Orthogonal Curvilinear Coordinates

Let us now introduce one more system of coordinates.

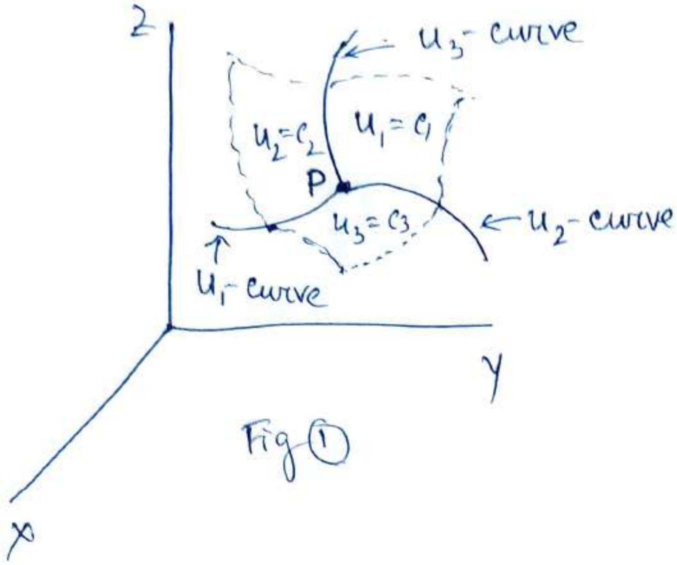


Fig (1)

Let P be a point in space whose cartesian coordinates are (x, y, z).

Let (x, y, z) of the pt. (Point) P be expressed as function of (u_1, u_2, u_3) so that

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3) \qquad \text{--- (1)}$$

Suppose eq^n. (1) can be solved for u_1, u_2, u_3 in terms of x, y, z i.e.,

$$u_1 = u_1(x, y, z), \quad u_2 = u_2(x, y, z), \quad u_3 = u_3(x, y, z) \qquad \text{--- (2)}$$

The functions in eq^{n}s. (1) and (2) are assumed to be single valued and to have continuous derivatives.

Then these (u_1, u_2, u_3) are called curvilinear coordinates of the pt. $P(x, y, z)$.

Polar coordinates (r, θ)

$x = r\cos\theta \Rightarrow x = x(r, \theta)$

$y = r\sin\theta \Rightarrow y = y(r, \theta)$

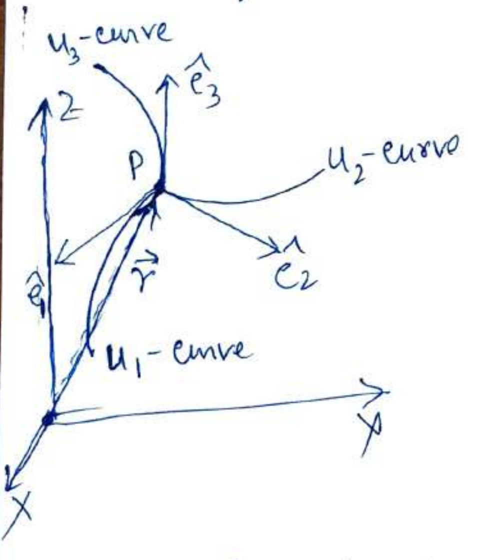
Again $\left.\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{array}\right\} \Rightarrow \begin{array}{l} r = r(x, y) \\ \theta = \theta(x, y) \end{array}$

Orthogonal curvilinear coordinates

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$, (where c_1, c_2, c_3 are constants) are called coordinate surfaces and each pair of these surfaces intersect in curves called coordinate curves or lines (see fig ①). If the coordinate surfaces intersect at right angles the curvilinear coordinate system is called orthogonal. The u_1, u_2 and u_3 coordinate curves of a curvilinear system are analogous to the x, y, z coordinate axes of a rectangular system.

