Lecture - 7

1.3.8 Solved Problems on Circular Convolution :

We can obtain the circular convolution of two sequences using two methods as follows :

A) Using Graphical Method B) Using Matrix Method.

A) Circular convolution using Graphical Method :

Prob. 1 : Given the two sequence of length 4 are :

$$x(n) = \{0, 1, 2, 3\}$$
$$h(n) = \{2, 1, 1, 2\}$$

Find the circular convolution.

Soln. : According to the definition of circular convolution,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N \quad ...(1)$$

Here given sequences are x (n) and h (n). The length of sequence is 4 that means N = 4. Thus Equation (1) becomes,

$$y(m) = \sum_{n=0}^{3} x(n)\, h((m-n))_4 \quad ...(2)$$

Step I : Draw x (n) and h (n) as shown in Fig. F-11 (a) and (b).

Note that x (n) and h (n) are plotted in anticlockwise direction.

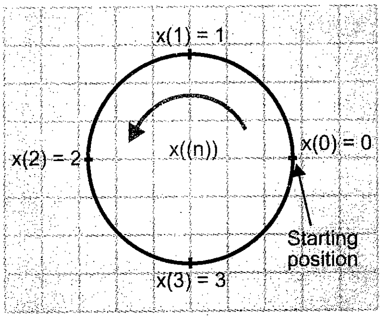


Fig. F-11 (a) : x (n) = {0, 1, 2, 3}

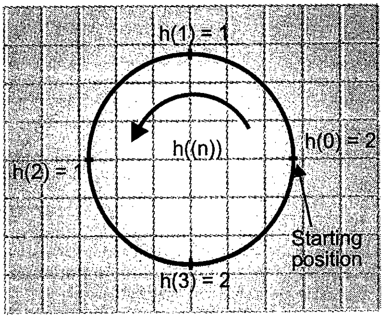


Fig. F-11 (b) : h (n) = {2, 1, 1, 2}

Now we will calculate different values of y (m) by putting m = 0 to m = 3 in Equation (2).

Step II : Calculation of y (0) :

Putting m = 0 in Equation (2) we get,

$$y(0) = \sum_{n=0}^{3} x(n)\, h((-n))_4 \,. \qquad \text{...(3)}$$

Equation (3) shows that we have to obtain the product of x (n) and h ((– n)), and then we have to take the summation of product elements. Using graphical method this calculation is done as follows. The sequence $h((-n))_4$ indicates circular folding of h (n). This sequence is obtained by plotting h (n) in a clockwise direction as shown in Fig. F-11(c).

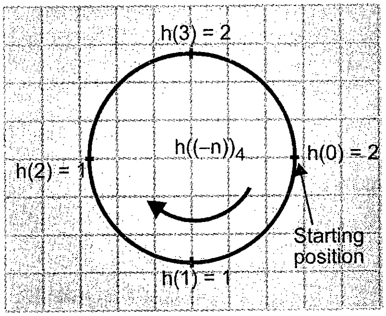


Fig. F-11(c) : h (n) is plotted in clockwise direction

To do the calculations; plot x (n) a nd h ((– n)) on two concentric circles as shown in Fig. F-11(d). x (n) is plotted on the inner circle and h ((– n)) is plotted on the outer circle. Now according to Equation (2); individual values of product x (n) and h ((– n) are obtained by multiplying two sequences point by point. Then y (0) is obtained by adding all product terms.

$$\therefore \quad y(0) = (0\times 2)+(1\times 2)+(1\times 2)+(3\times 1) = 0+2+2+3$$

