PORTFOLIO THEORY

The Benefits of Diversification

PORTFOLIO EXPECTED RETURN

$$E(R_P) = \sum_{i=1}^{n} w_i \, E(R_i)$$

where $E(R_P)$ = expected portfolio return

w_i = weight assigned to security i

$E(R_i)$ = expected return on security i

n = number of securities in the portfolio

Example A portfolio consists of four securities with expected returns of 12%, 15%, 18%, and 20% respectively. The proportions of portfolio value invested in these securities are 0.2, 0.3, 0.3, and 0.20 respectively.

The expected return on the portfolio is:

$$E(R_P) = 0.2(12\%) + 0.3(15\%) + 0.3(18\%) + 0.2(20\%)$$

$$= 16.3\%$$

PORTFOLIO RISK

The risk of a portfolio is measured by the variance (or standard deviation) of its return. Although the expected return on a portfolio is the weighted average of the expected returns on the individual securities in the portfolio, portfolio risk is not the weighted average of the risks of the individual securities in the portfolio (except when the returns from the securities are uncorrelated).

MEASUREMENT OF COMOVEMENTS IN SECURITY RETURNS

- To develop the equation for calculating portfolio risk we need information on weighted individual security risks and weighted comovements between the returns of securities included in the portfolio.

- Comovements between the returns of securities are measured by covariance (an absolute measure) and coefficient of correlation (a relative measure).

COVARIANCE

$$\text{Cov}\,(R_i\,, R_j) = p_1\,[R_{i1} - E(R_i)]\;[\,R_{j1} - E(R_j)]$$

$$+\, p_2\,[R_{i2} - E(R_j)]\;[R_{j2} - E(R_j)]$$

$$+$$

$$\vdots$$

$$+\, p_n\,[R_{in} - E(R_i)]\;[R_{jn} - E(R_j)]$$

ILLUSTRATION

The returns on assets 1 and 2 under five possible states of nature are given below

State of nature	Probability	Return on asset 1	Return on asset 2
1	0.10	-10%	5%
2	0.30	15	12
3	0.30	18	19
4	0.20	22	15
5	0.10	27	12

The expected return on asset 1 is :

$$E(R_1) = 0.10\,(-10\%) + 0.30\,(15\%) + 0.30\,(18\%) + 0.20\,(22\%) + 0.10\,(27\%) = 16\%$$

The expected return on asset 2 is :

$$E(R_2) = 0.10\,(5\%) + 0.30\,(12\%) + 0.30\,(19\%) + 0.20\,(15\%) + 0.10\,(12\%) = 14\%$$

The covariance between the returns on assets 1 and 2 is calculated below :

State of nature (1)	*Probability* (2)	*Return on asset 1* (3)	*Deviation of the return on asset 1 from its mean* (4)	*Return on asset 2* (5)	*Deviation of the return on asset 2 from its mean* (6)	*Product of the deviations times probability* (2) x (4) x (6)
1	0.10	-10%	-26%	5%	-9%	23.4
2	0.30	15%	-1%	12%	-2%	0.6
3	0.30	18%	2%	19%	5%	3.0
4	0.20	22%	6%	15%	1%	1.2
5	0.10	27%	11%	12%	-2%	-2.2
						Sum = 26.0

Thus the covariance between the returns on the two assets is 26.0.

CO EFFIENT OF CORRELATION

$$\text{Cor}(R_i, R_j) \text{ or } \rho_{ij} = \frac{\text{Cov}(R_i, R_j)}{\sigma(R_i, R_j)}$$

$$= \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

$$\sigma_{ij} = \rho_{ij} \cdot \sigma_i \cdot \sigma_j$$

where ρ_{ij} = correlation coefficient between the returns on securities *i* and *j*

σ_{ij} = covariance between the returns on securities *i* and *j*

σ_i, σ_j = standard deviation of the returns on securities *i* and *j*

PORTFOLIO RISK : 2 – SECURITY CASE

$$\sigma_p = [w_1^2\ \sigma_1^2 + w_2^2\ \sigma_2^2 + 2w_1w_2\ \rho_{12}\ \sigma_1\ \sigma_2]^{½}$$