Unit vectors in curvilinear systems

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x(u_1, u_2, u_3)\hat{i} + y(u_1, u_2, u_3)\hat{j} + z(u_1, u_2, u_3)\hat{k}$ be the position vector of a point P. Then using eqn ① we can write



$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

A tangent vector to the u_1-curve at point P (for which u_2, u_3 are constants) is $\frac{\partial \vec{r}}{\partial u_1}$. Then a unit tangent vector in this direction is $\hat{e}_1 = \dfrac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}$ so that

$$\frac{\partial \vec{r}}{\partial u_1} = \hat{e}_1 h_1 \text{ where } h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|.$$

Similarly if $\hat{e}_2$ and $\hat{e}_3$ are unit tangent vectors to u_2 and u_3 curves at P respectively, $\frac{\partial \vec{r}}{\partial u_2} = \hat{e}_2 h_2$ and $\frac{\partial \vec{r}}{\partial u_3} = \hat{e}_3 h_3$ where $h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|$, $h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$.

The quantities h_1, h_2, h_3 are called scale factors.

Thus the three unit tangent vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ in the directions of increasing u_1, u_2, u_3 curves are given by

$$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}, \quad \hat{e}_2 = \frac{\frac{\partial \vec{r}}{\partial u_2}}{\left|\frac{\partial \vec{r}}{\partial u_2}\right|} \quad \text{and} \quad \hat{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left|\frac{\partial \vec{r}}{\partial u_3}\right|}.$$

Since $\vec{\nabla} u_1$ is a vector at P normal to the surface $u_1 = c_1$, a unit vector in this direction is given by $\hat{E}_1 = \frac{\vec{\nabla} u_1}{|\vec{\nabla} u_1|}$. Similarly the unit vectors $\hat{E}_2 = \frac{\vec{\nabla} u_2}{|\vec{\nabla} u_2|}$ and $\hat{E}_3 = \frac{\vec{\nabla} u_3}{|\vec{\nabla} u_3|}$ at P are normals to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively. A vector in terms of $\hat{e}_1, \hat{e}_2, \hat{e}_3$ can be represented as $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$.

The Gradient, Divergence and Curl can be expressed in terms of curvilinear coordinates. If Φ is a scalar function i.e. $\Phi = \Phi(u_1, u_2, u_3)$ and $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3 i.e. $\vec{A} = \vec{A}(u_1, u_2, u_3)$, then the following results are valid.

1. $\vec{\nabla}\Phi = \frac{1}{h_1}\frac{\partial \Phi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \Phi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \Phi}{\partial u_3}\hat{e}_3$

2. $\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$

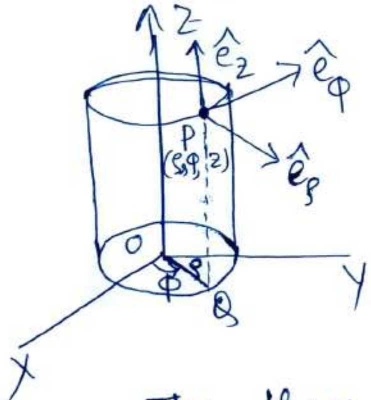
3. $\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$

If $h_1 = h_2 = h_3 = 1$ and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are replaced by $\hat{i}, \hat{j}, \hat{k}$ these reduce to the usual expression in rectangular coordinates where u_1, u_2, u_3 are replaced by x, y, z.

The scale factor gives a measure of how a change in the coordinates changes the position of a point.

Special cases of orthogonal curvilinear coordinate system

Cylindrical coordinate system (ρ, ϕ, z)



A cylindrical coordinate system is a special case of orthogonal curvilinear coordinate system. In cylindrical coordinate system the three coordinates are denoted by (ρ, ϕ, z) instead of $(u_1, u_2 u_3)$.

The three coordinates (ρ, ϕ, z) of a point P are defined as:

- The axial distance or radial distance ρ is the Euclidean distance from the z-axis to the pt. P.
- The azimuth ϕ is the angle between the reference direction (x-direction say) on the chosen plane (x-y-plane here) and the line from the origin to the projection Q of the point P on the plane.
- The height z is the signed distance from the chosen plane (xy plane here) to the point P.

