University Institute of Engineering & Technology CSJMU KANPUR

Department of Electronics & Communication Engineering

Course Name- Network Analysis and Synthesis (ECE 202)

Branch B.Tech Electronics & Communication

UNIT 2: Network Transient and steady state analysis

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3.12 APPLICATION OF LAPLACE TRANSFORMATION IN ANALYSING NETWORKS

3.12.1 Step and Impulse Responses of Series R-L Circuit

Step Response

In the series RL circuit shown in Fig. 3.3, let the switch S be closed at time $t = 0$. For the step response, the input excitation is $x(t) = V_0 \cdot u(t)$. Applying Kirchoff's voltage law to the circuit, we get the following differential equation.

$$L \frac{di(t)}{dt} + Ri(t) = V_0 \cdot u(t) \quad (3.25)$$

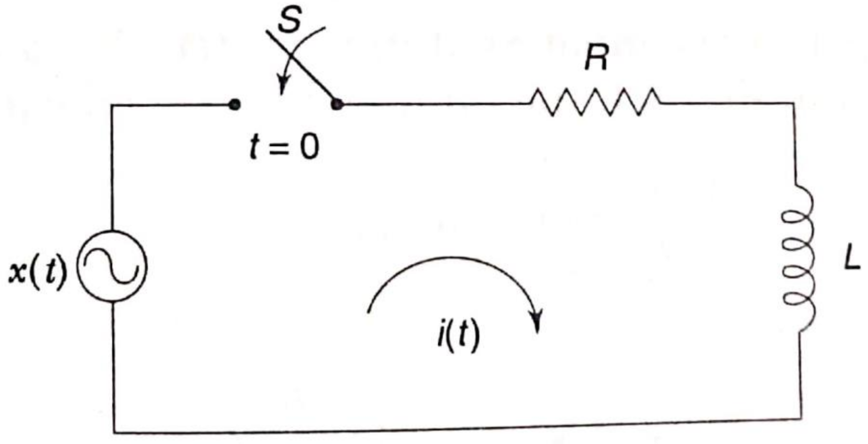

Fig. 3.3 *Series RL Circuit*

Taking Laplace transform, the above equation becomes,

$$L\{sI(s) - i(o^+)\} + RI(s) = \frac{V_0}{s} \quad (3.26)$$

Because of the presence of inductance L, $i(o^+) = 0$, i.e. the current through an inductor cannot change instantaneously due to the conservation of flux linkages.

Therefore, $I(s)[Ls + R] = \frac{V_0}{s}$

Hence, $$I(s) = \frac{V_0}{L} \cdot \frac{1}{s\left(s + \frac{R}{L}\right)}$$

$$= \frac{V_0}{L} \cdot \frac{L}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right] = \frac{V_0}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right]$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{V_0}{R} \cdot \left[1 - e^{-\frac{R}{L}t}\right] \tag{3.27}$$

Impulse Response

For the impulse response, the input excitation is $x(t) = \delta(t)$. Hence, the differential equation becomes

$$L\frac{di(t)}{dt} + Ri(t) = \delta(t)$$

$$L\{sI(s) - i(0^+)\} + RI(s) = 1$$

Since $i(0^+) = 0$,

$$I(s) = \frac{1}{R + Ls} = \frac{1}{L} \cdot \frac{1}{s + R/L}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{1}{L} \cdot e^{-(R/L)t}\, u(t)$$

3.12.2 Step and Impulse Responses of Series R-C Circuit

Step Response

For the step response, the input excitation is $x(t) = V_0 \cdot u(t)$. In the series RC circuit shown in Fig. 3.4, the integro-differential equation is

$$\frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + Ri(t) = V_0\, u(t) \tag{3.28}$$

