Lecture - 6

1.3 Properties of DFT :

In this section we will study some important properties of DFT. We know that, the DFT of discrete time sequence, x(n) is denoted by X(k). And the DFT and IDFT pair is denoted by,

$$x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$$

Here 'N' indicates 'N' point DFT.

1.3.1 Linearity :

Statement : If $x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$ then,

$$a_1 x_1(n) + a_2 x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} a_1 X_1(k) + a_2 X_2(k)$$

Here a_1 and a_2 are some constants.

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad ...(1)$$

Here $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{kn}$$

Since a_1 and a_2 are constants; we can take it out of the summation sign.

$$\therefore \quad X(k) = a_1 \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + a_2 \sum_{n=0}^{N-1} x_2(n) W_N^{kn} \qquad \text{...(2)}$$

Comparing Equation (2) with the definition of DFT,

$$X(k) = a_1 X_1(k) + a_2 X_2(k)$$

Meaning : DFT of linear combination of two or more signals is equal to the sum of linear combination of DFT of individual signal.

1.3.2 Periodicity :

Statement : If $x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$ then

$$x(n+N) = x(n) \qquad \text{for all n}$$

$$\text{and} \quad X(k+N) = X(k) \qquad \text{for all k.}$$

Proof : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \text{...(1)}$$

Replacing k by k + N we get,

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{(k+N)n}$$

$$\therefore \quad X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} W_N^{Nn} \qquad \text{...(2)}$$

We know that W_N is a twiddle factor and it is given by,

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\therefore \quad W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}}\right)^{Nn} = e^{-j\frac{2\pi}{N}\cdot Nn} = e^{-j2\pi n}$$

$$\therefore \quad W_N^{Nn} = \cos 2\pi n - j \sin 2\pi n \qquad \text{...(3)}$$

But 'n' is an integer $\quad \therefore \cos 2\pi n = 1$ and $\sin 2\pi n = 0$

$$\therefore \quad W_N^{Nn} = \text{`}1 \qquad \text{...(4)}$$

Putting this value in Equation (2)

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ...(5)$$

Comparing Equations (1) and (5),

$$\boxed{X(k+N) = X(k)}$$ Hence proved

Meaning : DFT of a finite length sequence results in a periodic sequence.

1.3.3 Circular Symmetries of a Sequence :

We have studied the periodicity property of DFT. Suppose input discrete time sequence is $x(n)$ then, the periodic sequence is denoted by $x_p(n)$. The period of $x_p(n)$ is 'N' which means after 'N' the sequence $x(n)$ repeats itself. Now we can write the sequence $x_p(n)$ as,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \quad ...(1)$$

We will consider one example. Let $x(n) = \{1, 2, 3, 4\}$. This sequence is shown in Fig. F-8(a). The periodic sequence $x_p(n)$ is shown in Fig. F-8(b).

We will delay the periodic sequence $x_p(n)$ by two sample as shown in Fig. F-8(c). This sequence is denoted by $x_p(n-2)$. Now the original signal is present in the range $n = 0$ to $n = 3$. In the same range we will write the shifted signal as shown in Fig. F-8(d). This signal is denoted by $x'(n)$.

Now from Fig. F-8 we can write every sequence as follows :