$$\therefore \quad y(0) = 7$$

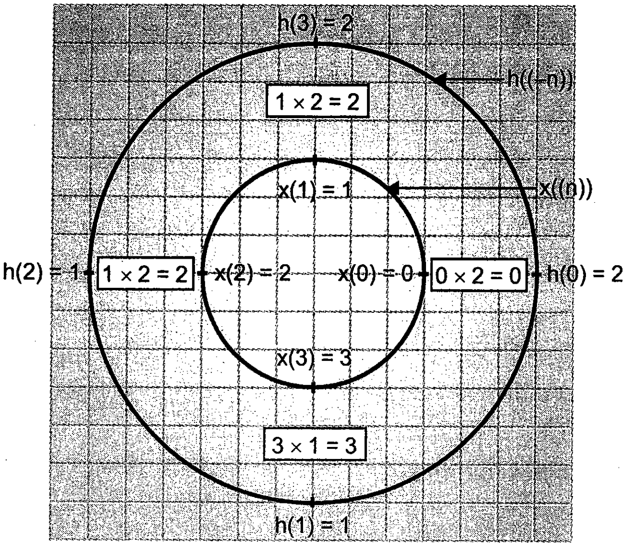


Fig. F-11(d) : $\sum_{n=0}^{3} x(n)\, h((-n))_4$

Step III : Calculation of y (1) : Putting m = 1 in Equation (2),

$$y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4 \qquad ...(4)$$

Here $h((1-n))_4$ is same as $h((-n+1))_4$. This indicates delay of $h((-n))$ by 1 sample. This is obtained by shifting $h(-n))$ in anticlockwise direction by 1 sample, as shown in Fig. F-11(e).

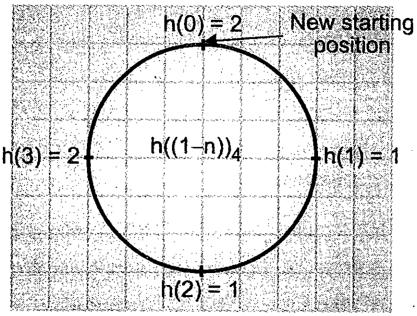


Fig. F-11(e) : $h((-n+1))_4$

We have already drawn the sequence x (n) as shown in Fig. F-11(a). To do the calculations, according to Equation (4), two sequences x (n) and $h((1-n))_4$ are plotted on two concentric circles as shown in Fig. F-11(f). y (1) is obtained by adding the product of individual terms.

$$\therefore \quad y(1) = (0\times1)+(3\times1)+(2\times2)+(1\times2) = 0+3+4+2$$

$$\therefore \quad y(1) = 9$$

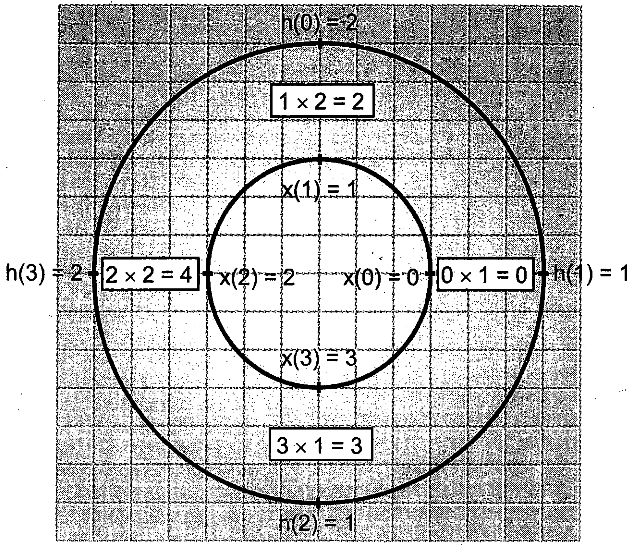


Fig. F-11(f) : $y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4$

Step IV : Calculation of y (2) : Putting m = 2 in Equation (2) we get,

$$y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4 \qquad ..(5)$$

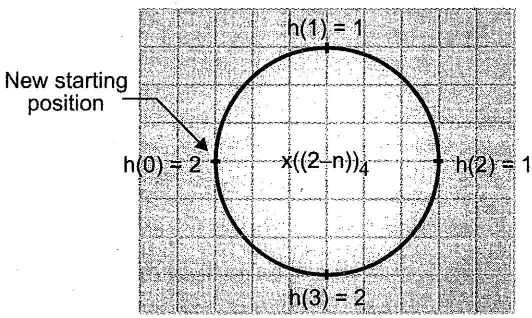


Fig. F-11(g) : $h((-n+2))_4$

Here $h((2-n))_4$ is same as $h((-n+2))_4$. It indicates delay of $h((-n))_4$ by 2 samples. It is obtained by shifting $h((-n))_4$ by two samples in anticlockwise direction as shown in Fig. F-11(g).

According to Equation (4) the value of y (2) is obtain by adding individual product terms as shown in Fig. F-11(h).