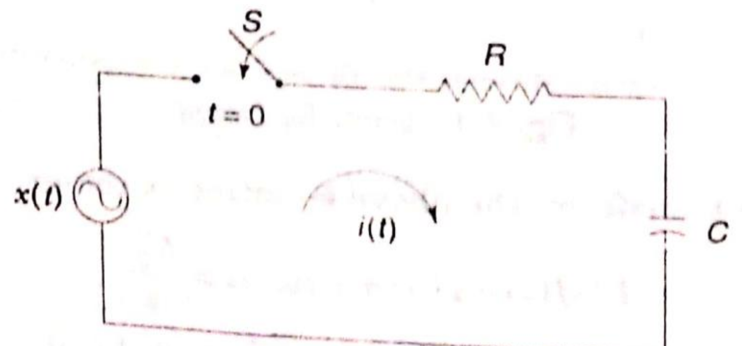

Fig. 3.4 *Series R-C circuit*

This may be written as

$$\frac{1}{C}\int_0^t i(t)\,dt + \frac{1}{C}\int_{-\infty}^0 i(t)\,dt + Ri(t) = V_0 u(t)$$

Taking Laplace transform, the above equation becomes

$$\frac{1}{C}\left[\frac{I(s)}{s}\right] + \frac{1}{C}\mathcal{L}\left[q(0^+)\right] + RI(s) = \frac{V_0}{s}$$

$$\frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^+)}{s}\right] + RI(s) = \frac{V_0}{s}$$

Now $q(0^+)$ is the charge on the capacitor C at time $t = 0^+$. If the capacitor is initially uncharged, then $q(0^+) = 0$.

Therefore, $$I(s)\left[\frac{1}{Cs} + R\right] = \frac{V_0}{s}$$

Hence, $$I(s) = \frac{V_0 / R}{s + \dfrac{1}{RC}}$$

Therefore, $$i(t) = \frac{V_0}{R} e^{-\frac{t}{RC}} \tag{3.29}$$

Impulse Response

For the impulse response, the input excitation is $x(t) = \delta(t)$. Hence, the differential equation becomes

$$\frac{1}{C}\int_{-\infty}^t i(t)\,dt + Ri(t) = \delta(t)$$

$$\frac{1}{C}\int_{-\infty}^0 i(t)\,dt + \frac{1}{C}\int_0^t i(t)\,dt + Ri(t) = \delta(t)$$

Taking Laplace transform, the above equation becomes

$$\frac{1}{C}\left[\frac{I(s)}{s}\right] + \frac{1}{C}\mathcal{L}\left[q(0^+)\right] + RI(s) = 1$$

$$\frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^+)}{s}\right] + RI(s) = 1$$

Since $$q(0^+) = 0,\ I(s)\left[\frac{1}{Cs} + R\right] = 1$$

Therefore, $$I(s) = \frac{1}{R\left(1 + \dfrac{1}{RCs}\right)}$$

$$= \frac{s}{R\left(s+\frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{\left(s+\frac{1}{RC}\right)-\frac{1}{RC}}{s+\frac{1}{RC}}\right]$$

$$= \frac{1}{R}\left[1-\frac{1}{RC}\,\frac{1}{(s+1/RC)}\right]$$

Its inverse Laplace transform is

$$i(t) = \frac{1}{R}\left[\delta(t)-\frac{1}{RC}\cdot e^{-t/RC}\,u(t)\right]$$

3.12.3 Step Response of Series R-L-C Circuit

In the series RLC circuit shown in Fig. 3.5, the integro-differential equation is

$$L\,\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = V_0\,u(t) \tag{3.30}$$

