Faraday's Law, Displacement Current & Maxwell's Equation

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Faraday’s Law

- Faraday discovered that the **induced emf Vemf**(in volts), in any closed **circuit** is equal to the time rate of change of the magnetic flux linkage by the circuit.
- This is called ***Faraday's law***, and it can be expressed as

$$V_{\text{emf}} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

Faraday's Law

- where *N* is the number of turns in the circuit and Ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as ***Lenz's law*** and it emphasizes the fact that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field.

Transformer and Motional EMF

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$$V_{emf} = -\frac{d\Psi}{dt}$$

In terms of **E** and **B**, eq. (can be written as

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

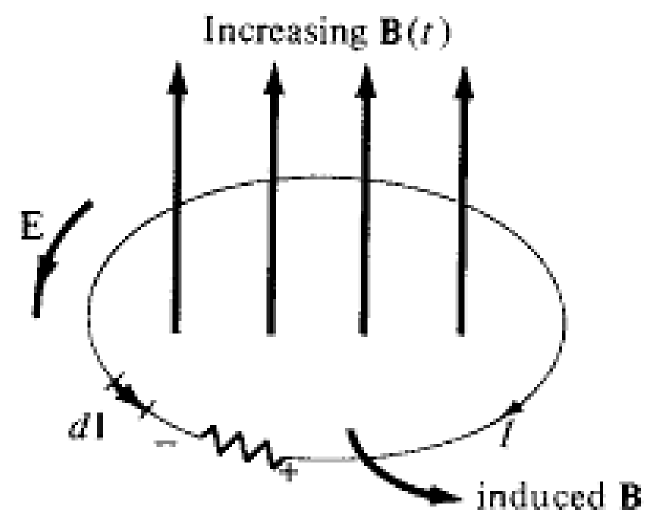
The variation of flux with time may be caused in three ways:

1. By having a stationary loop in a time-varying B field
2. By having a time-varying loop area in a static B field
3. By having a time-varying loop area in a time-varying B field.

Stationary loop in a time varying B field

- This is the case portrayed in Figure below where a stationary conducting loop is in a time varying magnetic B field. Above equation becomes

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$



Stationary loop in a time varying B field

- This emf induced by the time-varying current (producing the time-varying B field) in a stationary loop is often referred to as ***transformer emf*** in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. , we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Stationary loop in a time varying B field

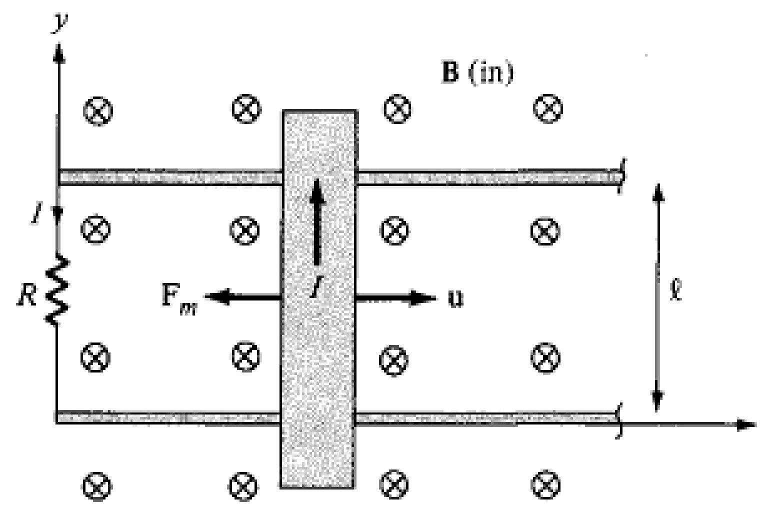
- For the two integrals to be equal, their integrands must be equal; that is,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Moving loop in stationary B field

- When a conducting loop is moving in a static B field, an emf is induced in the loop.
- We recall that the force on a charge moving with uniform velocity u in a magnetic field B is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$



Moving loop in a time varying B field

- This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining above equations gives the total emf as

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

Moving loop in stationary B field

We define the *motional electric field* $\mathbf{E}_m$ as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

