Lecture - 9

1.3.9 Time Reversal of Sequence :

Statement : If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$

then $x((-n))_N = x(N-n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k)$

Proof : According to the definition of DFT,

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j\,2\pi kn/N} \qquad ..(1)$$

$$\therefore \quad DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j\,2\pi kn/N} \qquad ...(2)$$

put $l = N - n$, the limits will change as follows :

when $n = 0 \Rightarrow l = N - 0 \quad \therefore \quad l = N$

and when $n = N - 1 \Rightarrow l = N - N + 1 \quad \therefore \quad l = 1$

Thus Equation (2) becomes,

$$\text{DFT}\{x(N-n)\} = \sum_{l=N}^{1} x(l)\, e^{-j2\pi k(N-l)/N} \quad ...(3)$$

Here $x((-n))_N$ indicates circularly folded sequence. It can also be represented as $x(N-n)$. That means the sequence $x(N-n)$ is circular in nature and we know that the DFT is periodic. As given by Equation (2); the original limits of summation are from $n = 0$ to $N-1$. That means here summation is calculated for the period 'N'. Since the DFT is periodic in nature; if we calculate the DFT for next period then the result remains same. Now the next period is, $n = 0 + N$ to $n = N - 1 + N$. That means $n = N$ to $2N - 1$. But the sequence is circular; so this period is same as $n = N$ to $n = 1$. Thus even if we change the index; the limits of summation will remain same.

Basic limits of DFT are 0 to $N-1$ as per Equation (2).

$$\therefore \quad \text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi k(N-l)/N}$$

$$= \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi k} \cdot e^{(j2\pi kl)/N} \quad ...(4)$$

Now we have,

$$e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k$$

Since k is an integer, $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

$$\therefore \quad e^{-j2\pi k} = 1 \quad ...(5)$$

Putting this value in Equation (4),

$$\text{DFT}\{x(N-n)\} = \sum_{l=0}^{N-1} x(l)\, e^{j2\pi kl/N} \quad ...(6)$$

Similar to Equation (5) we can write,

$$e^{-j2\pi l} = 1 \qquad \therefore \quad e^{\frac{-j2\pi lN}{N}} = 1 \quad ...(7)$$

We can multiply R.H.S. of Equation (6) by Equation (7); since its value is 1.

$$\therefore \quad \text{DFT}\left\{x(N-n)\right\} = \sum_{l=0}^{N-1} x(l)\, e^{j2\pi kl/N} \cdot e^{-j2\pi\frac{lN}{N}}$$

$$\therefore \quad \text{DFT}\left\{x(N-n)\right\} = \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi l(N-k)/N} \quad ...(8)$$

Now according to the definition of DFT; R.H.S. of Equation (8) is $X(N-k)$

$$\therefore \text{DFT}\{x(N-n)\} = X(N-k) = X((-k))_N$$

Meaning : If a sequence is circularly folded; its DFT is also circularly folded.

1.3.10 Circular Time Shift of Sequence :

Statement : If $x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$ then,

$$x((n-l))_N \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)\, e^{-j2\pi kl/N}$$

$$\text{or} \quad x((n-l))_N \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

Proof : According to the definition of IDFT,

$$x(n) = \text{IDFT}\{X(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad \text{...(1)}$$

$$\therefore \text{IDFT}\left\{X(k)\, W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \cdot W_N^{kl}$$

$$\therefore \text{IDFT}\left\{X(k)\, W_N^{kl}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-k(n-l)} \qquad \text{...(2)}$$

Now we have,

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k) \qquad \text{...(3)}$$

Comparing R.H.S. of Equations (1) and (2) we can write,

$$x(n-l) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)\, W_N^{kl}$$

The sequence is circular and DFT is periodic in nature so we can write,

$$x((n-l))_N \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)\, W_N^{kl} \qquad \text{..(4)}$$

Hence the proof.

Meaning : Shifting the sequence in time domain by 'l' samples is equivalent to multiplying the sequence in frequency domain by W_N^{kl} or $e^{-j2\pi kl/N}$.

1.3.11 Circular Frequency Shift :

This property is also called as Quadrature Modulation Theorem.

Statement : If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$ then,

$$x(n)\, e^{j2\pi ln/N} \underset{N}{\overset{DFT}{\longleftrightarrow}} X((k-l))_N = X(k+l)$$

OR

$$x(n)\, e^{-j2\pi ln/N} \underset{N}{\overset{DFT}{\longleftrightarrow}} X((k+l))_N = X(k-l)$$

Meaning : Multiplication of sequence $x(n)$ by $e^{\pm j2\pi kl/N}$ is equivalent to the circular shift of DFT in time domain by 'l' samples.