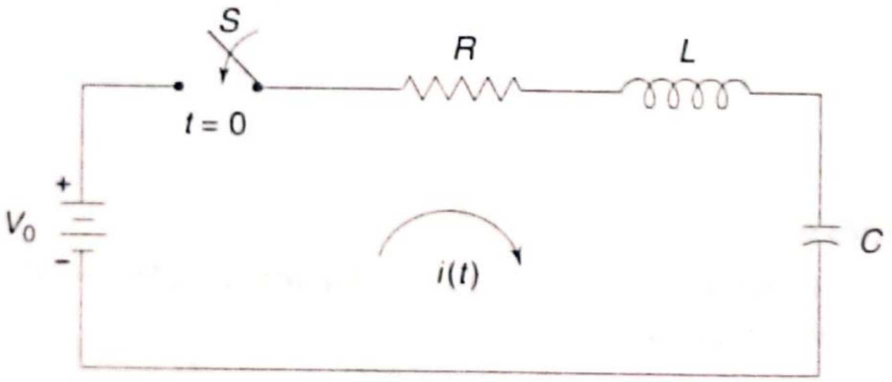

Fig. 3.5 *Series RLC circuit*

This may be written as

$$L\,\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt = V_0\,u(t)$$

Taking Laplace transform, the equation becomes

$$L\left[sI(s)-i(0^+)\right] + RI(s) + \frac{1}{C}\,\mathcal{L}\left[q(0^+)\right] + \frac{1}{C}\,\frac{I(s)}{s} = \frac{V_0}{s}$$

or
$$L\left[sI(s)-i(0^+)\right] + RI(s) + \frac{1}{C}\,\frac{q(0^+)}{s} + \frac{1}{C}\,\frac{I(s)}{s} = \frac{V_0}{s}$$

Because of the presence of inductor L, $i\ (0^+) = 0$. Also, $q\ (0^+)$ is the charge on the capacitor C at $t = 0^+$. If the capacitor is initially uncharged, then $q\ (0^+) = 0$. Substituting these two initial conditions, we get

$$LsI(s) + RI(s) + \frac{I(s)}{C_s} \doteq \frac{V_0}{s}$$

$$I(s)\left[Ls + R + \frac{1}{Cs}\right] = \frac{V_0}{s}$$

Therefore, $$I(s) = \frac{V_0}{Ls^2 + Rs + \frac{1}{C}} = \frac{V_0}{L(s - p_1)(s - p_2)}$$

where $$p_1, p_2 = \frac{-R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - 4\frac{L}{C}}$$

Therefore, $$i(t) = \frac{V_0/L}{(p_1 - p_2)}\left[e^{p_1 t} - e^{p_2 t}\right] \tag{3.31}$$

3.12.4 Step Response of Parallel RLC Circuit

In the parallel *RLC* circuit shown in Fig. 3.6, let the switch *S* be opened at time $t = 0$, thus connecting the d.c. current source I_0 to the circuit. Applying Kirchoff's current law to the circuit, we get the following integro-differential equation.

$$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{t} V\,dt = I_0 \cdot u(t) \tag{3.32}$$

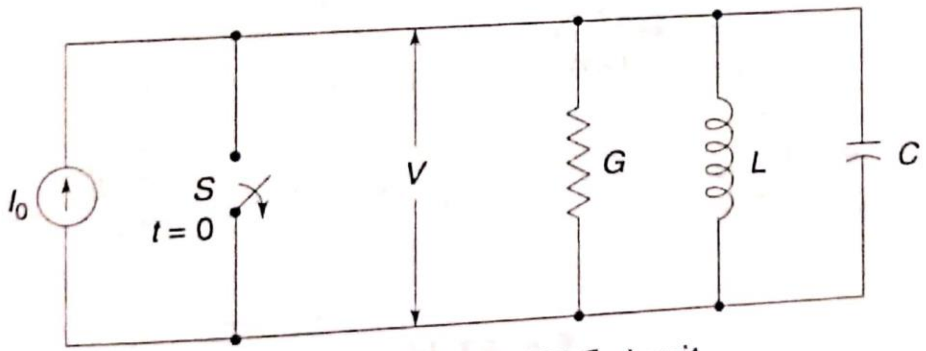

Fig. 3.6 *Parallel RLC circuit*

This may be written as

$$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{0} V\,dt + \frac{1}{L}\int_{0}^{t} V\,dt = I_0 \cdot u(t)$$

Taking Laplace transform, the equation becomes

$$C\left[sV(s) - V(0^+)\right] + GV(s) + \frac{1}{L}\mathcal{L}\left[\varphi(0^+)\right] + \frac{1}{L}\frac{V(s)}{s} = \frac{I_0}{s}$$

where $\varphi(0^+)$ is the flux linkage and equals $Li(0^+)$.

