given sequence. This method of adding zeros, to adjust the length of sequence is called as zero-padding.

Range of 'n' : The range of 'n' for y (n) is calculated as follows :

Lowest index of y (n) $\Rightarrow y_l = x_l + h_l = -2 + (-3) = -5$

and Highest index of y (n) $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (− 5) to y (3). Now using tabulation method the convolution is obtained as shown in Fig. A-60.

		x(n)			
	1	1	0	1	1
h(n) 1					
−2					
−3					
4					
0					

Fig. A-60(a) : Matrix of x (n) and h (n)

	1	1	0	1	1
1	1×1	1×1	1×0	1×1	1×1
−2	-2×1	-2×1	-2×0	-2×1	-2×1
−3	-3×1	-3×1	-3×0	-3×1	-3×1
4	4×1	4×1	4×0	4×1	4×1
0	0×1	0×1	0×0	0×1	0×1

Fig. A-60(b) : Convolution of x (n) and h (n)

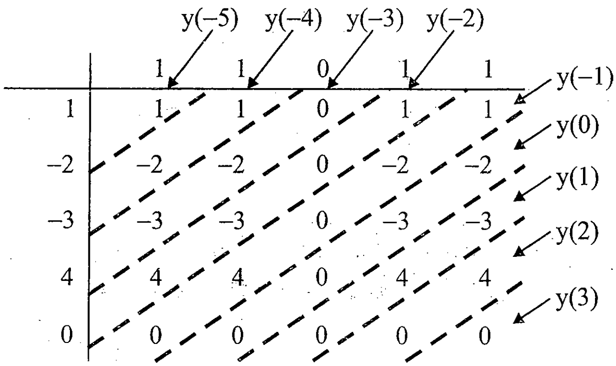


Fig. A-60(c) : Diagonally separation of elements

Different values of y (n) are calculated by adding corresponding elements as follows :

$$y(-5) = 1$$
$$y(-4) = -2+1 = -1$$
$$y(-3) = -3-2+0 = -5$$
$$y(-2) = 4-3+0+1 = 2 \qquad y(-1) = 0+4+0-2+1 = 3$$
$$y(0) = 0+0-3-2 = -5 \qquad y(1) = 0+4-3 = 1$$
$$y(2) = 0+4 = 4 \qquad y(3) = 0$$

Thus result of convolution y (n) is,

$$y(n) = \{y(-5), y(-4), y(-3), y(-2), y(-1), \underset{\uparrow}{y(0)}, y(1), y(2), y(3)\}$$

$$\therefore \quad y(n) = \{1, -1, -5, 2, 3, -5, 1, 4, 0\}$$
$$\uparrow$$

Neglecting the last '0' term we have,

$$\boxed{y(n) = \{1, -1, -5, 2, 3, -5, 1, 4\}}$$
$$\uparrow$$

Prob. 10 : The impulse response of linear time invarient is

$$h(n) = \{1, 2, 1, -1\}$$
$$\uparrow$$

Determine the response of system to the input

$$x(n) = \{1, 2, 3, 1\}$$
$$\uparrow$$

Soln. : The response of the system is,

$$y(n) = x(n) * h(n)$$

Range of n :

$$\text{Lowest of } y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$$

$$\text{Highest range of } y(n) \Rightarrow y_h = x_h + h_h = 3 + 3 = 6$$

Using tabulation method; the convolution is obtained as shown in Fig. A-61.

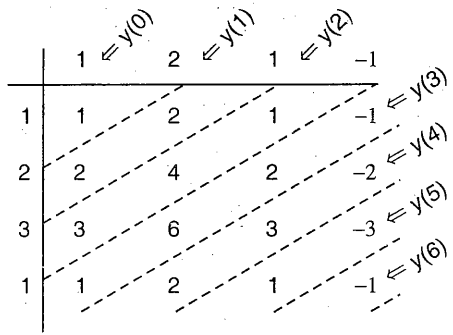


Fig. A-61 : Linear convolution of x (n) and h (n)

$$\therefore \quad y(n) = \{1, 4, 8, 8, 3, -2, -1\}$$
$$\uparrow$$

4) Multiplication method of linear convolution :

This is another easy method to obtain convolution of two sequences. This method is similar to multiplication of multidigit numbers. Write down the two sequences, x (n) and h (n) and obtain the multiplication by usual method. The result of multiplication will be equal to the convolution of two sequences. We will solve one example by this method.

Prob. 12 : Obtain linear convolution of following sequences :

$$x(n) = \{1, \underset{\uparrow}{2}, 1, 2\} \text{ and } h(n) = \{2, 2, \underset{\uparrow}{-1}, 1\}$$

Soln. : First we will decide the range of 'n' for y (n).

Lowest index $\Rightarrow y_l = x_l + h_l = -1 + (-2) = -3$

Highest index $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 3) to y (3).

We will solve this problem using multiplication method.

	x(n) :-			1	2	1	2	
Multiplication	h(n) :- ⊗			2	2	–1	1	
				1	2	1	2	⇐ x(n) is multiplied by (1)
Addition ® ⊕			–1	–2	–1	–2	0	⇐ x(n) is multiplied by (–1) ← Zero digit similar to normal multiplication
Addition ® ⊕		2	4	2	4	0	0	⇐ x(n) is multiplied by (2) ← Two zero digits similar to normal multiplication
Addition ® ⊕	2	4	2	4	0	0	0	⇐ x(n) is multiplied by (2) ← Three zero digits similar to normal multiplication
	2	6	5	5	5	–1	2	⇐ These values are obtained by adding all digits in corresponding columns.

This result of multiplication is same as linear convolution. We know that range of y (n) is from y (– 3) to y (3). So the first term from L.H.S. represents the value of y (– 3). Now mark the arrow at the position of y (0). Thus output y (n) can be written as,

$$\boxed{y(n) = \{2, 6, 5, \underset{\uparrow}{5}, 5, -1, 2\}}$$

Let us verify this result using tabulation method shown in Fig. A-65.

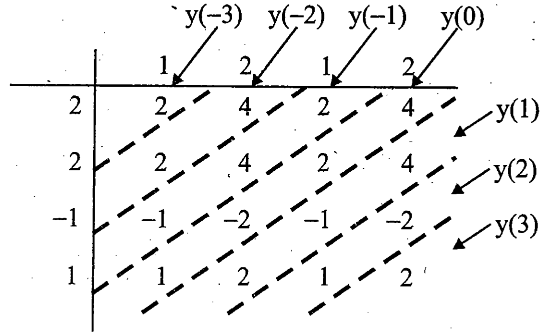
h(n) \ x(n)	1	2	1	2
2				
2				
-1				
1				

(a)

	1	2	1	2
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
– 1	(– 1 × 1)	(– 1 × 2)	(– 1 × 1)	(– 1 × 2)
1	(1 × 1)	(1 × 2)	(1 × 1)	(1 × 2)

(b)

Fig. A-65



(c)

Fig. A-65 : Convolution of $x(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$ and $h(n) = \{2, 2, -1, 1\}$

Thus using tabulation method we get,

$$y(-3) = 2 \qquad y(-2) = 2 + 4 = 6$$

$$y(-1) = -1 + 4 + 2 = 5 \qquad y(0) = 1 - 2 + 2 + 4 = 5$$

$$y(1) = 2 - 1 + 4 = 5 \qquad y(2) = 1 - 2 = -1$$

$$y(3) = 2$$

$$\therefore \quad \boxed{y(n) = \{2, 6, 5, \underset{\uparrow}{5}, 5, -1, 2\}}$$