Example : $w_1 = 0.6$, $w_2 = 0.4$,

$\sigma_1 = 10\%$, $\sigma_2 = 16\%$

$\rho_{12} = 0.5$

$$\sigma_p = [0.6^2 \text{ x } 10^2 + 0.4^2 \text{ x } 16^2 + 2 \text{ x } 0.6 \text{ x } 0.4 \text{ x } 0.5 \text{ x } 10 \text{ x } 16]^{½}$$

$$= 10.7\%$$

PORTFOLIO RISK : n – SECURITY CASE

$\sigma_p = [\ \sum \sum w_i w_j \rho_{ij} \sigma_i \sigma_j\ ]^{1/2}$

Example : $w_1 = 0.5$, $w_2 = 0.3$, and $w_3 = 0.2$

$\sigma_1 = 10\%$, $\sigma_2 = 15\%$, $\sigma_3 = 20\%$

$\rho_{12} = 0.3$, $\rho_{13} = 0.5$, $\rho_{23} = 0.6$

$\sigma_p = [w_1^2\ \sigma_1^2 + w_2^2\ \sigma_2^2 + w_3^2\ \sigma_3^2 + 2\ w_1\ w_2\ \rho_{12}\ \sigma_1\ \sigma_2$

$+ 2w_2\ w_3\ \rho_{13}\ \sigma_1\ \sigma_3 + 2w_2\ w_3\ \rho_{23} \sigma_2\ \sigma_3]^{1/2}$

$= [0.5^2 \text{ x } 10^2 + 0.3^2 \text{ x } 15^2 + 0.2^2 \text{ x } 20^2$

$+ 2 \text{ x } 0.5 \text{ x } 0.3 \text{ x } 0.3 \text{ x } 10 \text{ x } 15$

$+ 2 \text{ x } 0.5 \text{ x } 0.2 \text{ x } 05 \text{ x } 10 \text{ x } 20$

$+ 2 \text{ x } 0.3 \text{ x } 0.2 \text{ x } 0.6 \text{ x } 15 \text{ x } 20]^{1/2}$

$= 10.79\%$

RISK OF AN N - ASSET PORTFOLIO

$$\sigma^2_p = \Sigma\,\Sigma\, w_i w_j\, \rho_{ij} \sigma_i \sigma_j$$

n x *n* MATRIX

	1	2	3	…	*n*
1	$w_1^2\sigma_1^2$	$w_1 w_2 \rho_{12} \sigma_1 \sigma_2$	$w_1 w_3 \rho_{13} \sigma_1 \sigma_3$	…	$w_1 w_n \rho_{1n} \sigma_1 \sigma_n$
2	$w_2 w_1 \rho_{21} \sigma_2 \sigma_1$	$w_2^2\sigma_2^2$	$w_2 w_3 \rho_{23} \sigma_2 \sigma_3$	…	$w_2 w_n \rho_{2n} \sigma_2 \sigma_n$
3	$w_3 w_1 \rho_{31} \sigma_3 \sigma_1$	$w_3 w_2 \rho_{32} \sigma_3 \sigma_2$	$w_3^2\sigma_3^2$	…	
:	:				:
n	$w_n w_1 \rho_{n1} \sigma_n \sigma_1$				$w_n^2\sigma_n^2$

DOMINANCE OF COVARIANCE

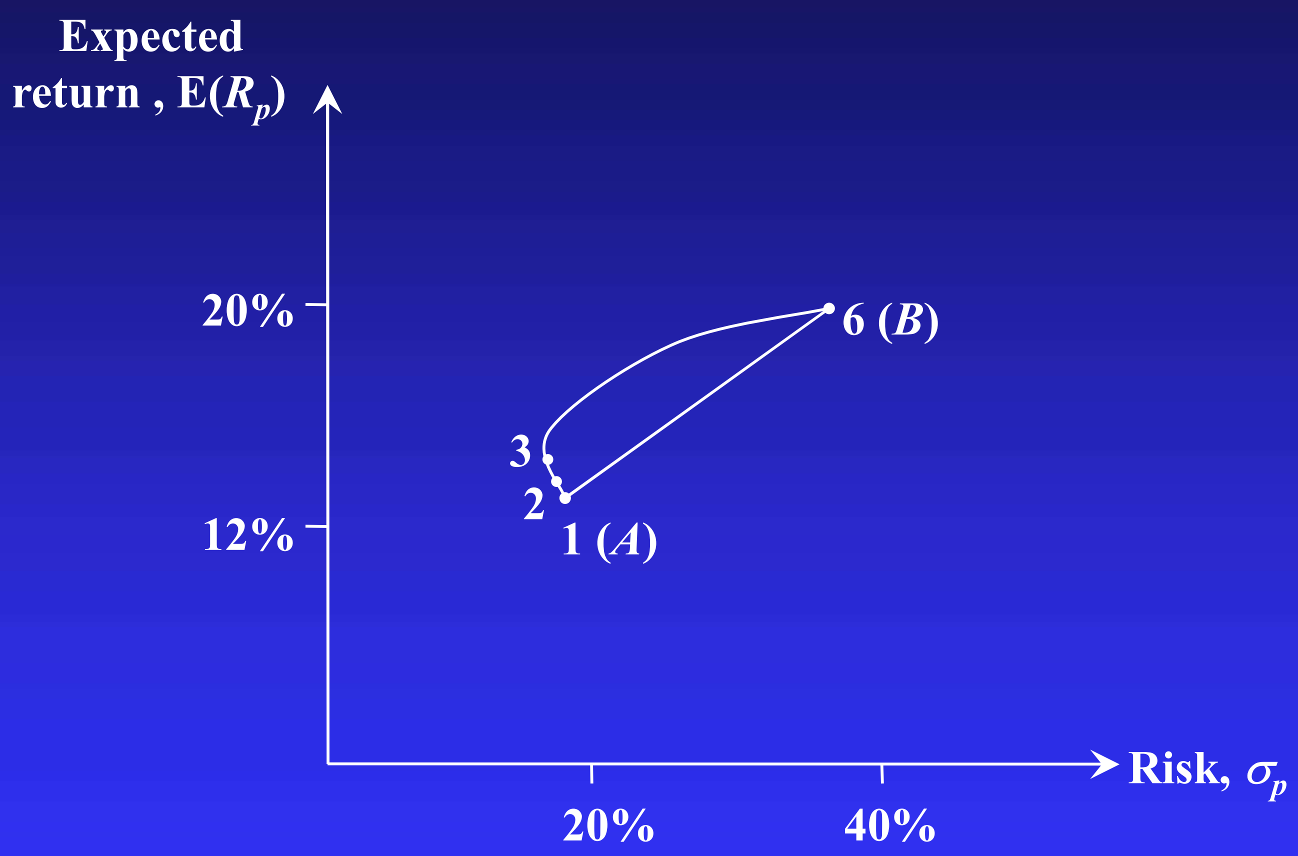
As the number of securities included in a portfolio increases, the importance of the risk of each individual security decreases whereas the significance of the covariance relationship increases