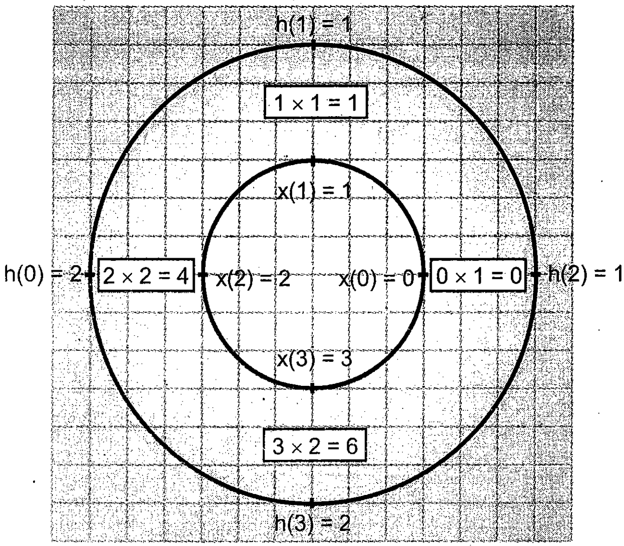


Fig. F-11(h) : $y(2) = \sum_{n=0}^{3} x(n)\,h((2-n))_4$

$$\therefore \quad y(2) = (0\times1)+(3\times2)+(2\times2)+(1\times1) = 0+6+4+1$$

$$\therefore \quad y(2) = 11$$

Step V : Calculation of y (3) :

Putting m = 3 in Equation (2) we get,

$$y(3) = \sum_{n=0}^{3} x(n)\,h((3-n))_4 \qquad \text{...(6)}$$

Here $h((3-n))_4$ is same as $h((-n+3))_4$. It indicates delay of $h((-n))_4$ by 3 samples. It is obtained by shifting $h((-n)_4$ by 3 samples in anticlockwise direction as shown in Fig. F-11(i).

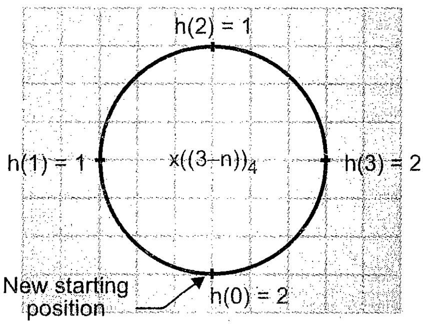


Fig. F-11(i) : $h((-n+3))_4$

According to Equation (5), y (3) is obtained by adding individual product terms as shown in Fig. F-11(j).

$$\therefore \quad y(3) = (0\times2)+(3\times2)+(2\times1)+(1\times1) = 0+6+2+1$$

$$\therefore \quad y(3) = 9$$

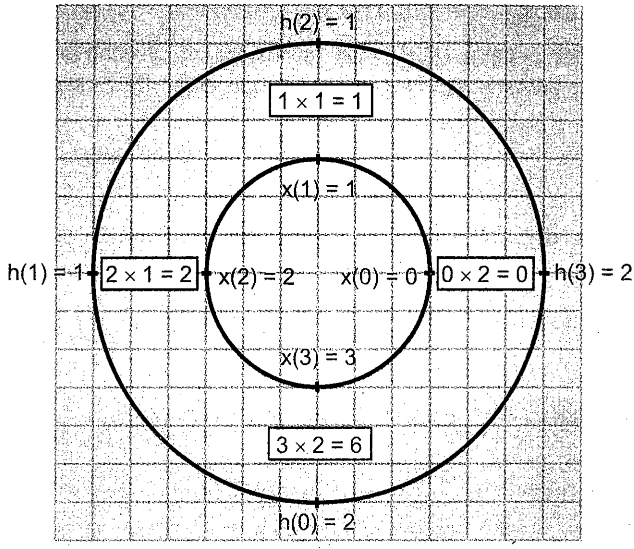


Fig. F-11(j) : $y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4$

Now the resultant sequence y (m) can be written as,

$$y(m) = \{y(0), y(1), y(2), y(3)\}$$

$$\therefore \; y(m) = \{7, 9, 11, 9\}$$

Prob. 2 : Using graphical method, obtain a 5-point circular convolution of two DT signals defined as,

$$x(n) = (1.5)^n, \qquad 0 \le n \le 2$$
$$y(n) = 2n - 3, \qquad 0 \le n \le 3$$

Does the circular convolution obtained is same to that of linear convolution ?

