NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

A number of problems in science and technology can be formulated into differential equations.

The analytical methods of solving differential equations are applicable only to a limited class of equations.

Solution of a differential equation

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x.

Such a solution of a differential equation is known as the closed or finite form of solution.

In the absence of such a solution, we have recourse to numerical methods of solution.

Let us consider the first order differential equation

$$dy/dx = f(x, y), \text{ given } y(x_0) = y_0$$

In the numerical methods, we replace the differential equation by a difference equation and then solve it.

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or a set of values of x and y.

The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in equation is approximated by a truncated series, each term of which is a function of x. These are referred to as single-step methods.

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth, etc. belong to the latter class of solutions.

In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations until sufficient accuracy is achieved. These methods are called step-by-step methods.

Euler and Runga-Kutta methods are used for computing y over a limited range of x- values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values.

Initial and boundary conditions

An ordinary differential equation of the *n*th order is of the form

$$F(x, y, dy/dx, d^2y/dx^2, \ldots.., d^ny/dx^n) = 0$$

Its general solution contains *n* arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \ldots\ldots., c_n) = 0 \ (3)$$

To obtain its particular solution, *n* conditions must be given so that the constants c_1, c_2 ……, c_n can be determined.

If these conditions are prescribed at one point only (say:x_0), then the differential equation together with the conditions constitute an initial value problem *of the nth order.*

If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem.

Picard's Method

Consider the first order equation $\frac{dy}{dx} = f(x, y)$

It is required to find that particular solution of equ. which assumes the value y_0 when $x = x_0$. Integrating equ. between limits, we get

$$\int_{y_0}^{y} dy = \int f(x,y)dx \quad \text{or } y - y_0 = \int f(x,y)dx$$

This is an integral equation equivalent to equ. (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in f(x, y) and integrate equ. (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

For a second approximation y_2, we put $y = y_1$ in f(x, y) and integrate(2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x\, y_1)\, dx$$

Similarly, *a third approximation is*

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

Continuing this process, we obtain y_4, y_5, …. y_n where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

Hence this method gives a sequence of approximations y_1, y_2, y_3 …….each giving a better result than the preceding one.

Taylor's Series Method

Consider the first order equation $\dfrac{dy}{dx} = f(x, y)$

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad i.e. \;\; y'' = f_x + f_y f'$$

Differentiating this successively, we can get y'',.. y^{iv} etc. Putting x = x_0 and y = 0, The Values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + ...$$

gives the values of y for every value of x for which above equ. converges.

On finding the value y1 for $x = x_i$ from equ., y', y'' etc. can be evaluated at $x = x_1$ by means of preceding equ etc.

y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series.

Euler's Method

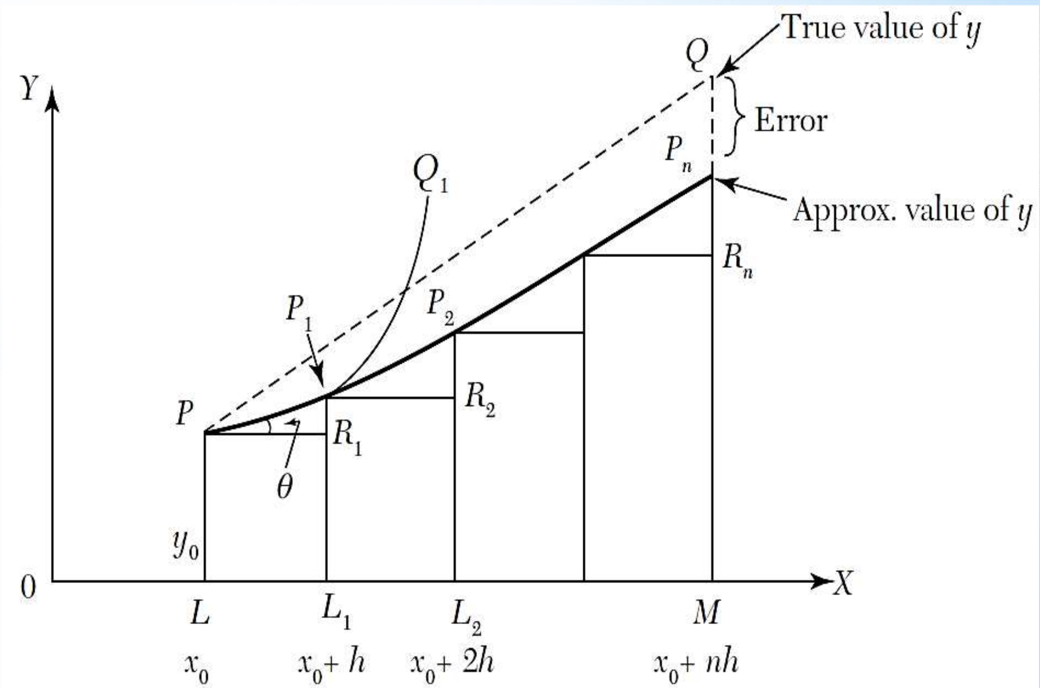
Consider the first order equation $\frac{dy}{dx} = f(x, y)$ (1)

given that $y(x_0) = y_0$.

