Mathematical Modelling

The control systems can be represented with a set of mathematical equations known as mathematical model. These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

The following mathematical models are mostly used.

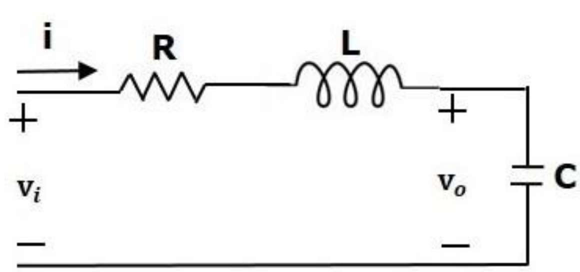
- Differential equation model
- Transfer function model
- State space model

Differential Equation Model

Differential equation model is a time domain mathematical model of control systems. systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

Example: Consider the following electrical system as shown in the following figure. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in series. The input voltage applied to this circuit is and the voltage across the capacitor is the output voltage.



Mesh equation for this circuit is-

$$v_i = Ri + L + di/dt + v_o$$

Substitute, the current passing through capacitor, in the above equation.

$$i = c\frac{\mathrm{d}v_o}{\mathrm{d}t}$$

$$\Rightarrow\ v_i = RC\frac{\mathrm{d}v_o}{\mathrm{d}t} + LC\frac{\mathrm{d}^2 v_o}{\mathrm{d}t^2} + v_o$$

$$\Rightarrow \frac{\mathrm{d}^2 v_o}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}v_o}{\mathrm{d}t} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i$$

The above equation is a second order **differential equation**.

Transfer Function Model

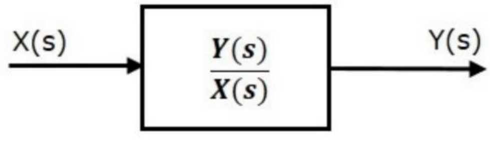
Transfer function model is an s-domain mathematical model of control systems. systems. The Transfer function of a Linear Time Invariant (LTI) system is defined as the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

If x(t) and y(t) are the input and output of an LTI system, then the corresponding Laplace transforms are X(s) and Y(s).

Therefore, the transfer function of LTI system is equal to the ratio of Y(s) and X(s).

i.e. Transfer function = Y(s) / X(s)

The transfer function model of an LTI system is shown in the following figure.



Here, we represented an LTI system with a block having transfer function inside it. And this block has an input X(s) & output Y(s).

Example

Previously, we got the differential equation of an electrical system as

$$\frac{d^2 v_o}{dt^2} + \left(\frac{R}{L}\right)\frac{dv_o}{dt} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i$$

Applying Laplace transform on both sides,

$$s^2 V_o(s) + \left(\frac{sR}{L}\right)V_o(s) + \left(\frac{1}{LC}\right)V_o(s) = \left(\frac{1}{LC}\right)V_i(s)$$

$$\Rightarrow \left\{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}\right\}V_o(s) = \left(\frac{1}{LC}\right)V_i(s)$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$

Where,

- $V_i(s)$ the Laplace transform of the input voltage V_i.
- $V_o(s)$ is the Laplace transform of the output voltage V_o.
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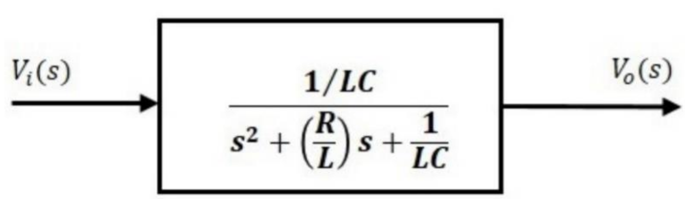
The above equation is a transfer function of the second order electrical system. system. The transfer function model of this system is shown below.



Here, we show a second order electric electrical system with a block having the transfer function inside it. And this block has an input $V_i(s)$ & an output $V_o(s)$.

Modelling of Mechanical Systems

There are two types of mechanical systems based on the type of motion.

- Translational mechanical systems
- Rotational mechanical systems

Modeling of Translational Mechanical Systems

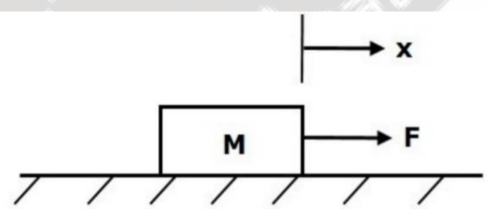
Translational mechanical systems move along a straight line. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero.

Let us now see the force opposed by these three elements individually.