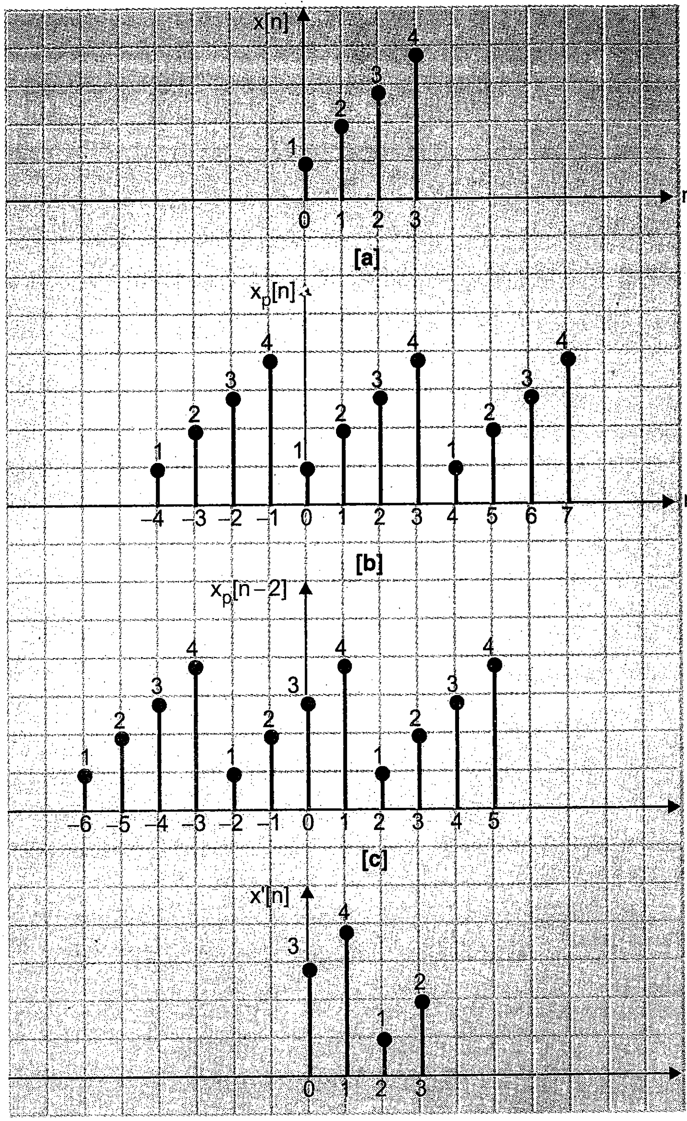
$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \quad ...(2)$$

$$x_p(n) = \{ .... 1, 2, 3, 4, \underset{\uparrow}{1}, 2, 3, 4, 1, 2, 3, 4, ...\} \quad ...(3)$$

$$x_p(n-2) = \{... 1, 2, 3, 4, 1, 2, \underset{\uparrow}{3}, 4, 1, 2, 3, 4, ...\} \quad ....(4)$$

$$\text{and} \quad x'(n) = \{\underset{\uparrow}{3}, 4, 1, 2\} \quad ...(5)$$

Now observe Equations (2) and (5). We can say that the sequence $x'(n)$ is obtained by circularly shifting sequence $x(n)$, by two samples. That means $x'(n)$ is related to $x(n)$ by circular shift.



[d]

Fig. F-8 : Shifting of sequence x (n)

Notation :

This relation of circular shift is denoted by,

$$x'(n) = x(n-k, \text{modulo } N) \qquad \text{...(6)}$$

It means that divide (n – k) by N and retain the remainder only. We can also use the short hand notation as follows :

$$x'(n) = x((n-k))_N \qquad \text{...(7)}$$

Here 'k' indicates the number of samples by which x (n) is delayed and 'N' indicates N-point DFT. In the present example, the sequence x (n) is delayed by two samples; thus k = 2. Since there are four samples in x (n); this is four point DFT. Thus N = 4. Now for this example Equation (6) becomes,

$$x'(n) = x((n-2))_4 \qquad ...(8)$$

Graphical Representation :

The circular shifting of a sequence is plotted graphically as follows :

(1) Circular plot of s equence x (n) : Here we have considered,

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Circular plot of x (n) is denoted by $x((n))_4$. This plot is obtained by writing the samples of x (n) circularly anticlockwise. It is shown in Fig. F-9(a).

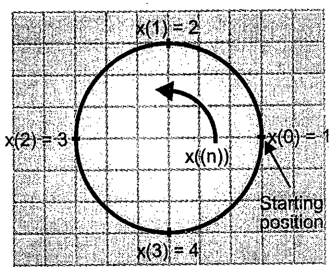


Fig. F-9(a) : $x((n))_4$ – The samples of x (n) are plotted circularly anticlockwise

(2) Circular delay by one sample : To delay sequence x (n) circularly by one sample; shift every sample circularly in anticlockwise direction by 1. This is shown in Fig. F-9(b). This operation is denoted by x ((n – 1)).