Soln. : First we will obtain the sequences x (n) and y (n) by putting values of n as follows :

Given, $x(n) = (1.5)^n, \quad 0 \le n \le 2$

For $n = 0 \Rightarrow x(0) = (1.5)^0 = 1$

For $n = 1 \Rightarrow x(1) = (1.5)^1 = 1.5$

For $n = 2 \Rightarrow x(2) = (1.5)^2 = 2.25$

$$\therefore \; x(n) = \{1, 1.5, 2.25\} \qquad \text{...(1)}$$

Now $y(n) = 2n - 3, \; 0 \le n \le 3$

For $n = 0 \Rightarrow y(0) = 0 - 3 = -3$

For $n = 1 \Rightarrow y(1) = 2 - 3 = -1$

For $n = 2 \Rightarrow y(2) = 4 - 3 = 1$

For $n = 3 \Rightarrow y(3) = 6 - 3 = 3$

$$\therefore \quad y(n) = \{-3, -1, 1, 3\} \qquad ...(2)$$

It is asked to calculate 5-point DFT. That means length of each sequence should be 5. This length is adjusted by adding zeros at the end of each sequence as follows (zero padding) :

$$x(n) = \{1, 1.5, 2.25, 0, 0\} \qquad ...(3)$$

$$\text{and} \quad y(n) = \{-3, -1, 1, 3, 0\} \qquad ...(4)$$

Now according to the definition of circular convolution we have,

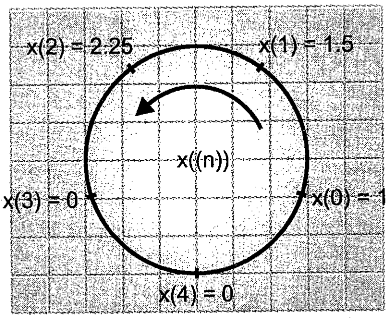
$$y(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad ...(5)$$

Here the given sequences are x (n) and y (n) and length N = 5.

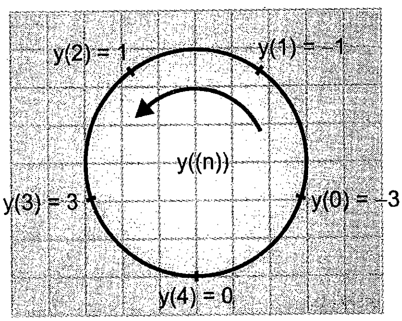
$$\therefore \quad y(m) = \sum_{n=0}^{4} x(n)\, y((m-n))_5 \qquad ...(6)$$

Step I : Draw x (n) and y (n) as shown in Fig. F-12(a) and (b) respectively.

Here x (n) and y (n) are plotted in anticlockwise direction.



(a) x (n) = {1, 1.5, 2.25, 0, 0}



(b) y (n) = {– 3, – 1, 1, 3, 0}

Fig. F-12

Now we will obtain values of y (m) by putting m = 0 to m = 4 in Equation (6).

Step II : Calculation of y (0) :

Putting m = 0 in Equation (6) we get, $$y(0) = \sum_{n=0}^{4} x(n)\, y((-n))_5 \quad \text{...(7)}$$

Here $y((-n))$ indicates circular 'folding of $y(n)$. It is obtained by plotting $y(n)$ in clockwise direction as shown in Fig. F-12(c).

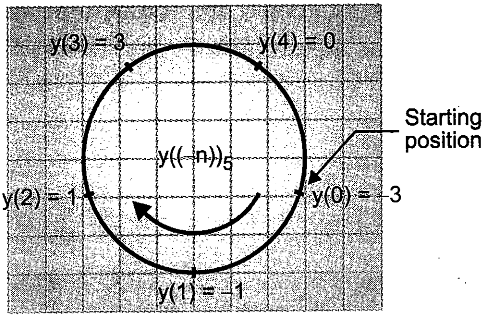


Fig. F-12(c) : y ((– n))

According to Equation (7), $y(0)$ is obtained by adding all product terms as shown in Fig. F-12(d). Here $x(n)$ and $y((-n))_5$ is drawn on two concentric circles.