1.3.12 Solved Examples using Circular Properties of DFT :

Prob. 1 : A four point sequence $x(n) = \{1, 2, 3, 4\}$ has DFT $X(k)$, $0 \le k \le 3$, without performing DFT or IDFT. Find the signal values which has DFT $X(k-1)$.

Soln. : According to the circular frequency shifting property,

$$x(n) \cdot e^{-j2\pi ln/N} \underset{N}{\overset{DFT}{\longleftrightarrow}} X((k+l))_N = X(k-l)$$

Here $l = 1$. Let the signal whose DFT is $X(k-1)$ be denoted by $x_1(n)$.

$$\therefore \quad x_1(n) = x(n)\, e^{-j2\pi \cdot 1 \cdot n/4}$$

Since $N = 4$ in this case.

The given sequence is $x(n) = \{1, 2, 3, 4\}$

$$\therefore \quad x(0) = 1, \quad x(1) = 2, \quad x(2) = 3 \quad \text{and} \quad x(3) = 4$$

We will find the sequence $x_1(n)$ as follows :

For $n = 0 \Rightarrow x_1(0) = x(0) \cdot e^0 = 1$

For $n = 1 \Rightarrow x_1(1) = x(1)\, e^{-\frac{j2\pi}{4}} = 2\, e^{-\frac{j\pi}{2}} = 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$

$$\therefore \quad x_1(1) = -2j$$

For $n = 2 \Rightarrow x_1(2) = x(2)\, e^{-\frac{j4\pi}{4}} = 3\, e^{-j\pi} = 3[\cos\pi - j\sin\pi]$

$$\therefore \quad x_1(2) = -3$$

For $\quad n = 3 \Rightarrow x_1(3) = x(3)\, e^{-\frac{j6\pi}{4}} = 4\, e^{-\frac{j3\pi}{2}} = 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$$\therefore \quad x_1(3) = 4j$$

$$\therefore \quad x_1(n) = \{1, -2j, -3, 4j\}$$

Prob. 2 : Consider a real finite length sequence,

$$x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$$

(i) $y(n)$ is a sequence related to $x(n)$ such that,

$Y(k) = W_8^{4k} X(k)$ where $X(k)$ is 8 point DFT of $x(n)$. Obtain $y(n)$.

(ii) Also obtain finite length sequence $q(n)$ related to $x(n)$ such that its 8 point DFT is $Q(k) = R_e\{X(k)\}$.

Soln. :

(i) Given $\quad Y(k) = W_8^{4k} X(k) \quad$ ...(1)

According to circular time shifting property,

$$x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)\, W_N^{kl} \quad ...(2)$$

Here $\quad N = 8 \quad$ and $\quad l = 4$

Thus comparing Equations (1) and (2).

$$y(n) = x((n-4))_8$$

That means $y(n)$ represents circular delay of sequence $x(n)$ by 4 samples. It is represented in Fig. F-17(a).

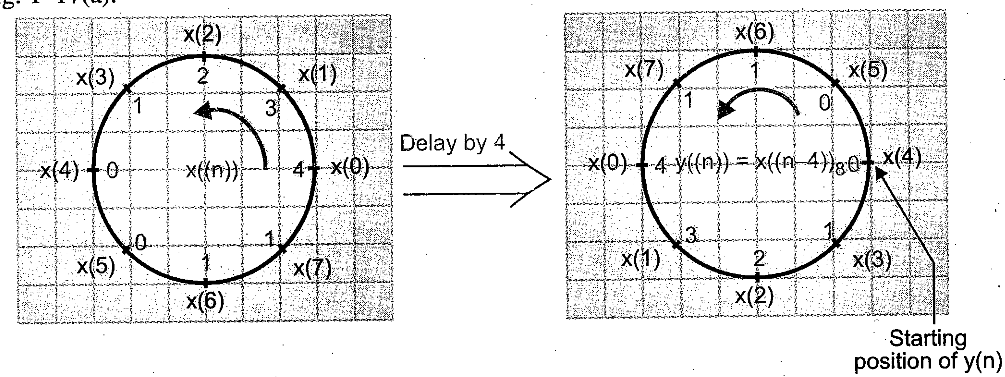

Fig. F-17(a)

$$\therefore \quad y(n) = \{0, 0, 1, 1, 4, 3, 2, 1\}$$

(ii) Given, $\quad Q(k) = R_e\{X(k)\} \quad$ ...(3)

Let $\quad X(k) = M(k) + j\,N(k) \quad$ ...(4)

Thus M (k) represents real part of X (k).

$$\text{Now} \quad X^*(k) = M(k) - jN(k) \quad \text{...(5)}$$

Adding Equations (4) and (5),

$$X(k) + X^*(k) = 2M(k)$$

$$\therefore \quad M(k) = \frac{X(k) + X^*(k)}{2} \quad \text{...(6)}$$

Taking IDFT of both sides,

$$m(n) = \frac{x(n) + x^*(-n)}{2} \quad \text{...(7)}$$

$$\text{As} \quad x^*(-n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X^*(k)$$

$$\text{We have,} \quad x(n) = \{4, 3, 2, 1, 0, 0, 1, 1\} \quad \text{...(8)}$$

$$\therefore \quad x^*(n) = \{4, 3, 2, 1, 0, 0, 1, 1\}$$

$X^*(-n)$ represents circular folding of $x^*(n)$. It is shown in Fig. F-17(b).