The point $P(x_0, y_0)$ on x-y solution curve is shown in Figure. Now we have to find the ordinate of any other point Q on this curve.

Let us divide *LM* into *n* sub-intervals each of width *h* at L_1, L_2 ....so that *h* is quite small

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 = h, y_1)$, then



$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$
$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 = 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is Euler's method of finding an approximate solution of (1).

Modified Euler's Method

In Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P such that at P_1, we have

$$y_1 = y_0 + h\,f(x_0, y_0)$$

Then the slope of the curve of solution through P_1

$$[\text{i.e., } (dy/dx)_{P_1} = f(x_0 + h, y_1)]$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 = 2h, y_2)$.

Now we find a better approximation (1) y_1 of $y(x_0 = h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, i.e.,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1(1)$

Again (2) is applied and we find a still better value $y_1(2)$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, until two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to a desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation (1) y_2 is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the modified Euler's method which gives great improvement in accuracy over the original method.

Runge's Method

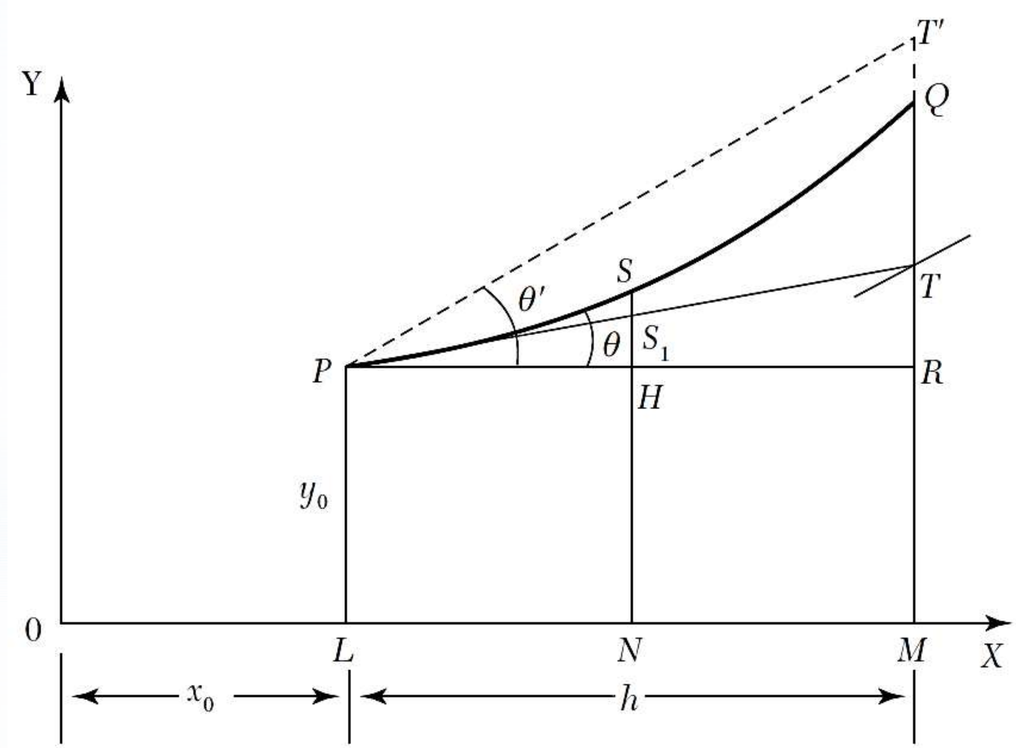
Consider the differential equation, $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ shown in Figure.

Integrating both sides of equ. from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x,y)dx$$

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of f(x, y) (i.e., dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$.



For this purpose, we first determine the values of *y* at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_S is given by the point S_1

$$y_s = NS_1 = LP + HS_1 = y_0 + PH.\tan\theta$$

$$= y_0 + h(dy/dx)_p = y_0 + \frac{h}{2} f(x_0, y_0)$$

$$y_T = MT = LP + RT = y_0 + PR.\tan\theta = y_0 + hf(x_0 + y_0).$$

Now the value of y_Q at $x_0 + h$ is given by the point T″ where the line through P draw with slope at $T(x_0 + h, y_T)$ meets MQ.

Slope at $T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$y_Q = R + RT = y_0 + PR\ .\tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0,\ y_0)]$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $S = f(x_0 + h/2, y_S)$

and the value of $f(x, y)$ at $Q = (x_0 + h, y_Q)$

where y_S and y_Q are given by equations above.