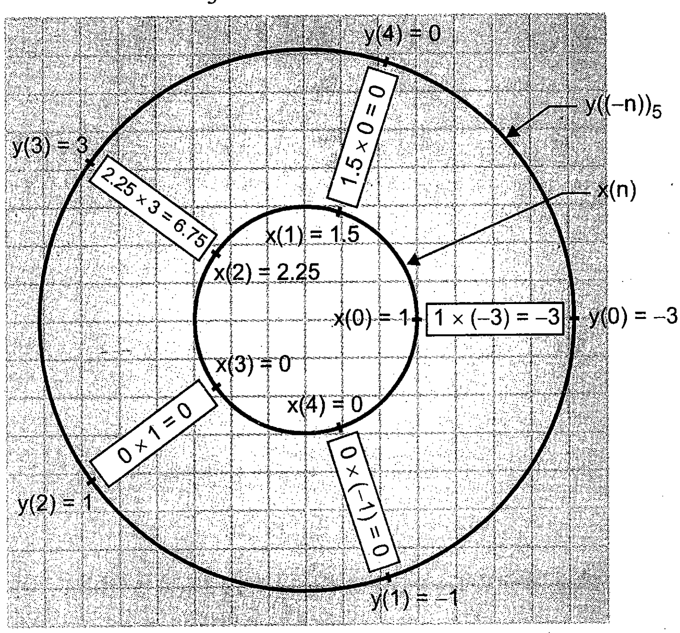


Fig. F-12(d) : $y(0) = \sum_{n=0}^{4} x(n)\, y((-n))$

$$\therefore \quad y(0) = [1 \times (-3)] + [0 \times (-3)] + [0 \times 1] + [2.25 \times 3] + [1.5 \times 0]$$
$$= 3 + 0 + 0 + 6.75 + 0$$

$\therefore\ y(0) = 3.75$

Step III : Putting m = 1 in Equation (6),

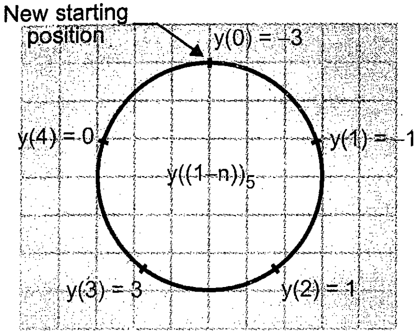


Fig. F-12(e) : $y((1-n))_5$

$$y(1) = \sum_{n=0}^{4} x(n)\, y((1-n))_5 \qquad ...(8)$$

Here $y((1-n))_5$ indicates delay of $y((-n))$ by 1 sample. It is obtained by shifting $y((-n))$ by 1 sample in anticlockwise direction as shown in Fig. F-12(e).

According to Equation (8), value of y (1) is obtained a shown in Fig. F-12(f).

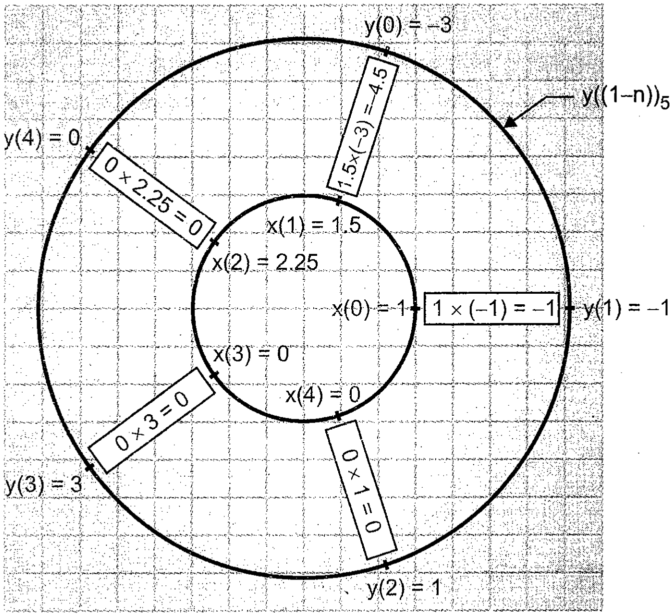


Fig. F-12(f) : $y(1) = \sum_{n=0}^{4} x(n)\, y((1-n))_5$

$$\therefore \quad y(1) = [1 \times (-1)] + [0 \times 1] + [0 \times 3] + [0 \times 2.25] + [1.5 \times (-3)] = -1 + 0 + 0 + 0 - 4.5$$

$$\therefore \quad y(1) = -5.5$$

Step IV : Putting m = 2 in Equation (6),

$$y(2) = \sum_{n=0}^{4} x(n)\, y((2-n))_5 \qquad ...(9)$$

Here $y((2-n))_5$ is delay of $y((-n))_5$ by 2 samples. It is obtained by shifting $y((-n))_5$ by 2 samples in anticlockwise direction as shown in Fig. F-12(g).