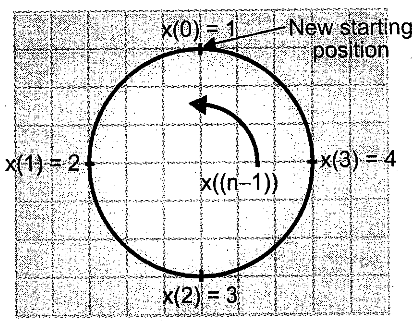


Fig. F-9(b) : x ((n – 1)) shift every sample by 1 in anticlockwise direction

Remember that delay by 'k' samples means shift the sequence circularly in anticlockwise direction by k.

(3) Circular advance by one sample : To advance sequence x (n) circularly by one sample; shift every sample circularly in clockwise direction by 1 sample. This sequence is denoted by x ((n + 1)). It is shown in Fig. F-9(c).

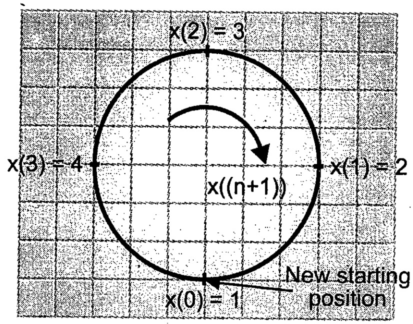


Fig. F-9(c) : x ((n + 1)) shift every sample by one in clockwise direction

Remember that advance by k samples means shift the sequence circularly in clockwise direction by k.

(4) Circularly folded sequence : A circularly folded sequence is denoted by x ((− n)). We have plotted sequence x ((n)) in anticlockwise direction. So folded sequence x ((− n)) is plotted in clockwise direction. It is shown in Fig. F-9(d).

Remember that circular folding means plot the samples in clockwise direction.

Now recall Equation (7) it is,

$$x'(n) = x((n-2))_4$$

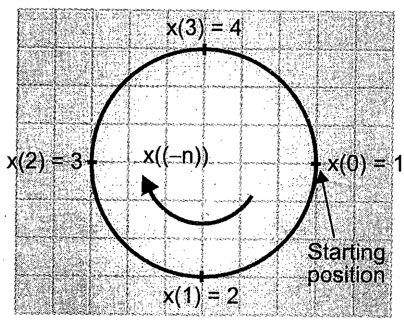


Fig. F-9(d) : x ((− n)) samples are plotted circularly clockwise

It indicates delay of sequence x (n) by two samples. It is obtained by rotating samples of Fig. F-9(a) in anticlockwise direction by two samples. This sequence is shown in Fig. F-9(e).

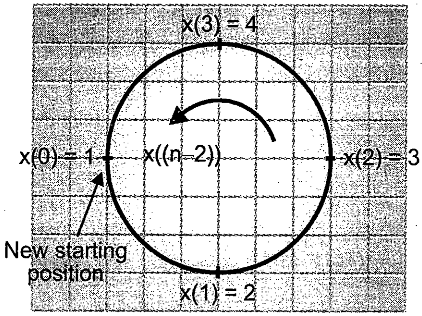


Fig. F-9(e) : Plot of $x((n-2))_4$

(5) Circularly even sequence :

The N-point discrete time sequence is circularly even if it is symmetric about the point zero on the circle.

That means,

$$x(N-n) = x(n), \quad 1 \le n \le N-1$$

Consider the sequence,

$x(n) = \{1, 4, 3, 4\}$. It is plotted as shown in Fig. F-9(f).