EFFICIENT FRONTIER FOR A TWO-SECURITY CASE

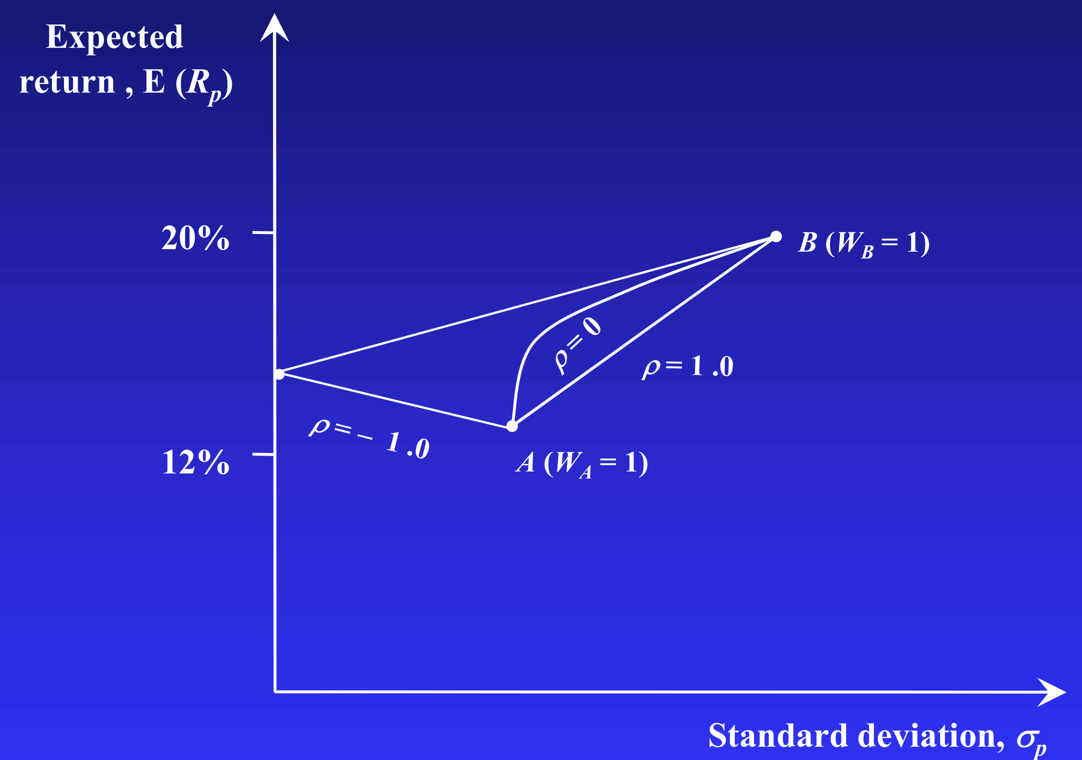
	Security A	Security B
Expected return	12%	20%
Standard deviation	20%	40%
Coefficient of correlation	-0.2	

