Lecture – 4

Correlation

1.14 Correlation of Discrete Time Systems :

Basics of correlation :

The word co-relation means comparison. Basically it is comparison between two signals or sequences. A good application of correlation is the radar system.

In case of radar system the signal is transmitted from radar to the target. Let signal is $x(n)$. Now if target is present then the signal is reflected back from the target. We will denote this reflected signal by $y(n)$. This signal will be attenuated because of different losses taking place in the path of signal. We will denote this attenuation by factor β. Similarly this signal $y(n)$ will be delayed in time, compared to transmitted signal $x(n)$. Thus we can write,

$$y(n) = \beta x(n - n_0) + N \qquad \text{...(1)}$$

Here β = Attenuation factor

n_0 = Amount by which signal is delayed.

N = Noise levels added in the signal.

Now correlation means compare this output with the transmitted signal $x(n)$. That means check whether $y(n) = x(n)$ or not.

Applications of correlation :

(1) Detection of difference between two signals.

(2) Identification of signals in the presence of noise.

(3) Observing effects of inputs and outputs in control engineering.

(4) Determination of time delays between transmitted and received signals.
(5) Used in image processing to compare data from different images.
(6) Identification of specific pattern within the data stream.
(7) Computation of average power in the waveforms.
(8) For diagnosis of medical disorders.

Types of correlation :

There are two types of correlation as follows :

(1) Auto-correlation
(2) Cross-correlation.

(1) Auto-correlation :

Many times it is required to correlate the signal with itself then this correlation is called as auto-correlation. Thus if the two signals are generated from the same source then it is called as auto-correlation. Auto-correlation is denoted by $r_{xx}(l)$. Here 'l' is a variable and r_{xx} indicates the comparison between $x(n)$ and $x(n)$.

(2) Cross-correlation :

The correlation between $x(n)$ and $y(n)$ is called as cross-correlation. That if the two signals are generated from different sources then it is called as cross-correlation. It is denoted by $r_{xy}(l)$. Here 'l' is a variable and r_{xy} indicates the comparison between $x(n)$ and $y(n)$.

Mathematical Equations :

(1) Auto-correlation : It is defined as,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l) \qquad ...(1)$$

Here first sequence $x(n)$ is kept as it is; while the sequence $x(n)$ is delayed by 'l'. Equation (1) can also be written as,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,x(n) \qquad ...(2)$$

(2) Cross-correlation :

The cross-correlation between $x(n)$ and $y(n)$ is defined as,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad ...(3)$$

Here sequence $x(n)$ is kept as it is; while $y(n)$ is delayed by 'l'. Equation (3) can also be written as,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\,y(n) \qquad ...(4)$$

Here sequence $y(n)$ is kept as it is; while $x(n)$ is advanced by 'l'.

If we change the role of $x(n)$ and $y(n)$ then the cross-correlation is denoted by $r_{yx}(l)$.

. It is given by,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \qquad \text{...(5)}$$

Alternately Equation (5) can be written as,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)\,x(n) \qquad \text{...(6)}$$

Solved Problems :

Prob. 1 : Compute the cross-correlation between

$$x(n) = \{1,\ 1,\ \underset{\uparrow}{0},\ 1\} \text{ and}$$

$$y(n) = \{4,\ -3,\ \underset{\uparrow}{-2},\ 1\}$$

Soln. : The cross-correlation of $x(n)$ and $y(n)$ is given by,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad \text{...(1)}$$

Range of 'n' :

In Equation (1) sequence $x(n)$ is not changed but sequence $y(n)$ is delayed by 'l'. Thus range of summation ('n') will be same as $x(n)$.

Thus range of 'n' is from -2 to 1.

Putting these values in Equation (1) we get,

$$r_{xy}(l) = \sum_{n=-2}^{1} x(n)\,y(n-l) \qquad \text{...(2)}$$

Now we will decide range of 'l' as follows :

Range of 'l' :

Observe the given sequences. Here maximum value of $y(n)$ is $y(1)$. So in Equation (2) $y(n-l)$ should have maximum value equals to $y(1)$. If we get $y(n-l) = y(2)$ then, result of summation will be zero.