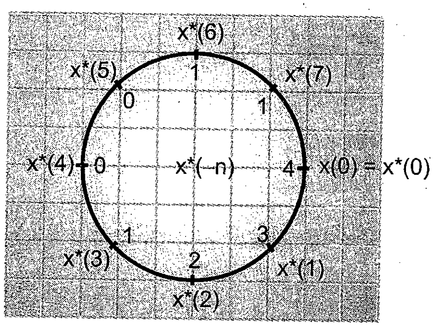

Fig. F-17(b)

$$\therefore \quad x^*(-n) = \{4, 1, 1, 0, 0, 1, 2, 3\} \quad \text{...(9)}$$

Putting Equations (8) and (9) in Equation (7) we can find sequence m (n) as follows :

$$\text{For} \quad n = 0 \Rightarrow m(0) = \frac{x(0) + x^*(0)}{2} = \frac{4+4}{2} = 4$$

$$\text{For} \quad n = 1 \Rightarrow m(1) = \frac{x(1) + x^*(1)}{2} = \frac{3+1}{2} = 2$$

$$\text{For} \quad n = 2 \Rightarrow m(2) = \frac{x(2) + x^*(2)}{2} = \frac{2+1}{2} = \frac{3}{2}$$

$$\text{For} \quad n = 3 \Rightarrow m(3) = \frac{x(3) + x^*(3)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\text{For} \quad n = 4 \Rightarrow m(4) = \frac{x(4) + x^*(4)}{2} = \frac{0+0}{2} = 0$$

$$\text{For} \quad n = 5 \Rightarrow m(5) = \frac{x(5) + x^*(5)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$\text{For} \quad n = 6 \Rightarrow m(6) = \frac{x(6) + x^*(6)}{2} = \frac{1+2}{2} = \frac{3}{2}$$

$$\text{For} \quad n = 7 \Rightarrow m(7) = \frac{x(7) + x^*(7)}{2} = \frac{1+3}{2} = 2$$

$$\therefore \quad m(n) = q(n) = \left\{ 4, 2, \frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2 \right\}$$

1.3.13 Multiplication of Two Sequences :

Statement : If $x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$ then,

$$x_1(n) \cdot x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \frac{1}{N} \left[X_1(k) \text{ \textcircled{N} } X_2(k) \right]$$

Meaning : The multiplication of two sequences in time domain is equivalent to its circular convolution in the frequency domain.

1.3.14 Circular Correlation :

Statement : If $x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$ and $y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$ then,

$$r_{xy}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} R_{xy}(k) = X(k)\, Y^*(k)$$

Meaning : The circular crosscorrelation of two sequences in time domain is equivalent to the multiplication of DFT of one sequence with the complex conjugate DFT of other sequence.

Prob. 1 : DFT of a sequence x (n) is given by

$X(k) = \{4, 1+2j, j, 1-3j\}$

Using DFT property only find DFT of $x^*(n)$ if $x^*(n)$ is complex conjugate of x (n).

Soln. : According to complex conjugate property,

$$\text{If} \quad x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k) \text{ then}$$

$$x^*(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X^*(-k)_N$$

$$\text{We have,} \quad X(k) = \{4, 1+2j, j, 1-3j\}$$

$$\therefore \quad X^*(k) = \{4, 1-2j, -j, 1+3j\}$$

But DFT of $x^*(n)$ is $X^*(-k)_N$

Here $X^*(-k)_N$ indicates circular folding of $X^*(k)$. That means sequence $X^*(-k)_N$ is obtained by plotting the samples of $X^*(k)$ in clockwise direction.

$$\therefore \quad X^*(-k)_N = \{4, 1+3j, -j, 1-2j\}$$

This is the DFT of $x^*(n)$.

Prob. 2 : DFT of a sequence x (n) is given by,

$X(k) = \{6, 0, -2, 0\}$

(i) Determine x (n)

(ii) Plot $x_1(n)$ if $X_1(k)$ is $X(k) \cdot e^{-j2\pi k/2}$

(iii) Determine circular autocorrelation of x (n) using DFT and IDFT only.