Mass

Mass is the property of a body, which stores kinetic energy. If a force is applied on a body having mass M, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.



$$F_m \propto a$$

$$\Rightarrow F_m = Ma = M\frac{d^2x}{dt^2}$$

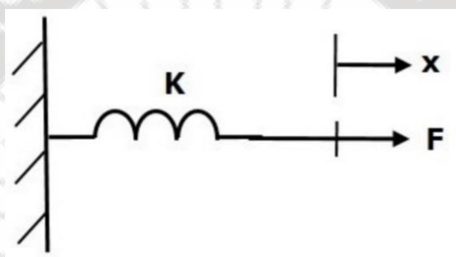
$$F = F_m = M\frac{d^2x}{dt^2}$$

Where,

F is the applied force
$\mathbf{F_m}$ is the opposing force due to masses
M is masses
a is acceleration
x is displacement

Spring

Spring is an element, which stores potential energy. If a force is applied on spring K, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume masses and friction are negligible.



$$F \propto x$$

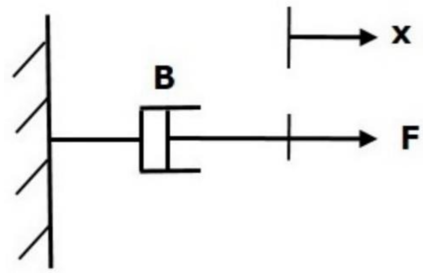
$$\Rightarrow F_k = Kx$$

$$F = F_k = Kx$$

Where,
F is the applied force
$\mathbf{F_k}$ is the opposing force due to elasticity of spring
K is spring constant
x is displacement

Dashpot

If a force is applied on dashpot B, then it is opposed by an opposing force due to friction of the dashpot. This opposing force is proportional to the velocity of the body. Assume masses and elasticity are negligible.



$$F_b \propto \nu$$

$$\Rightarrow F_b = B\nu = B\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$F = F_b = B\frac{\mathrm{d}x}{\mathrm{d}t}$$

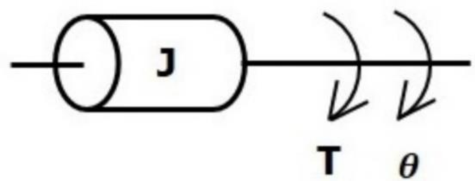
Where,
F_b is the opposing force due to friction off dashpot
B is the frictional coefficient
v is velocity
x is displacement

Modeling of Rotational Mechanical Systems

Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basics elements. Those are moment off inertia, torsional spring and dashpot.
If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero.
Let us now see the torque opposed by these three elements individually.

Moment of Inertia

In translational mechanical system, masses store kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores kinetic energy.
If a torque is applied on a body having moment of inertia J, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.



$$T_j \propto \alpha$$

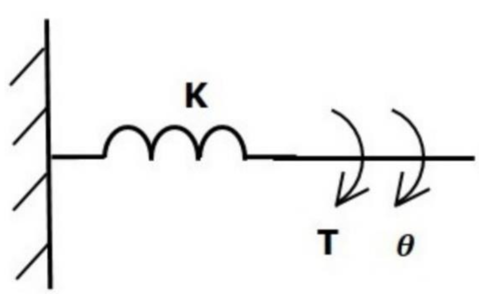
$$\Rightarrow T_j = J\alpha = J\frac{d^2\theta}{dt^2}$$

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

Where,
T is the applied torque
T_j is the opposing torque due to moment off inertia
J is moment off inertia
α is angular acceleration
θ is angular displacement

Torsional Spring

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores potential energy.
If a torque is applied on torsional spring K, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.



$$T_k \propto \theta$$

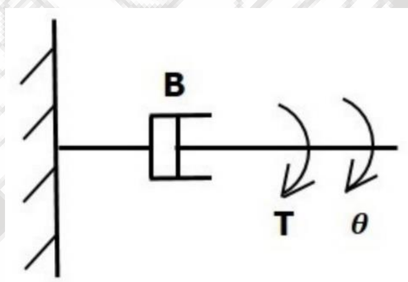
$$\Rightarrow T_k = K\theta$$

$$T = T_k = K\theta$$

Where,
T is the applied torque
T_k is the opposing torque due to elasticity of torsional spring
K is the torsional spring constant
θ is angular displacement

Dashpot

If a torque is applied on dashpot B, then it is opposed by an opposing torque due to the rotational friction of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



$$T_b \propto \omega$$

$$\Rightarrow T_b = B\omega = B\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$T = T_b = B\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Where,
T_b is the opposing torque due to the rotational friction of the dashpot
B is the rotational friction coefficient
ω is the angular velocity
θ is the angular displacement