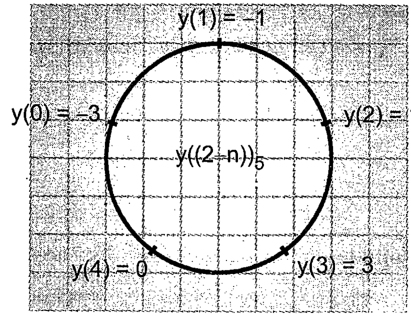


Fig. F-12(g) : $y((2-n))_5$

y (2) is obtained by adding all product terms as shown in Fig. F-12(h).

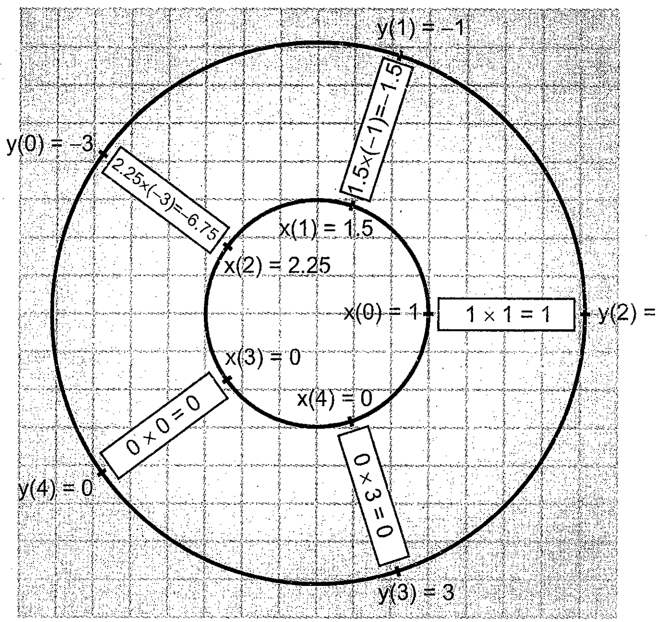


Fig. F-12(h) : $y(2) = \sum_{n=0}^{4} x(n)\, y((2-n))_5$

$\therefore \quad y(2) = [1 \times 1] + [0 \times 3] + [0 \times 0] + [2.25 \times (-3)] + [1.5 \times (-1)] = 1 + 0 + 0 - 6.75 - 1.5$

$\therefore \quad y(2) = -7.25$

Step V : Calculation of **y (3)** :

Putting m = 3 in Equation (6) we get,

$$y(3) = \sum_{n=0}^{4} x(n)\, y((3-n))_5 \qquad ...(10)$$

Here $y((3-n))_5$ is delay of $y((-n))_5$ by 3 samples. It is obtained by shifting $y((-n))_5$ by 3 samples in anticlockwise direction as shown in Fig. F-12(i).

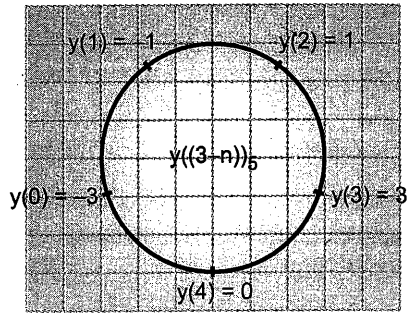


Fig. F-12(i) : $y((3-n))_5$

y (3) is obtained by adding all product terms as shown in Fig. F-12(j).