Now starting value of 'n' in summation is $n = -2$.

Thus $y(n-l) = y(1)$ maximum $\Rightarrow n-l = 1$ maximum that means,

$$-2-l = 1 \Rightarrow l = -2-1 = -3.$$

So we will start from $l = -3$

Now from the given sequence; lowest value of $y(n)$ is $y(-2)$. Thus from Equation (2), we have $y(n-l) = y(-2)$. Now according to Equation (2) we should stop the summation at $n = 1$.

$$\text{Thus} \quad y(n-l) = y(-2) \Rightarrow 1-l = -2 \Rightarrow l = 1+2 = 3$$

Thus we will stop at $l = +3$.

So range of l is from -3 to $+3$.

Calculation of $r_{xy}(-3)$:

Putting $l = -3$ in Equation (2) we get,

$$r_{xy}(-3) = \sum_{n=-2}^{1} x(n)\, y(n+3) \qquad \text{...(3)}$$

Expanding the summation,

$$r_{xy}(-3) = x(-2)\,y(-2+3) + x(-1)\,y(-1+3) + x(0)\,y(0+3) + x(1)\,y(1+3)$$

$$\therefore \; r_{xy}(-3) = x(-2)\,y(1) + x(-1)\,y(2) + x(0)\,y(3) + x(1)\,y(4) \qquad \text{...(4)}$$

Now we have given sequences,

$$x(n) = \{1,\, 1,\, \underset{\uparrow}{0},\, 1\} \quad \text{and} \quad y(n) = \{4,\, -3,\, \underset{\uparrow}{-2},\, 1\}$$

$$\begin{array}{rclcrcl} x(-2) & = & 1 & \text{and} & y(-2) & = & 4 \\ x(-1) & = & 1 & \text{and} & y(-1) & = & -3 \\ x(0) & = & 0 & \text{and} & y(0) & = & -2 \\ x(1) & = & 1 & \text{and} & y(1) & = & 1 \end{array}$$

Putting these values in Equation (4) we get,

$$r_{xy}(-3) = (1 \times 1) + (1 \times 0) + (0 \times 0) + (1 \times 0) = 1+0+0+0$$

$$\boxed{\therefore \; r_{xy}(-3) = 1}$$

Calculation of $r_{xy}(-2)$:

Putting $l = -2$ in Equation (2) we get,

$$r_{xy}(-2) = \sum_{n=-2}^{1} x(n)\, y(n+2) \qquad \text{...(5)}$$

Expanding the summation,

$$r_{xy}(-2) = x(-2)\,y(-2+2) + x(-1)\,y(-1+2) + x(0)\,y(2) + x(1)\,y(1+2)$$

$$\therefore \; r_{xy}(-2) = x(-2)\,y(0) + x(-1)\,y(1) + x(0)\,y(2) + x(1)\,y(3)$$

$$\therefore \; r_{xy}(-2) = (1 \times (-2)) + (1 \times 1) + (0 \times 0) + (1 \times 0) = -2+1$$

$$\boxed{\therefore \; r_{xy}(-2) = -1}$$

Calculation of $r_{xy}(-1)$:

Putting $l = -1$ in Equation (2) we get,

$$r_{xy}(-1) = \sum_{n=-2}^{1} x(n)\,y(n+1) \qquad \text{...(6)}$$

Expanding the summation,

$$r_{xy}(-1) = x(-2)\,y(-2+1) + x(-1)\,y(-1+1) + x(0)\,y(0+1) + x(1)\,y(1+1)$$

$$\therefore\ r_{xy}(-1) = x(-2)\,y(-1) + x(-1)\,y(0) + x(0)\,y(1) + x(1)\,y(2)$$

$$\therefore\ r_{xy}(-1) = [1\times(-3)] + [1\times(-2)] + [0\times 1] + [1\times 0] = -3-2$$

$$\boxed{\therefore\ r_{xy}(-1) = -5}$$

Calculation of $r_{xy}(0)$:

Putting $l = 0$ in Equation (2) we get,

$$r_{xy}(0) = \sum_{n=-2}^{1} x(n)\,y(n) \qquad \text{...(7)}$$

Expanding the summation,

$$\therefore\ r_{xy}(0) = x(-2)\,y(-2) + x(-1)\,y(-1) + x(0)\,y(0) + x(1)\,y(1)$$

$$= [1\times 4] + [1\times(-3)] + [0\times(-2)] + [1\times 1] = 4-3+0+1$$

$$\boxed{\therefore\ r_{xy}(0) = 2}$$

Calculation of $r_{xy}(1)$:

Putting $l = 1$ in Equation (2) we get,

$$r_{xy}(1) = \sum_{n=-2}^{1} x(n)\,y(n-1) \qquad \text{...(8)}$$

Expanding the summation,

$$r_{xy}(1) = x(-2)\,y(-2-1) + x(-1)\,y(-1-1) + x(0)\,y(0-1) + x(1)\,y(1-1)$$

$$= x(-2)\,y(-3) + x(-1)\,y(-2) + x(0)\,y(-1) + x(1)\,y(0)$$

$$= [1\times 0] + [1\times 4] + [0\times(-3)] + [1\times(-2)] = 0+4+0-2$$

$$\boxed{\therefore\ r_{xy}(1) = 2}$$

Calculation of $r_{xy}(2)$:

Putting $l = 2$ in Equation (2) we get,

$$r_{xy}(2) = \sum_{n=-2}^{1} x(n)\,y(n-2) \qquad \text{...(9)}$$

Expanding the summation,

$$r_{xy}(2) = x(-2)y(-2-2)+x(-1)y(-1-2)+x(0)y(0-2)+x(1)y(1-2)$$

$$= x(-2)y(-4)+x(-1)y(-3)+x(0)y(-2)+x(1)y(-1)$$

$$= [1\times 0]+[1\times 0]+[0\times 4]+[1\times(-3)] = 0+0+0-3$$

$$\boxed{\therefore \; r_{xy}(2) = -3}$$

Calculation of $r_{xy}(3)$:

Putting $l = 3$ in Equation (2) we get,

$$r_{xy}(3) = \sum_{n=-2}^{1} x(n)\,y(n-3) \qquad \text{...(10)}$$

Expanding the summation,

$$r_{xy}(3) = x(-2)y(-2-3)+x(-1)y(-1-3)+x(0)y(0-3)+x(1)y(1-3)$$

$$= x(-2)y(-5)+x(-1)y(-4)+x(0)y(-3)+x(1)y(-2)$$

$$= [1\times 0]+[1\times 0]+[0\times 0]+[1\times 4] = 0+0+0+4$$

$$\boxed{\therefore \; r_{xy}(3) = 4}$$

Now the final sequence is written as,

$$r_{xy}(l) = \{\, r_{xy}(-3),\, r_{xy}(-2),\, r_{xy}(-1),\, \underset{\uparrow}{r_{xy}(0)},\, r_{xy}(1),\, r_{xy}(2),\, r_{xy}(3) \,\}$$

$$\boxed{\therefore \; r_{xy}(l) = \{\, 1,\ -1,\ -5,\ \underset{\uparrow}{2},\ 2,\ -3,\ 4 \,\}}$$

Note that arrow is marked at the value of $r_{xy}(0)$.

This is a tedious method. We will solve some examples using simple method.

Simple method to calculate correlation :

We will use one of the important properties of correlation. It is as follows :

$$r_{xy}(l) = x(n) * y(-n) \qquad \text{...(11)}$$

Observe Equation (11). The L.H.S. indicates cross correlation of $x(n)$ and $y(n)$. R.H.S. term indicates the convolution ('*') operation. It is the convolution of $x(n)$ and $y(-n)$. Here $y(-n)$ is the folded version of $y(n)$.

So this equation implies :

(i) Take signal $x(n)$ as it is.

(ii) Fold sequence $y(n)$ to obtain $y(-n)$

(iii) Obtain the convolution of $x(n)$ and $y(-n)$

(iv) The result of convolution will be same as the correlation.