Soln. :

(i) According to the definition of IDFT

$$x(n) = \frac{1}{N}[W_N^*] \cdot X_N$$

$$\text{Here} \quad [W_N^*] = [W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore \quad x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\therefore \quad x(n) = \frac{1}{4}\begin{bmatrix} 6+0-2+0 \\ 6+0+2+0 \\ 6+0-2+0 \\ 6+0+2+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad x(n) = \{1, 2, 1, 2\}$$

(ii) Given $X_1(k) = X(k)\, e^{-j2\pi k/2}$

According to circular time shift property,

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)\, e^{-j2\pi kl/N} \quad \text{...(1)}$$

Given term can be expressed as,

$$X(k)\, e^{-j2\pi k/2} = X(k)\, e^{-\frac{j2\pi k \cdot 2}{4}} \quad \text{...(2)}$$

Comparing Equations (1) and (2),

$$x((n-2))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k) = X(k)\, e^{-\frac{j2\pi k \cdot 2}{4}}$$

That means $x_1(n) = x((n-2))_N$

Here $x((n-2))_N$ indicates circular delay of x (n) by 2 samples. It is plotted as shown in Fig. F-18.

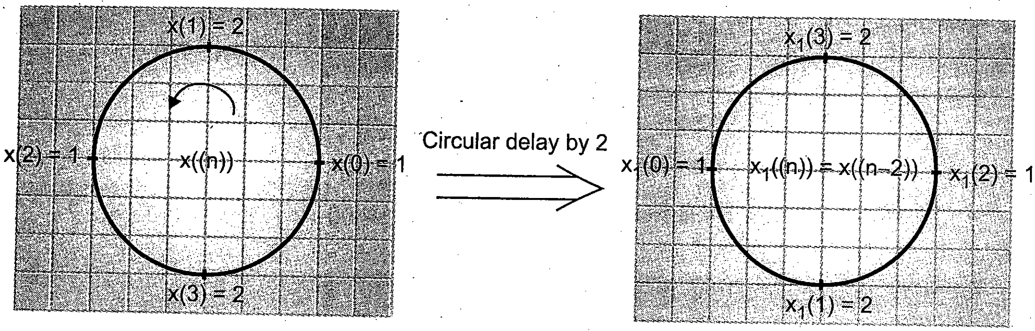

Fig. F-18

(iii) According to circular correlation property we have

$$r_{xx}(l) \underset{N}{\overset{DFT}{\longleftrightarrow}} = R_{xx}(k) = X(k) \cdot X^*(k)$$

We have $X(k) = \{6, 0, -2, 0\}$

$X^*(k)$ is complex conjugate of $X(k)$

$$\therefore \quad X^*(k) = \{6, 0, -2, 0\}$$

$$\therefore \quad X(k) \cdot X^*(k) = \{36, 0, 4, 0\}$$

Now $r_{xx}(l)$ is obtained by taking IDFT of $X(k) \cdot X^*(k)$.

$$\therefore \quad r_{xx}(l) = \frac{1}{4}[W_4^*] \cdot X(k) \cdot X^*(k)$$

$$\therefore \quad r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 36 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad r_{xx}(l) = \frac{1}{4}\begin{bmatrix} 36+0+4+0 \\ 36+0-4+0 \\ 36+0+4+0 \\ 36+0-4+0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 40 \\ 32 \\ 40 \\ 32 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 10 \\ 8 \end{bmatrix}$$

$$\therefore \quad r_{xx}(l) = \{10, 8, 10, 8\}$$

1.6 Relationship between DFT and Z-Transform :

The Z-transform of sequence x(n) is,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \text{...(1)}$$

We know that at $Z = e^{j\omega}$,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{...(2)}$$

It means that X(Z) is evaluated on unit circle.

Now suppose X(Z) is sampled at 'N' equally spaced points on the unit circle. Then we have

$$\omega = \frac{2\pi K}{N} \qquad \text{...(3)}$$

Now if X(Z) is evaluated at $Z = e^{j\omega K/N}$ then by putting Equation (3) in Equation (2) we get,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi Kn/N} \quad \text{at } Z = e^{j2\pi K/N} \qquad \text{...(4)}$$

In Equation (4), if $x(n)$ is causal sequence and has 'N' number of samples then we can write Equation (4) as,

$$X(Z) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad \text{at } Z = e^{j2\pi K/N} \qquad \text{...(5)}$$

But R.H.S. of Equation (5) is DFT of x(n).

$$\therefore \quad X(K) = X(Z) \text{ at } Z = e^{j2\pi K/N}$$

This means if Z-transform is evaluated on the unit circle at evenly spaced points only; then it becomes DFT.