Conversion between cylindrical and cartesian coords.

For the conversion between cylindrical and cartesian coordinates, it is convenient to assume that the reference plane of the cylindrical coordinate system is the cartesian xy-plane (with equation $z=0$), and the cylindrical axis is the cartesian z-axis. Then the z-coordinate is the same in both systems, and the correspondence between cylindrical coordinates (ρ, φ, z) and cartesian coordinates (x, y, z) are given by

$$x = \rho\cos\varphi \qquad \rho = \sqrt{x^2+y^2}$$

$$y = \rho\sin\varphi \qquad \frac{y}{x} = \tan\varphi, \quad \varphi = \tan^{-1}(y/x)$$

$$z = z \qquad z = z$$

where $\rho \geq 0$, $0 \leq \varphi < 2\pi$, $-\infty < z < \infty$.

For cylindrical coordinates
$u_1 = \rho$, $u_2 = \varphi$, $u_3 = z$
The three unit tangent vectors are denoted by
$\hat{e}_1 = \hat{e}_\rho$ for the curve ρ
$\hat{e}_2 = \hat{e}_\varphi$ for the curve φ
$\hat{e}_3 = \hat{e}_z$ for the curve z

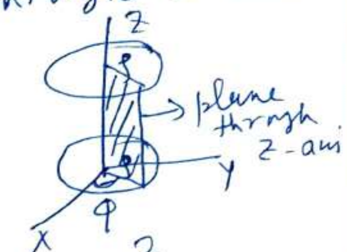
<u>The coordinate curves are</u>

1. Intersection of the surface $\varphi = c_2$ and $z = c_3$ (ρ-curve) is a straight line. i.e ρ-curve $\rightarrow$ a straight line
2. Intersection of $\rho = c_1$ and $z = c_3$ (φ-curve) is a circle (or point)
3. Intersection of $\rho = c_1$ and $\varphi = c_2$ (z-curve) is a straight line.

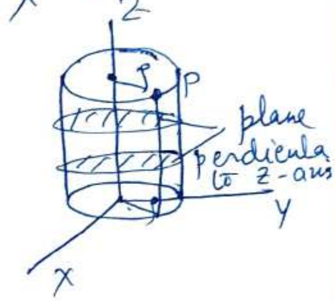
The coordinate surfaces are

1. $\rho = c_1$, $x^2+y^2 = c_1^2 \Rightarrow$ the curved surfaces of the cylinders ie cylinders coaxial with z axis (or z axis if $c_1=0$)

2. $\varphi = c_2$, $y = (\tan c_2)x$ ~~ie~~ $\Rightarrow$ plane through z-axis



3. $z = c_3$ planes perpendicular to the z-axis



Three unit tangent vectors $\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$ along the three coordinate curves ie, ρ-curve, φ-curve and z-curve respectively :

The position vector of any point in cylindrical coordinate is
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = \rho\cos\varphi\,\hat{i} + \rho\sin\varphi\,\hat{j} + z\hat{k}$$

The tangent vector to the ρ curve: $\left(\frac{\partial \vec{r}}{\partial \rho}\right)$

$$\therefore \frac{\partial \vec{r}}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\rho\cos\varphi\,\hat{i} + \rho\sin\varphi\,\hat{j} + z\hat{k}\right)$$
$$= \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}$$

$\therefore$ unit tangent vector to the ρ-curve is

$$\hat{e}_\rho = \frac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|} = \frac{\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}}{\underbrace{\sqrt{\cos^2\varphi + \sin^2\varphi}}_{1}} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}$$

with $\underbrace{h_\rho}_{\text{scale factor}} = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$.