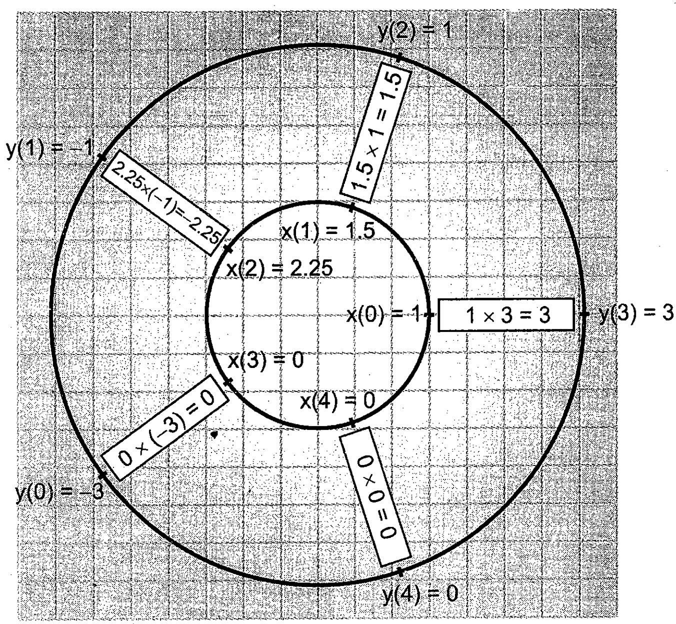


Fig. F-12(j) : $y(3) = \sum_{n=0}^{4} x(n)\, y((3-n))_5$

$$\therefore \quad y(3) = [1\times 3]+[0\times 0]+[0\times(-3)]+[2.25\times(-1)]+[1.5\times 1]$$

$$= 3+0+0-2.25+1.5$$

$$\therefore\ y(3) = 2.25$$

Step VI : Calculation of y (4) :

Putting m = 4 in Equation (6) we get,

$$y(4) = \sum_{n=0}^{4} x(n)\, y((4-n))_5 \qquad ...(11)$$

Here $y((4-n))_5$ is delay of $y((-n))_5$ by 4 samples. It is obtained by shifting $y((-n))_5$ by 4 samples in anticlockwise direction as shown in Fig. F-12(k).

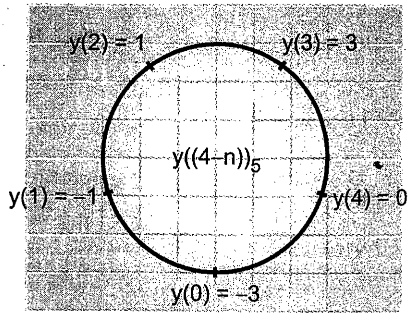


Fig. F-12(k) : $y((4-n))_5$

y (4) is obtained as shown in Fig. F-12(*l*)

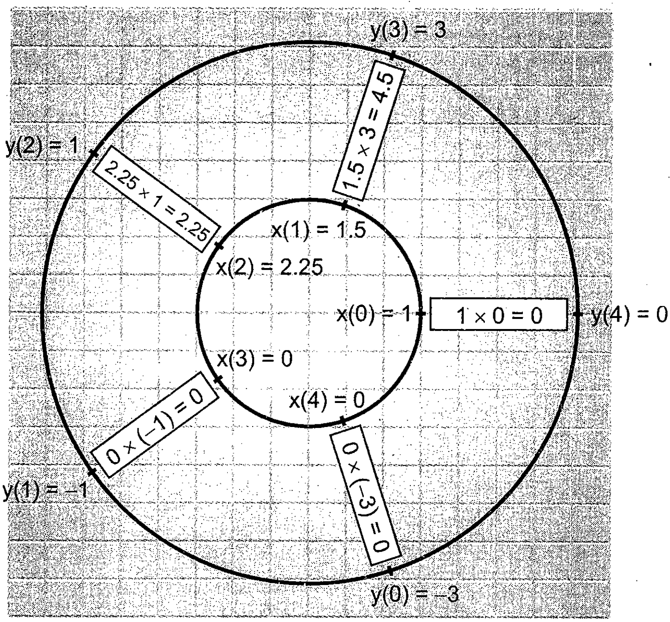


Fig. F-12(*l*) : $y(4) = \sum_{n=0}^{4} x(n)\, y((4-n))_5$

$$\therefore \quad y(4) = [1 \times 0] + [0 \times (-3)] + [0 \times (-1)] + [2.25 \times 1] + [1.5 \times 3]$$

$$= 0 + 0 + 0 + 2.25 + 4.5$$

$\therefore y(4) = 6.75$

Now the result of circular convolution can be expressed as,

$$y(m) = \{y(0), y(1), y(2), y(3), y(4)\}$$

$$\therefore \quad y(m) = \{3.75, -5.5, -7.25, 2.25, 6.75\} \quad ...(12)$$

A) Comparison with linear convolution :