If we consider a conducting loop, moving with uniform velocity $\mathbf{u}$ as consisting of a large number of free electrons, the emf induced in the loop is

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}}$$

Moving loop in stationary B field

- This type of emf is called ***motional emf*** or ***flux-cutting emf*** because it is due to motional action.
- By applying Stokes's theorem

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

or

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})}$$

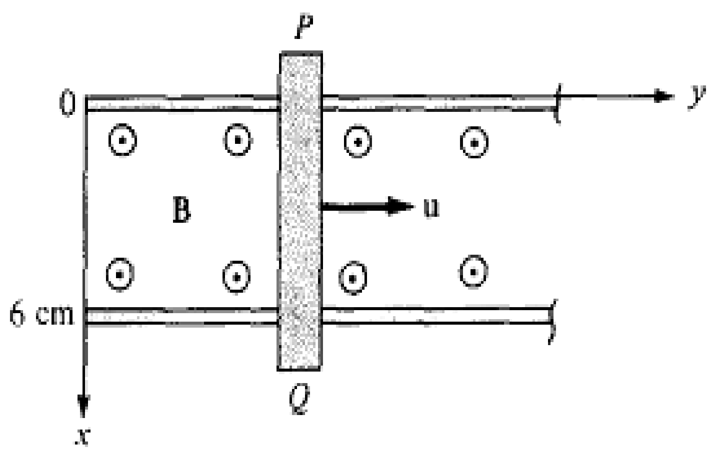
Example 1

A conducting bar can slide freely over two conducting rails as shown in Figure Calculate the induced voltage in the bar

(a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4 \cos 10^6 t\, \mathbf{a}_z$ mWb/m^2

(b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z$ mWb/m^2

(c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4 \cos (10^6 t - y)\, \mathbf{a}_z$ mWb/m^2



Figure

Solution

(a) In this case, we have transformer emf given by

$$V_{emf} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^{6}) \sin 10^{6}t \, dx \, dy$$

$$= 4(10^{3})(0.08)(0.06) \sin 10^{6}t$$

$$= 19.2 \sin 10^{6}t \text{ V}$$

The polarity of the induced voltage (according to Lenz's law) is such that point P on the bar is at lower potential than Q when $\mathbf{B}$ is increasing.

Solution

(b) This is the case of motional emf:

$$V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^{0} (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x$$

$$= -uB\ell = -20(4.10^{-3})(0.06)$$

$$= -4.8 \text{ mV}$$

Solution (c)

$$V_{emf} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int_{x=0}^{0.06} \int_{0}^{y} 4.10^{-3}(10^6) \sin(10^6 t - y')dy'\, dx$$

$$+ \int_{0.06}^{0} [20\mathbf{a}_y \times 4.10^{-3} \cos(10^6 t - y)\mathbf{a}_z] \cdot dx\, \mathbf{a}_x$$

$$= 240 \cos(10^6 t - y')\Big|_0^y - 80(10^{-3})(0.06) \cos(10^6 t - y)$$

$$= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos(10^6 t - y)$$

$$\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t$$

Displacement Current

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J}$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. To do this, we add a term to eq. so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

Displacement Current

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}}$$

Substituting

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Displacement Current

- This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term Jd = *dD/dt* is known as and J is the conduction current density.

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

Example

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin $10^3 t$ V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_o$.

Solution

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S \frac{d\rho_s}{dt} = S \frac{dD}{dt} = \varepsilon S \frac{dE}{dt} = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$
$$= 147.4 \cos 10^3 t \text{ nA}$$

Generalised Maxwell's Equation

Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

Constitutive Equations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_o (\mathbf{H} + \mathbf{M})$$

$$\mathbf{J} = \sigma \mathbf{E} + \rho_v \mathbf{u}$$

In general, a phasor could be scalar or vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the *phasor form* of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as

$$\boxed{\mathbf{A} = \text{Re}\,(\mathbf{A}_s e^{j\omega t})}$$

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s$$

Similarly,

$$\int \mathbf{A}\, \partial t \rightarrow \frac{\mathbf{A}_s}{j\omega}$$

Time-Harmonic Maxwell's Equations

Time-Harmonic Maxwell's Equations
Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$