Lecture - 5

1.2.3 Cyclic Property of Twiddle Factor :

The twiddle factor is denoted by W_N and is given by,

$$W_N = e^{-j\,2\pi/N} \qquad \ldots(1)$$

Now the discrete time sequence x (n) can be denoted by x_N. Here 'N' stands for 'N' point DFT.

While in case of 'N' point DFT; the range of 'n' is from 0 to N – 1. Now the sequence x_N is represented in the matrix form as follows :

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ . \\ x(N-1) \end{bmatrix}_{N \times 1} \qquad \text{...(2)}$$

This is a "N × 1" matrix and 'n' varies from 0 to N – 1. Now the DFT of x (n) is denoted by X (k). We have denoted x (n) by x_N; similarly we can denote X (k) by x_N. In the matrix form X_k can be represented as follows,

$$X_k = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ . \\ . \\ X(N-1) \end{bmatrix}_{N \times 1} \qquad \text{...(3)}$$

This is also "N × 1" matrix and 'k' varies from 0 to N – 1. Now recall the definition of DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(4)}$$

We can also represent W_N^{kn} in the matrix form. Remember that 'k' varies from 0 to N – 1 and 'n' also varies from 0 to N – 1.

$$\therefore \quad W_N^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & \ldots\ldots & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N \times N} \end{array} \qquad \text{...(5)}$$

Note that each value is obtained by taking multiplication of k and n.

For example if k = 2, n = 2, then we get $W_N^{kn} = W_N^4$.

Thus DFT can be represented in the matrix form as,

$$X_N = [W_N] x_N \quad ...(6)$$

Similarly, IDFT can be represented in the matrix form as,

$$x_N = \frac{1}{N}\left[W_N^*\right]X_N \quad ...(7)$$

Here W_N^* is complex conjugate of W_N.

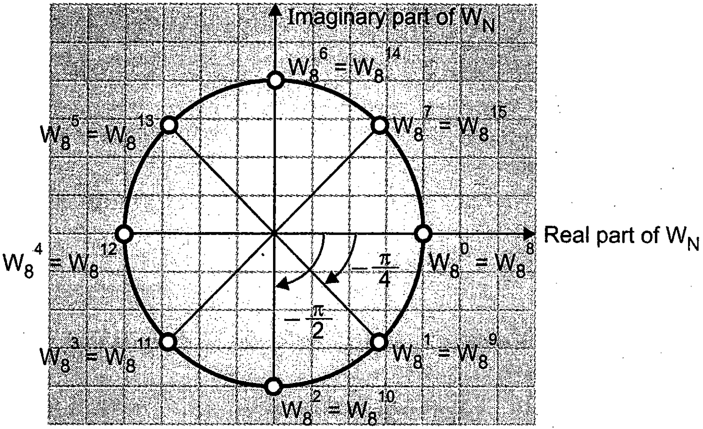

Fig. F-4 : Cyclic property of twiddle factor

1.2.4 Solved Problems on DFT :

Prob. 1 : Determine 2-point and 4-point DFT of a sequence,

$$x(n) = u(n) - u(n-2)$$

Sketch the magnitude of DFT in both the cases.

$$\therefore \quad x(n) = \{1, 1\} \qquad ...(1)$$

Determination of 2-point DFT :

For 2-point DFT, N = 2

We have, $W_N = e^{-j\frac{2\pi}{N}}$ $\qquad \therefore \quad W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$

$$\therefore \quad W_2^{kn} = e^{-j\pi kn} \qquad ...(2)$$

We know that 'n' is from 0 to N – 1. In this case, 'n' is from 0 to 1. Similarly, 'k' is from 0 to N – 1. In this case 'k' is from 0 to 1.

Now the matrix $W_N = W_2^{kn} = e^{-j\pi kn}$ can be written as,

$$W_2^{kn} = \begin{matrix} & n=0 & n=1 \\ k=0 & W_2^0 & W_2^0 \\ k=1 & W_2^0 & W_2^1 \end{matrix} \qquad ...(3)$$

According to Equation (2) we have,

$$W_2^{kn} = e^{-j\pi kn}$$

$$\text{For } kn = 0 \Rightarrow W_2^0 = e^{-j\pi \times 0} = e^0 = 1$$

$$\text{For } kn = 1 \Rightarrow W_2^1 = e^{-j\pi \times 1} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

Putting these values in Equation (3),

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad ...(4)$$

Now given sequence x (n) = {1, 1}. In the matrix form it can be written as,

$$\therefore \quad x_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad ...(5)$$

Now DFT matrix is given by,

$$X_N = [W_N]\, x_N$$

$$\therefore \quad X_N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (1 \times 1) \\ (1 \times 1) + (1 \times -1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus 2-point DFT is,

$$X(k) = \{2, 0\} \qquad ...(6)$$

Determination of 4-point DFT :

For 4-point DFT, N = 4.

We have, $$W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$\therefore \quad W_N^{kn} = e^{-j\frac{\pi}{2}kn} \qquad ...(7)$$

The range of K and n is from 0 to N – 1. That means 0 to 3.

$$[W_4] = W_4^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \qquad ...(8)$$

Now using Equation (7) we get,

$$W_4^0 = e^{-j\frac{\pi}{2}\times 0} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j\frac{\pi}{2}\times 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

According to cyclic property of DFT.

$$W_4^0 = W_4^4 = 1$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

and $$W_4^3 = W_4^7 = W_4^{11} = +j$$

Putting these values in Equation (8) we get,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \text{...(9)}$$

Now given sequence is,

$x(n) = \{1, 1\}$. We want the length of this sequence equal to 4. It is obtained by adding zeros at the end of sequence. This is called as zero padding.

$$\therefore \quad x(n) = \{1, 1, 0, 0\}$$

$$\therefore \quad x_N = x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(10)}$$