Similarly the unit tangent vector to the φ-curve (for which ρ, z are constant)

$$\hat{e}_2 = \hat{e}_\varphi = \frac{\frac{\partial \vec{r}}{\partial \varphi}}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} = \frac{-\rho\sin\varphi\,\hat{i} + \rho\cos\varphi\,\hat{j}}{\sqrt{\rho^2\sin^2\varphi + \rho^2\cos^2\varphi}} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

with scale factor $h_2 = h_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = \rho$

And the unit tangent vector to the z-curve (for which ρ, φ are constants) is given by

$$\hat{e}_3 = \hat{e}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \frac{\hat{k}}{1} = \hat{k}$$

with scale factor $h_3 = h_z = \sqrt{1} = 1$

Now we can see that the three unit tangent vectors are given by

$$\boxed{\begin{aligned} \hat{e}_\rho &= \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \\ \hat{e}_\varphi &= -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \\ \hat{e}_z &= \hat{k} \end{aligned}}$$

In matrix notation

$$\begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\varphi \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

with $h_\rho = 1$, $h_\varphi = \rho$, $h_z = 1$.

We will now show that the cylindrical coordinate system is orthogonal ie, $\hat{e}_\rho \cdot \hat{e}_\varphi = \hat{e}_\varphi \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_\rho = 0$ just like $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$ in cartesian coordinate system.

Let us now calculate

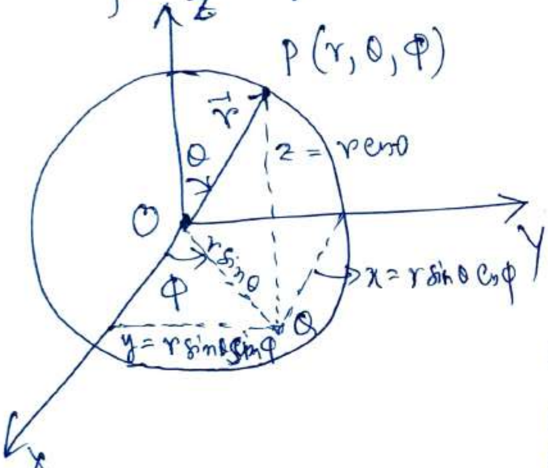
$$\hat{e}_\rho \cdot \hat{e}_\varphi = (\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) \cdot (-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})$$
$$= -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0$$

$$\hat{e}_\varphi \cdot \hat{e}_z = (-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}) \cdot \hat{k} = 0$$
$$\hat{e}_z \cdot \hat{e}_\rho = \hat{k} \cdot (\cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}) = 0$$

Thus we see that the three unit tangent vectors $\hat{e}_\rho, \hat{e}_\varphi$ and $\hat{e}_z$ are mutually perpendicular to each other which implies that cylindrical coordinate system is orthogonal.

Spherical Polar Coordinate System (r, θ, ϕ)

Spherical polar coordinate system is another example of orthogonal curvilinear coordinate system. The coordinates of spherical polar coordinate system are denoted by (r, θ, ϕ) as shown in the figure. r is known as radial distance, θ is the polar angle and ϕ is the azimuthal angle.



Let P be a point whose spherical polar coord. (r, θ, ϕ)

- The radial distance r is the Euclidean distance from origin O to P.
- The polar angle θ is the angle between the zenith (z-axis) direction and the line segment OP.
- The azimuthal angle ϕ is the signed angle measured from the azimuth reference direction (ie, x-axis) to the orthogonal projection (OQ) of the line segment OP on the reference plane (xy-plane)

The correspondence between (r, θ, ϕ) and (x, y, z) are given below

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

where $r \geq 0$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$

In spherical polar coordinate system:

$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$

$h_1 = h_r, \quad h_2 = h_\theta, \quad h_3 = h_\phi$

unit tangent vectors are denoted as $\hat{e}_1 = \hat{e}_r, \ \hat{e}_2 = \hat{e}_\theta, \ \hat{e}_3 = \hat{e}_\phi$

The coordinate curves

1. r-curve : Intersection of $\theta = c_2$ and $\phi = c_3$ surfaces
$\Rightarrow$ straight line