Let us solve the same example using the method. Given sequences are,

$$x(n) = \{1, \underset{\uparrow}{1}, 0, 1\} \quad \text{and} \quad y(n) = \{4, -3, \underset{\uparrow}{-2}, 1\}$$

Step I : To obtain $y(-n)$; write down sequence $y(n)$ in the reverse order. Here arrow is marked at sample (-2). Do not change the arrow.

Thus $y(-n) = \{1, \underset{\uparrow}{-2}, -3, 4\}$

Step II : We will decide the range of n for convolution of $x(n)$ and $y(-n)$

Lowest index 'l' $\Rightarrow -2+(-1) = -3$

Highest index 'l' $\Rightarrow 1 + 2 = 3$

Thus output sequence will be from -3 to $+3$.

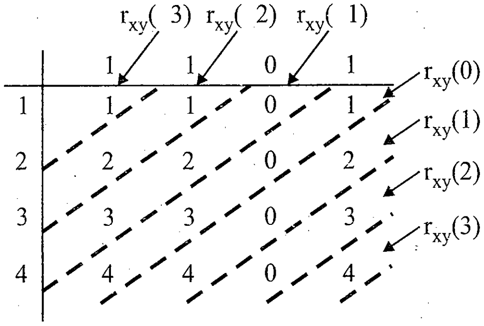
Step III : The convolution of $x(n)$ and $y(-n)$ using tabulation method is shown in Fig. A-72.

		x(n)		
	1	1	0	1
y(n) 1				
2				
3				
4				

(a)

	1	1	0	1
1	(1×1)	(1×1)	(1×0)	(1×1)
-2	(-2×1)	(-2×1)	(-2×0)	(-2×1)
-3	(-3×1)	(-3×1)	(-3×0)	(-3×1)
4	(4×1)	(4×1)	(4×0)	(4×1)

(b)



(c)

Fig. A-72 : Convolution of x (n) and y (− n)

$$\therefore \quad r_{xy}(-3) = 1 \qquad r_{xy}(-2) = -2+1 = -1$$

$$r_{xy}(-1) = -3-2+0 = -5 \qquad r_{xy}(0) = 4-3+0+1 = 2$$

$$r_{xy}(1) = 4+0-2 = 2 \qquad r_{xy}(2) = 0-3 = -3$$

$$\text{and} \quad r_{xy}(3) = 4$$

Thus we get $r_{xy}(l) = x(n) * y(-n) = \{1,\ -1,\ -5,\ \underset{\uparrow}{2},\ 2\ -3,\ 4\}$.

This is the same sequence, we obtained earlier using definition of correlation.

Prob. 2 : Determine the auto-correlation of the following signals :

(i) $x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$ (ii) $y(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$

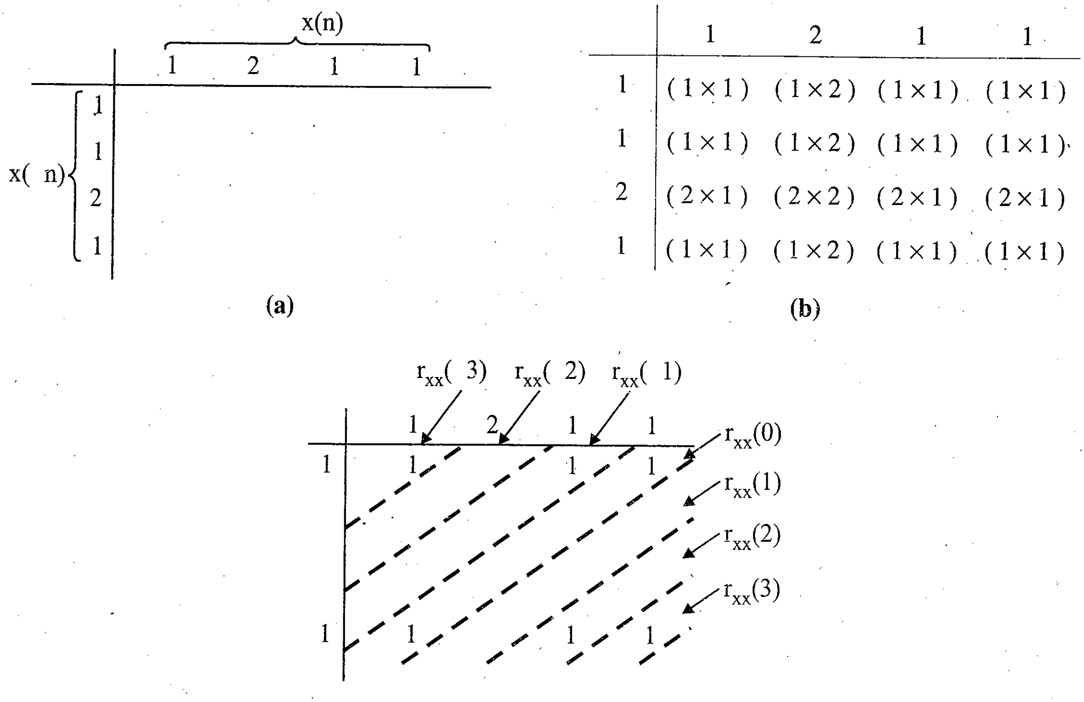
What is your conclusion ?