Now the DFT is given by, $X_N = [W_N] x_N$

$$\therefore \quad X_N = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X_N = X_4 = \begin{bmatrix} 1+1+0+0 \\ 1-j+0+0 \\ 1-1+0+0 \\ 1+j+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$\therefore \quad X_4 = \{2, \quad 1-j, \quad 0, \quad 1+j\}$$

This DFT sequence can also be written as,

$$X_4 = \{2 + j\,0, 1-j, 0 + j\,0, 1 + j\}$$

$\uparrow$

$k = 0$

Prob. 3 : Calculate 8 point DFT of :

$$x(n) = \{1, 2, 1, 2\}$$

Soln. : First we will make length of given sequence '8' by doing zero padding.

$$\therefore \quad x(n) = \{1, 2, 1, 2, 0, 0, 0, 0\} \quad ...(1)$$

We have, $W_N = e^{-j\frac{2\pi}{N}}$

$$\therefore \quad W_8^{kn} = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}kn} \quad ...(2)$$

Here the range of K and n is from 0 to N – 1 that means from 0 to 7.

Now the matrix W_8^{kn} is as follows,

$$[W_8] = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \\ k=5 \\ k=6 \\ k=7 \end{array} \begin{array}{c} \begin{matrix} n=0 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 \end{matrix} \\ \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \end{array} \quad ...(3)$$

In Table F-1 we have already obtained different values of W_8^{kn}.

$$\therefore \quad W_8^0 = W_8^8 = W_8^{16} = W_8^{24} = W_8^{32} = W_8^{40} = ... = 1$$

$$W_8^1 = W_8^9 = W_8^{17} = W_8^{25} = W_8^{33} = W_8^{41} = W_8^{49} = ... = 0.707 - j\,0.707$$

$$W_8^2 = W_8^{10} = W_8^{18} = W_8^{26} = W_8^{34} = W_8^{42} = ... = -j$$

$$W_8^3 = W_8^{11} = W_8^{19} = W_8^{27} = W_8^{35} = W_8^{43} = ... = -0.707 - j\,0.707$$

$$W_8^4 = W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} = W_8^{44} = ... = -1$$

$$W_8^5 = W_8^{13} = W_8^{21} = W_8^{29} = W_8^{37} = W_8^{45} = ... = -0.707 + j\,0.707$$

$$W_8^6 = W_8^{14} = W_8^{22} = W_8^{30} = W_8^{38} = W_8^{46} = ... = j$$

$$W_8^7 = W_8^{15} = W_8^{23} = W_8^{31} = W_8^{39} = W_8^{47} = ... = 0.707 + j\,0.707$$

Now the DFT is given by,

$$X_8 = [W_8]\,x_n \quad ...(4)$$

Putting values of W_8^{kn} in Equation (3) and write x_n in matrix form we get,

$$X_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707 - j\,0.707 & -j & -0.707 - j\,0.707 & -1 & -0.707 + j\,0.707 & j & 0.707 + j\,0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707 - j\,0.707 & j & 0.707 - j\,0.707 & -1 & 0.707 + j\,0.707 & -j & -0.707 + j\,0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707 + j\,0.707 & -j & 0.707 + j\,0.707 & -1 & 0.707 - j\,0.707 & j & -0.707 - j\,0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707 + j\,0.707 & j & -0.707 + j\,0.707 & -1 & -0.707 - j\,0.707 & -j & 0.707 - j\,0.707 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X_8 = \begin{bmatrix} 1+2+1+2+0+0+0+0 \\ 1+1.414-j\,1.414-j-1.414-j\,1.414+0+0+0+0 \\ 1-j\,2-1+j\,2+0+0+0+0 \\ 1-1.414-j\,1.414+j+1.414-j\,1.414+0+0+0+0 \\ 1-2+1-2+0+0+0+0 \\ 1-1.414+j\,1.414-j+1.414+j\,1.414+0+0+0+0 \\ 1+j\,2-1-j\,2+0+0+0+0 \\ 1+1.414+j\,1.414+j-1.414+j\,1.414+0+0+0+0 \end{bmatrix}$$

$$\therefore \quad X_8 = \begin{bmatrix} 6 \\ 1-j\,2.414 \\ 0 \\ 1-j\,1.828 \\ -2 \\ 1+j\,1.828 \\ 0 \\ 1+j\,3.828 \end{bmatrix}$$

This is the required DFT.

Prob. 4 : Determine the length-4 sequence from its DFT.

$$X(k) = \{4,\ 1-j,\ -2,\ 1+j\}$$

Soln. : The IDFT in matrix form is given by,

$$\text{IDFT} = x(n) = x_N = \frac{1}{N}\left[W_N^*\right] \cdot X_N \quad \text{...(1)}$$

Here X_N is the given DFT matrix. '*' indicates complex conjugate. To obtain the complex conjugate, we have to change the sign of j term. For example, complex conjugate of $1 - j1$ is $1 + j1$.

Now we have already obtained the matrix W_4 in problem (1). It is,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad \text{...(2)}$$

$$\therefore \quad [W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad \text{...(3)}$$

Given matrix of DFT is,

$$X_N = X_4 = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \quad \text{...(4)}$$

Putting Equations (3) and (4), and putting N = 4 in Equation (1) we get,

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\therefore \quad x_N = \frac{1}{4}\begin{bmatrix} 4+1-j-2+1+j \\ 4+j-j^2+2-j-j^2 \\ 4-1+j-2-1-j \\ 4-j+j^2+2+j+j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4+2+1+1 \\ 4-4 \\ 4+2-2 \end{bmatrix} \quad \text{....as } j^2 = -1$$

$$\therefore \quad x_N = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore x(n) = \{1,\ 2,\ 0,\ 1\}$$