Now, the initial conditions are inserted.

Due to the presence of capacitor *C*, $V(0^+) = 0$ since the voltage across a capacitor changes instantaneously. Also, the current in the inductor *L* during the time interval $-\infty$ to 0 is zero. Hence, $\varphi(0^+) = 0$.

$$CsV(s) + GV(s) + \frac{1}{Ls} V(s) = \frac{I_0}{s}$$

or $$V(s)\left[Cs + G + \frac{1}{Ls}\right] = \frac{I_0}{s}$$

Therefore, $$V(s) = \frac{I_0}{\left(Cs^2 + Gs + \frac{1}{L}\right)} = \frac{I_0}{C\left(s^2 + \frac{G}{C}s + \frac{1}{LC}\right)}$$

$$= \frac{I_0}{C(s - p_1)(s - p_2)}$$

where $$p_1, p_2 = \frac{-G}{2C} \pm \frac{1}{2C}\sqrt{G^2 - 4\frac{C}{L}}$$

Therefore, $$v(t) = \frac{I_0/C}{(p_1 - p_2)}\left[e^{p_1 t} - e^{p_2 t}\right] \tag{3.33}$$

This is the same as the equation obtained through classical analysis.

Example 3.16 In the series RL circuit shown in Fig. E3.16, find the current $i(t)$.

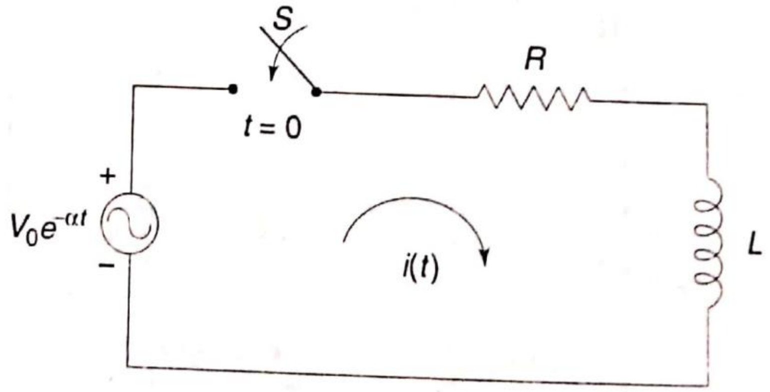

Fig. E3.16

Solution Applying Kirchoff's voltage law, we get

$$L\frac{di(t)}{dt} + Ri(t) = V_0 e^{-\alpha t}$$

Taking Laplace transform, we have

$$L\{sI(s) - i(0^+)\} + RI(s) = V_0\frac{1}{s + \alpha}$$

Since there is no initial current through inductor before closing the switch S, the current remains zero at time $t = 0^+$, i.e. $i(0^+) = 0$. Hence, the above equation becomes

$$I(s)[Ls + R] = \frac{V_0}{s + \alpha}$$

i.e. $$I(s) = \frac{V_0}{L} \cdot \frac{1}{(s+\alpha)\left(s + \frac{R}{L}\right)}$$

Case (i) When $\alpha \neq \frac{R}{L}$

$$i(t) = \frac{V_0}{R - \alpha L}\left[e^{-\alpha t} - e^{-\frac{R}{L}t}\right]$$

Case (ii) When $\alpha = \frac{R}{L}$

$$I(s) = \frac{V_0}{L} \cdot \frac{1}{(s+\alpha)^2}$$

Therefore, $$i(t) = \frac{V_0}{L} t e^{-\alpha t}.$$

Example 3.17 In the circuit of Fig. E3.17, (a) find the currents $i_1(t)$ and $i_2(t)$ and the output voltage across 5Ω resistor when the switch is closed, and (b) also determine the initial and final values of current.