Soln. : Auto-correlation means correlation of sequence with itself.

(i) Thus $r_{xx}(l) = x(n) * x(-n)$

Here $x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$ and $x(-n) = \{1, 1, 2, \underset{\uparrow}{1}\}$

The convolution $x(n) * x(-n)$ is shown in Fig. A-73.



	1	2	1	1
1	(1×1)	(1×2)	(1×1)	(1×1)
1	(1×1)	(1×2)	(1×1)	(1×1)
2	(2×1)	(2×2)	(2×1)	(2×1)
1	(1×1)	(1×2)	(1×1)	(1×1)

Fig. A-73 : Convolution of $x(n) * x(-n)$

Here the range of output sequence is,

$$\text{Lowest value} = 0 + (-3) = -3$$

$$\text{Highest value} = 3 + 0 = 3$$

Thus output sequence will be from -3 to $+3$

$$\therefore \quad r_{xx}(-3) = 1 \qquad r_{xx}(-2) = 1+2=3$$

$$r_{xx}(-1) = 2+2+1=5 \qquad r_{xx}(0) = 1+4+1+1=7$$

$$r_{xx}(1) = 2+2+1=5 \qquad r_{xx}(2) = 2+1=3$$

$$r_{xx}(3) = 1$$

$$\therefore \quad r_{xx}(l) = \{ r_{xx}(-3), r_{xx}(-2), r_{xx}(-1), \underset{\uparrow}{r_{xx}(0)}, r_{xx}(1), r_{xx}(2), r_{xx}(3) \}$$

$$\boxed{\therefore \; r_{xx}(l) = 1, 3, 5, \underset{\uparrow}{7}, 5, 3, 1}$$

(ii) Here $y(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$ Thus $y(-n) = \{1, 2, 1, \underset{\uparrow}{1}\}$

$$\therefore r_{yy}(l) = y(n) * y(-n)$$

Observe the given sequences. In part (i) we have obtained the convolution of same sequences. Thus the result of this convolution will be same as the previous part.

But the position of arrow in the sequences is changed. We will check the range of output sequence. It is,

$$\text{Lowest value} = 0+(-3) = -3$$

$$\text{Highest value} = 0+3 = 3.$$

Thus the position of arrow in the output sequence will be same.

$$\therefore \quad r_{yy}(l) = \{1, 3, 5, \underset{\uparrow}{7}, 5, 3, 1\}$$

Conclusion :

The auto-correlation sequences are same. That means $r_{xx}(l) = r_{yy}(l)$.

This is because in both convolutions the range of output sequence is same. And the magnitude of two sequences in both cases is same.

Properties of correlation :

Some important properties of auto-correlation and cross-correlation are as follows :

(1) The result of auto-correlation is maximum when signal matches with itself and there is no phase shifting.

(2) Auto-correlation is an even function.

$$\therefore \quad r_{xx}(l) = r_{xx}(-l)$$

(3) The cross-correlation is not commutative

That means $r_{xy}(l) = r_{yx}(-l)$