Hence from equ., we obtain

$$k = \int_{x_0}^{x_0+h} f(x,y)dx = \frac{h}{6}\left[f_P + 4f_S + f_Q\right] \qquad \text{by Simpson's rule}$$

$$= \frac{h}{6}\left[f(x_0 + y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)\right]$$

Which gives a sufficiently accurate value of k and also $y = y_0 + k$

The repeated application of process gives the values of y for equi-spaced points.

Working rule to solve by Runge's method:

Calculate successively $k_1 = hf(x_0,\, y_0),$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right)$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right)$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right)$$

$$k = \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$$

Runge-Kutta Method

The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points.

These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the *order of that method.*

First order R-K method. We have seen that Euler's method gives y_1

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x, y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \cdots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h. Hence, Euler's method is the Runge-Kutta method of the first order.

Second order R-K method. The modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right]$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right-hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + hf_0)\right] \quad \text{where} \quad f_0 = (x_0, y_0)$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)^{**}\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O\left(h^3\right)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3)$$

Comparing equations, it follows that the modified Euler's method agrees with the Taylor's series solution upto the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

The second order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

Third order R-K method. Similarly, it can be seen that Runge's method agrees with the Taylor's series solution upto the term in h^3.

As such, Runge's method is the Runge-Kutta method of the third order.

The third order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where $k_1 = hf(x_0, y_0), k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

And $k_3 = hf\left(x_0 + h, y_0 + k'\right)$ *where* $k' = k_3 = hf(x_0 + h, y_0 + k_1)$

Fourth order R-K method. This method is most commonly used and is often referred to as the *Runge-Kutta method* only.

Working rule

For finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), y(x_0)$$

is as follows:

Calculate successively

$$k_1 = hf(x_0, y_0),$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

Finally compute

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

which gives the required approximate value as $y_1 = y_0 + k$.

Boundary-value Problems

Introduction

In order to determine a unique solution to an initial-value problem for an Nth-order ODE, we have seen that N initial conditions must be specified.

- The N initial conditions are all given at the same point in time.

Boundary-value problems differ from initial-value problems in that the N conditions on the problem are provided at *more than one point*.

- Typically they are given at starting and finishing points at, say, $t = 0$ and $t = T$.

As a simple example of a boundary-value problem,

A moving object hit with an arrow. let's assume that the arrow moves in a single dimension, vertically, under the influence only of gravity no air drag or other forces will be kept. Then the position of the arrow, $y(t)$., satisfies

$$\frac{d^2y}{dt^2} = -g,$$

where g =9.8 m/s^2 is the acceleration of gravity. The arrow starts at y = 0 at time t = 0, and must be at y = H at t = T in order to hit an object that passes overhead at that instant. Therefore, the *boundary conditions* on the problem are

$$y(0) = 0, \qquad y(T) = H.$$

Integrating the ODE twice, yielding the following solution for y = t.:

$$y(t) = \left(\frac{H}{T} - \frac{gT}{2}\right)t - \frac{gt^2}{2}.$$

Finding the solution of this boundary-value problem seems to be no different than finding the solution of an initial-value problem.

However, there is a fundamental difference between these two types of problems: unlike solutions to initial-value problems that satisfy the conditions of Theorem,

[If the derivative of function in each of its arguments is continuous over some domain encompassing this initial condition, the solution to this problem exists and is unique for some length of time around the initial time.]

The solutions to boundary-value problems need not exist, and if they exist they need not be unique.

The two numerical methods are exist for solving such boundary value problems.

The first one is known as the finite difference method which makes use of finite difference equivalents of derivatives.

The second one is called the shooting method which makes use of the techniques for solving initial value problems.

Finite-Difference Method

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure.

These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under:

If y(x) and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots$$

and $$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \cdots$$

First Equation gives

$$y'(x) = \frac{1}{h}\left[y(x+h) - y(x)\right] - \frac{h}{2}y''(x) - \cdots$$

i.e.,

$$y'(x) = \frac{1}{h}\left[y(x+h) - y(x)\right] + O(h)$$

which is the forward difference approximation of y′(x) with an error of the order h.

Similarly Second equ. gives

$$y'(x) = \frac{1}{h}\left[y(x) - y(x-h)\right] + O(h)$$

which is the backward difference approximation of y′(x) with an error of the order h.

Subtracting second from first eqation, we obtain

$$y'(x) = \frac{1}{2h}\left[y(x+h) - y(x-h)\right] + O(h^2)$$

which is the central-difference approximation of $y'(x)$ with an error of the order h^2.

Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding first and second equation, we get

$$y''(x) = \frac{1}{h^2}\left[y(x+h) - 2y(x) + y(x-h)\right] + O(h^2)$$

which is the central difference approximation of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence the working expressions for the central difference approximations to the first four derivatives of y_i are as under:

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2})$$

The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation.

As we reduce h, the accuracy improves but the number of equations to be solved also increases.