We will obtain linear convolution of two sequences using tabular method. We have,

$$x(n) = \{1, 1.5, 2.25\} = \{1, 1.5, 2.25, 0\}$$

$$\text{and} \quad y(n) = \{-3, -1, 1, 3\}$$

$$\text{Let} \quad y_1(n) = x(n) * y(n)$$

The linear convolution of x (n) and y (n) is shown in Fig. F-12(m).

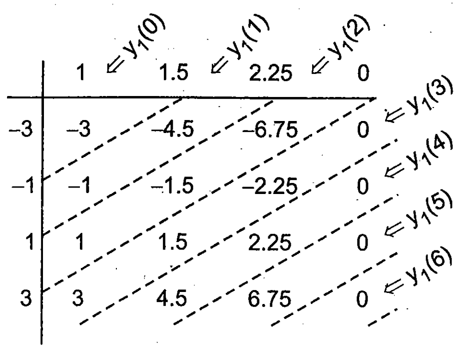


Fig. F-12(m) : x (n) * y (n)

From Fig. F-12(m),

$$y_1(0) = -3$$

$$y_1(1) = -1 - 4.5 = -5.5$$

$$y_1(2) = 1 - 1.5 - 6.75 = -7.25$$

$$y_1(3) = 3 + 1.5 - 2.25 = 2.25$$

$$y_1(4) = 4.5 + 2.25 = 6.75$$

$$y_1(5) = 6.75 + 0 = 6.75$$

$$y_1(6) = 0$$

$$\text{Thus} \quad x(n) * y(n) = \{-3, -5.5, -7.25, 2.25, 6.75, 6.75, 0\} \quad ..(13)$$

Equations (12) and (13) show that circular convolution and linear convolution are not same.

B) Circular convolution using matrix method :

The graphical method which we have studied is tedious, especially when many samples are present. While the matrix method is more convenient. In the matrix method, one sequence is repeated via circular shifting of samples. It is represented as follows :

we have $y(m) = x(n) \textcircled{N} h(n) = h(n) \textcircled{N} N x(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ y(N-2) \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h(0) & h(N-1) & h(N-2) & \ldots & h(2) & h(1) \\ h(1) & h(0) & h(N-1) & \ldots & h(3) & h(2) \\ h(2) & h(1) & h(0) & \ldots & h(4) & h(3) \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ h(N-2) & h(N-3) & h(N-4) & \cdots & h(0) & h(N-1) \\ h(N-1) & h(N-2) & h(N-3) & \cdots & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ \cdot \\ x(N-2) \\ x(N-1) \end{bmatrix}$$

Prob. 3 : Determine the sequence

$$y(n) = x(n) \textcircled{N} h(n)$$

where $x(n) = \{1, 2, 3, 1\}$
$\uparrow$

and $h(n) = \{4, 3, 2, 2\}$
$\uparrow$

Soln. :

We have, $y(m) = x_2(n) \textcircled{N} x_1(n) = h(n) \textcircled{N} x(n)$

Using matrix method,

Here $x(0) = 1$, $x(1) = 2$, $x(3) = 3$, $x(3) = 1$

and $h(0) = 4$, $h(1) = 3$, $h(2) = 2$, $h(3) = 2$

Here $N = 4$

In the matrix form we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} h(0) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(3) \\ h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & 3 \\ 3 & 4 & 2 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} (4\times1)+(2\times2)+(2\times3)+(3\times1) \\ (3\times1)+(4\times2)+(2\times3)+(2\times1) \\ (2\times1)+(3\times2)+(4\times3)+(2\times1) \\ (2\times1)+(2\times2)+(3\times3)+(4\times1) \end{bmatrix} = \begin{bmatrix} 4+4+6+3 \\ 3+8+6+2 \\ 2+6+12+2 \\ 2+4+9+4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

$$\therefore \quad y(m) = x(n) \, \textcircled{N} \, h(n) = \{17, 19, 22, 19\}$$