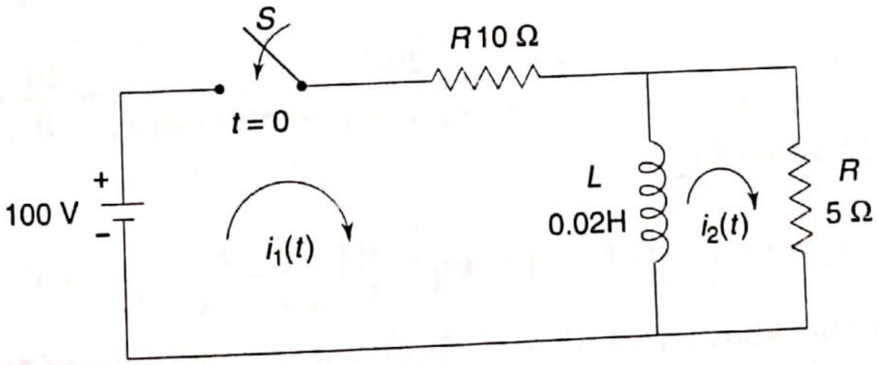

Fig. E3.17

Solution By applying Kirchhoff's voltage law to the circuit, we have the time-domain equations as

$$10i_1(t) + 0.02\frac{di_1(t)}{dt} - 0.02\frac{di_2(t)}{dt} = 100$$

$$0.02\frac{di_2(t)}{dt} + 5i_2(t) - 0.02\frac{di_1(t)}{dt} = 0$$

Taking Laplace transforms for the above equations, we get

$$(10 + 0.02s)I_1(s) - 0.02\, sI_2(s) = 100/s$$

$$(5 + 0.02s)I_2(s) - 0.02s I_1(s) = 0$$

Solving the equations, we have

$$I_2(s) = I_1(s)\left(\frac{s}{s+250}\right)$$

$$I_1(s) = \frac{20}{3}\left\{\frac{s+250}{s(s+500/3)}\right\} = \frac{10}{s} - \frac{10/3}{s+500/3}$$

$$I_2(s) = \frac{20}{3}\left(\frac{1}{s+500/3}\right)$$

Taking inverse transform, we get

$$i_1(t) = 10 - \frac{10}{3}e^{-(500/3)t}$$

$$i_2(t) = \frac{20}{3}e^{-(500/3)t}$$

The voltage across 5Ω resistor is $5i_2(t) = \frac{100}{3}e^{-(500/3)t}$ volts.

To find the initial and final value of the currents

The initial value of $i_1(t)$ is

$$i_1(0^+) = \lim_{s\to\infty}[s\,I_1(s)] = \lim_{s\to\infty}\left[\frac{20}{3}\left(\frac{s+250}{s+(500/3)}\right)\right] = \frac{20}{3}A$$

The final value of $i_1(t)$ is

$$i_1(\infty) = \lim_{s\to 0}[s\,I_1(s)] = \lim_{s\to 0}\left[\frac{20}{3}\left(\frac{s+250}{s+(500/3)}\right)\right] = 10\ A$$

The initial value of $i_2(t)$ is

$$i_2(0^+) = \lim_{s\to\infty}[s\,I_2(s)] = \lim_{s\to\infty}\left[\frac{20}{3}\left(\frac{s}{s+(500/3)}\right)\right] = \frac{20}{3}A$$

The final value of $i_2(t)$ is

$$i_2(\infty) = \lim_{s\to 0}[s\,I_2(s)] = \lim_{s\to 0}\left[\frac{20}{3}\left(\frac{s}{s+(500/3)}\right)\right] = 0$$

From the above initial and final values, it is clear that, at the instant of closing the switch S, the inductance gives an infinite impedance and hence the currents $i_1(t) = i_2(t) = 100/(10+5) = \frac{20}{3}A$. Then, in the steady state, the inductance becomes a short circuit and hence $i_1(t) = 10A$ and $i_2(t) = 0$.