LINEAR ALGEBRAIC EQUATIONS

In the previous method we determined the value x that satisfied a single equation, $f(x) = 0$.

Now, we deal with the case of determining the values $x_1, x_2, \ldots, x_n$ that simultaneously satisfy a set of equations

$$f_1(x_1, x_2, \ldots, x_n) = 0$$
$$f_2(x_1, x_2, \ldots, x_n) = 0$$
$$\text{-- -----------}$$
$$\text{--------------}$$
$$f_n(x_1, x_2, \ldots, x_n) = 0$$

The solution of this system consists of a set of x values that simultaneously result in all the equations equaling zero.

we deal with linear algebraic equations that are of the general form

$$a_{11}x_1 + a_{12}x_2 + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{nn}x_n = b_n$$

where
a's are constant coefficients,
b's are constants, and
n is the number of equations.

Graphical Method

A graphical solution is obtainable for two equations by plotting them on Cartesian coordinates with one axis corresponding to x_1 and the other to x_2.

Because we are dealing with linear systems, each equation is a straight line. This can be easily illustrated for the general equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

Cramer's rule or matrix mathod become tedious for large system of equation.

Direct Method of Solution

- Gauss Elimination method

The elimination of unknowns was used to solve a pair of simultaneous equations. The procedure consisted of two steps:

1. The equations were manipulated to eliminate one of the unknowns from the equations. The result of this elimination step was that we had one equation with one unknown. (The system is reduced to an upper triangular system)
2. Consequently, this equation could be solved directly and the result back-substituted into one of the original equations to solve for the remaining unknown.

Consider the equation

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Do partial or complete pivoting

- Eliminate x from second and third equation

Assuming $a_1 \neq 0$ eliminate the x from the second equation by subtracting (a_2/a_1) times the first equation

Similarly x from the third equation by subtracting (a_3/a_1) times the first equation.

$$a_1x + b_1y + c_1z = d_1$$
$$b'_2y + c'_2z = d'_2$$
$$b'_3y + c'_3z = d'_3$$

- Eliminate the y from third equation

Assuming $b_2 \neq 0$ eliminate the y from the third equation by subtracting (b'_3/b'_2) times the second equation

$$a_1x + b_1y + c_1z = d_1$$
$$b'_2y + c'_2z = d'_2$$
$$c''_3z = d''_3$$

Evaluate the unknown by back substitution

GAUSS-JORDAN

The Gauss-Jordan method is a variation of Gauss elimination. The major difference is that when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations rather than just the subsequent ones.

In addition, all rows are normalized by dividing them by their pivot elements. Thus, the elimination step results in an identity matrix (diagonal matrix) rather than a upper triangular matrix.

Consequently, it is not necessary to employ back substitution to obtain the solution.

Consider the equation

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Do partial or complete pivoting

- Eliminate x from second and third equation

Assuming $a_1 \neq 0$ eliminate the x from the second equation by subtracting (a_2/a_1) times the first equation

Similarly x from the third equation by subtracting (a_3/a_1) times the first equation.

$$a_1x + b_1y + c_1z = d_1$$
$$b'_2y + c'_2z = d'_2$$
$$b'_3y + c'_3z = d'_3$$

- Eliminate the y from First and third equation

Assuming $b'_2 \neq 0$ eliminate the y from the first equation by subtracting (b_1/b'_2) times the second equation

$$a_1x + \quad c'_1z = d'_1$$
$$b'_2y + c'_2z = d'_2$$
$$b'_3y + c'_3z = d'_3$$

Assuming $b'_2 \neq 0$ eliminate the y from the third equation by subtracting (b'_3/b'_2) times the second equation

$$a_1x + \qquad c'_1z = d'_1$$
$$b'_2y + c'_2z = d'_2$$
$$c''_3z = d''_3$$

- Eliminate the z from First and Second equation

Assuming $c''_{23} \neq 0$ eliminate the z from the first and second equation by subtracting (c'_1/c''_3) times of third equation to first equation and (c'_2/c''_3) times of third equation to second equation

$$a_1x \qquad = d''_1$$
$$b'_2y \qquad = d''_2$$
$$c''_3z = d''_3$$

Evaluate the unknown directly

Iterative Method of Solution

Iterative or approximate methods provide an alternative to the elimination methods. Such approaches are similar to the techniques used to obtain the roots of a single equation.

Those approaches consisted of guessing a value and then using a systematic method to obtain a refined estimate of the root.

Repeat the process if necessary to achieve a desired accuracy.

- **Jacobi's Iteration Method**

Consider the equation

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Do partial or complete pivoting

After rearrangement if coefficient a_1, b_2 and c_3 are large compared to other coefficent then

Solve these equation for x, y, and z respectively

$$x = k_1 - l_1y - m_1z$$
$$y = k_2 - l_2x - m_2z$$
$$z = k_3 - l_3x - m_3y$$

Start with the initial approximation of unknown x_0, y_0, z_0 put these value in equation and get the first approximation x_1, y_1 and z_1

$$x_1 = k_1 - l_1y_0 - m_1z_0$$
$$y_1 = k_2 - l_2x_0 - m_2z_0$$
$$z_1 = k_3 - l_3x_0 - m_3y_0$$

Similarly second approximation

$$x_2 = k_1 - l_1y_1 - m_1z_1$$
$$y_2 = k_2 - l_2x_1 - m_2z_1$$
$$z_2 = k_3 - l_3x_1 - m_3y_1$$

Repeat the process till the difference between two consecutive approximation is negligible

- ***Gauss-seidel iteration Method***

This is a modified Jacobi's Iteration method. Start with initial approximation x_0, y_0, z_0.

Put these value in the equation (getting after pivoting) discussed in Jacobi's method.

Substitute $y = y_0, z = z_0$ in the first equation and get x_1

$$x_1 = k_1 - l_1 y_0 - m_1 z_0$$

Then put $x = x_1$ and $z = z_0$ in the second equation and get y_1.

$$y_1 = k_2 - l_2 x_1 - m_2 z_0$$

Next substitute $x = x_1, y = y_1$ in the third equation and get z_1.

$$z_1 = k_3 - l_3 x_1 - m_3 y_0$$

Process is repeated till the convergence achieve the desired degree of accuracy.