Note that this sequence is symmetric about point zero on the circle. So it is circularly even sequence. We can also verify it using mathematical equation,

The sequence is $\quad x(n) = \{1, 4, 3, 4\}$

$\therefore\ x(0) = 1, \quad x(1) = 4, \quad x(2) = 3$ and $\quad x(3) = 4$

We have the condition for circularly even sequence,

$$x(N-n) = x(n) \qquad \ldots(1)$$

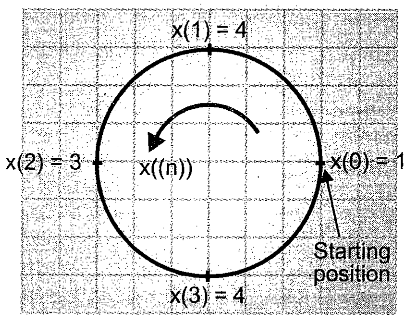


Fig. F-9(f) : $x(n) = \{1, 4, 3, 4\}$

Here N = 4. We will check this condition by putting different values of n as follows :

For $n = 1 \Rightarrow x(4-1) = x(1)$ that means $x(3) = x(1) = 4$

For n = 2 $\Rightarrow$ x (4 – 2) = x (2) that means x (2) = x (2) = 3

For n = 3 $\Rightarrow$ x (4 – 3) = x (3) that means x (1) = x (3) = 4.

Since for all values of 'n', Equation (1) is satisfied; the given sequence is circularly even.

(6) Circularly odd sequence :

A N-point sequence is called circularly odd if it is antisymmetric about point zero on the circle.

That means,

$$x(N-n) = -x(n), \quad 1 \le n \le N-1$$

Consider the sequence, x (n) = {2, – 3, 0, 3}

This sequence is plotted as shown in Fig. F-9.(g)

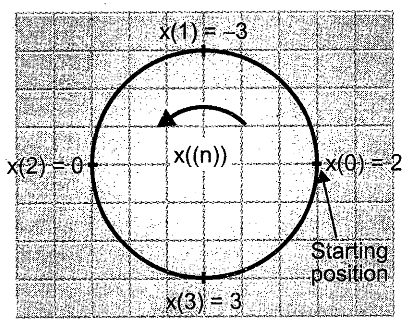


Fig. F-9(g) : Plot of x (n) = {2, – 3, 0, 3}

Here x (0) = 2, x (1) = – 3, x (2) = 0 and x (3) = 3.

We have the condition for circularly odd sequence,

$$x(N-n) = -x(n), \qquad \text{for } 1 \le n \le N-1 \qquad ...(1)$$

For n = 1 $\Rightarrow$ x (4 – 1) = – x (1) that means x (3) = – x (1)

For n = 2 $\Rightarrow$ x (4 – 2) = – x (2) that means x (2) = – x (2)

For n = 3 $\Rightarrow$ x (4 – 3) = – x (3) that means x (1) = – x (3)

Thus for all values of 'n', Equation (1) is satisfied. Hence the sequence is circularly odd.

1.3.7 Circular Convolution :

Statement : The multiplication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematical equation :

$$\text{If} \quad x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k) \text{ and } x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k) \text{ then,}$$

$$x_1(n) \circledN x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k) \cdot X_2(k) \qquad \text{...(1)}$$

Here Ⓝ indicates circular convolution.

Let the result of circular convolution of $x_1(n)$ and $x_2(n)$ be $y(m)$ then the circular convolution can also be expressed as,

$$y(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N, \quad m = 0, 1, \ldots\ldots N-1 \qquad \text{...(2)}$$

Here the term $x_2((m-n))_N$ indicates the circular convolution.

Proof : Consider two discrete time sequences $x_1(n)$ and $x_2(n)$. The DFT of $x_1(n)$ can be expressed as,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, \ldots N-1 \qquad \text{..(3)}$$

To avoid the confusion we will write the DFT of $x_2(n)$ using different index of summation.

$$\therefore \quad X_2(k) = \sum_{l=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kl}{N}}, \quad k = 0, 1, \ldots N-1 \qquad \text{...(4)}$$

Note that in Equation (4); instead of 'n' we have used 'l'.