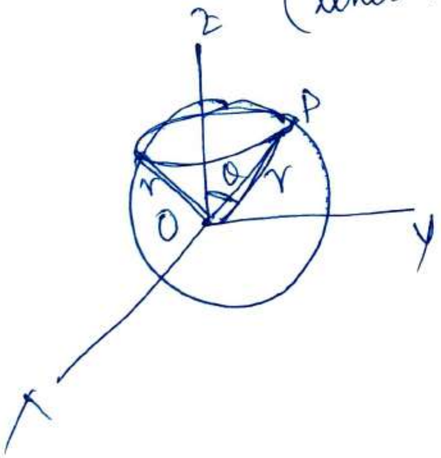
2. θ-curve : Intersection of $r = c_1$ and $\phi = c_3$ surfaces
$\Rightarrow$ is a semi circle, $0 \leq \theta \leq \pi$

3. ϕ-curve : Intersection of $r = c_1$ and $\theta = c_2$ surfaces
$\Rightarrow$ is a circle (or point), $0 \leq \phi < 2\pi$.

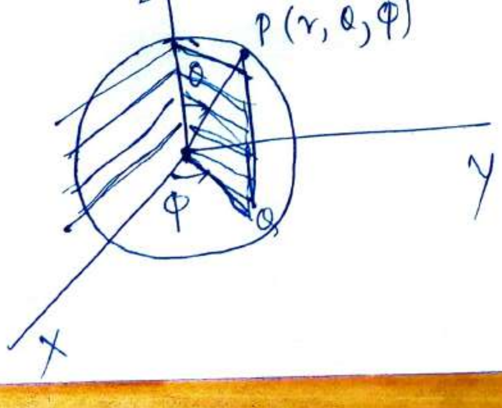
The coordinate surfaces

1. $r = c_1 \Rightarrow x^2+y^2+z^2 = c_1^2 = \text{const.} \Rightarrow$ spheres having centre at the origin (or origin if $c_1 = 0$)

2. $\theta = c_2 \Rightarrow$ cones having vertex at the origin (lines if $c_2 = 0$ or π and the xy plane if $c_2 = \pi/2$)



3. $\phi = c_3 \Rightarrow$ planes passing through the z-axis.



The unit tangent vectors along the three coordinate curves

We can write the position vector $\vec{r}$ as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

The tangent vector along the r-curve (for which θ, ϕ are constants)

$$\frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\left(r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}\right)$$

$$\underset{\uparrow}{\frac{\partial \vec{r}}{\partial r}} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

tangent vector to the r-curve

$\therefore$ Unit tangent vector to the r-curve is given by

$$\hat{e}_1 = \hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\underbrace{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}_{\Uparrow\ 1}}$$

$$\boxed{\therefore\ \hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \quad \text{with } h_1 = h_r = 1}$$

Similarly the tangent vector along θ-curve (for which r, ϕ constants)

$$\frac{\partial \vec{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}\right)$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$\therefore$ Unit tangent vector to the θ-curve is given by

$$\hat{e}_2 = \hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\underbrace{\sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}_{\Uparrow\ r}}$$

with $\boxed{h_\theta = r}$ $\boxed{\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}}$

The tangent vector along ϕ-curve (for which r, θ consts)

$$\frac{\partial \vec{r}}{\partial \phi} = \frac{\partial}{\partial \phi}\left(r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}\right)$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} \Rightarrow \text{tangent vector}$$

$\therefore$ The unit tangent vector along ϕ-curve is given by

$$\hat{e}_3 = \hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\underbrace{\sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}}_{r\sin\theta}}$$

$$\boxed{\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

with $h_3 = h_\phi = r\sin\theta$.

Thus the three unit tangent vectors are

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

In matrix notation:

$$\begin{pmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{pmatrix}$$

It can be shown that spherical polar coord. system is also orthogonal

ie $\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_r = 0$.