We will denote the multiplication of two DFTs; $X_1(k)$ and $X_2(k)$ by $Y(k)$.

$$\therefore \quad Y(k) = X_1(k) \cdot X_2(k) \quad \text{...(5)}$$

Let IDFT of $Y(k)$ be $y(m)$. Then using definition of IDFT,

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j 2\pi km}{N}} \quad \text{...(6)}$$

Putting Equation (5) in Equation (6),

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) e^{\frac{j 2\pi km}{N}} \quad \text{...(7)}$$

Putting the values of $X_1(k)$ and $X_2(k)$ from Equations (3) and (4) in Equation (7) we get,

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-\frac{j 2\pi kn}{N}} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-\frac{j 2\pi kl}{N}} \right] e^{\frac{j 2\pi km}{N}} \quad \text{...(8)}$$

Rearranging the summations and terms in Equation (8) we get,

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{\frac{-j2\pi kl}{N}} \cdot e^{\frac{-j 2\pi kn}{N}} \cdot e^{\frac{j 2\pi km}{N}} \right]$$

$$\therefore \quad y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{+j 2\pi k (m-n-l)/N} \right] \quad \text{..(9)}$$

Consider the last term of Equation (9); it can be written as,

$$e^{j 2\pi k (m-n-l)/N} = \left[e^{j 2\pi (m-n-l)/N} \right]^k \quad \text{...(10)}$$

Now use the standard summation formula,

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases} \quad \text{..(11)}$$

$$\text{Let here,} \quad a = e^{+j 2\pi \frac{(m-n-l)}{N}} \quad \text{...(12)}$$

Now according to Equation (11) we will consider two cases :

Case (i) : When a = 1

If $(m-n-l)$ is multiple of N which means,

$(m-n-l) = N, 2N, 3N, \ldots$ then Equation (12) becomes,

$$a = e^{+j2\pi} = e^{+j 2\pi (2)} = e^{+j 2\pi (3)} \ldots = 1$$

Thus when $(m-n-l)$ is multiple of N (that means $a = 1$) then according to Equation (11);

the third summation in Equation (9) becomes equal to N.

Case (ii) : When a ≠ 1 :

If $a \neq 1$ that means if $m - n - l$ is not multiple of N then according to Equation (11),

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a} \qquad ..(13)$$

Putting Equation (12) in Equation (13) we get,

$$\sum_{k=0}^{N-1} \left(e^{+j 2\pi(m-n-l)} \right)^k = \frac{1 - e^{+j 2\pi(m-n-l)}}{1 - e^{\frac{+j 2\pi(m-n-l)}{N}}} \qquad ..(14)$$

Here m, n and l are integers; thus $e^{+j 2\pi(m-n-l)} = 1$ always. Thus R.H.S. of Equation (14) becomes zero when $a \neq 1$. So to get the result of Equation (11) we have to consider the condition $a = 1$. That means when $m - n - l$ is multiple of N. For this condition, we have the result of summation equals to N. Thus Equation (9) becomes,

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N$$

$$\therefore \quad y(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \qquad ...(15)$$

We have obtained Equation (15) for the condition; $(m - n - l)$ is multiple of N. This condition can be expressed as,

$$m - n - l = -pN \qquad ...(16)$$

Here 'p' is an integer and an integer can be positive or negative. For the simplicity we have considered negative integer. Now from Equation (16) we get,

$$l = m - n + pN \qquad ...(17)$$

Putting this value in Equation (15) we get,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m - n + pN) \qquad ...(18)$$

Here we have not considered the second summation of Equation (15). Because this summation is in terms of 'l' and exponential term is absent in Equation (18).

Now the term $x_2(m - n + pN)$ indicates a periodic sequence with period N; this is because 'p' is an integer. This term also indicates that the periodic sequence is delayed by 'n' samples. We have studied that, if a sequence is periodic and delayed then it can be expressed as,

$$x_2(m - n + pN) = x_2((m - n))_N \qquad ...(19)$$

Here the R.H.S. term indicates circular shifting of $x_2(n)$. Putting this value in Equation (18). we get,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N \qquad m = 0, 1, \ldots N-1 \qquad \ldots(20)$$

This is the equation of circular convolution. Since in this equation, the sequence $x_2(n)$ is shifted circularly; this type of convolution is called as CircularConvolution.