Portfolio	*Proportion of A* w_A	*Proportion of B* w_B	*Expected return* $E(R_p)$	*Standard deviation* σ_p
1 (A)	1.00	0.00	12.00%	20.00%
2	0.90	0.10	12.80%	17.64%
3	0.759	0.241	13.93%	16.27%
4	0.50	0.50	16.00%	20.49%
5	0.25	0.75	18.00%	29.41%
6 (B)	0.00	1.00	20.00%	40.00%

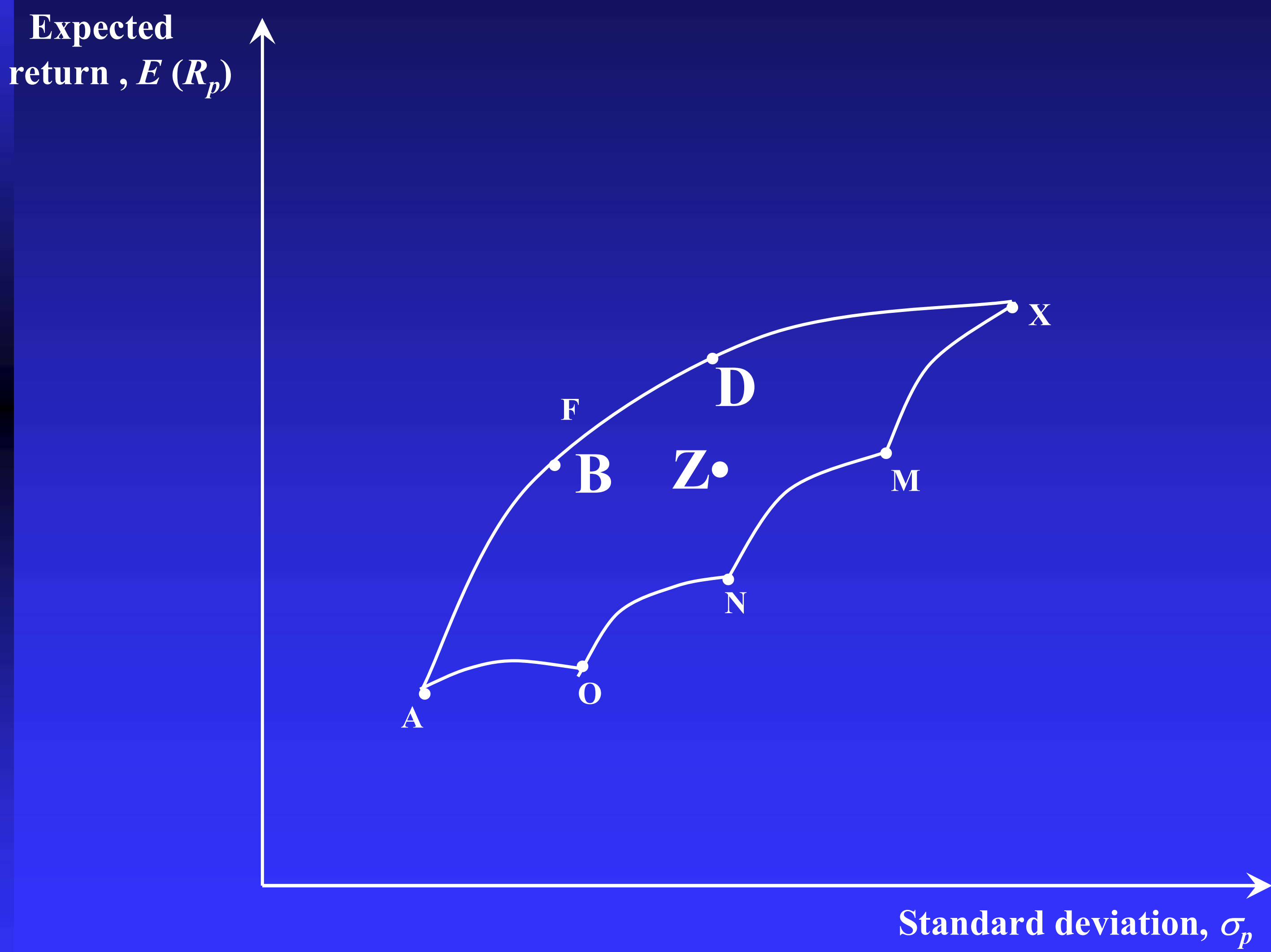
PORTFOLIO OPTIONS AND THE EFFICIENT FRONTIER



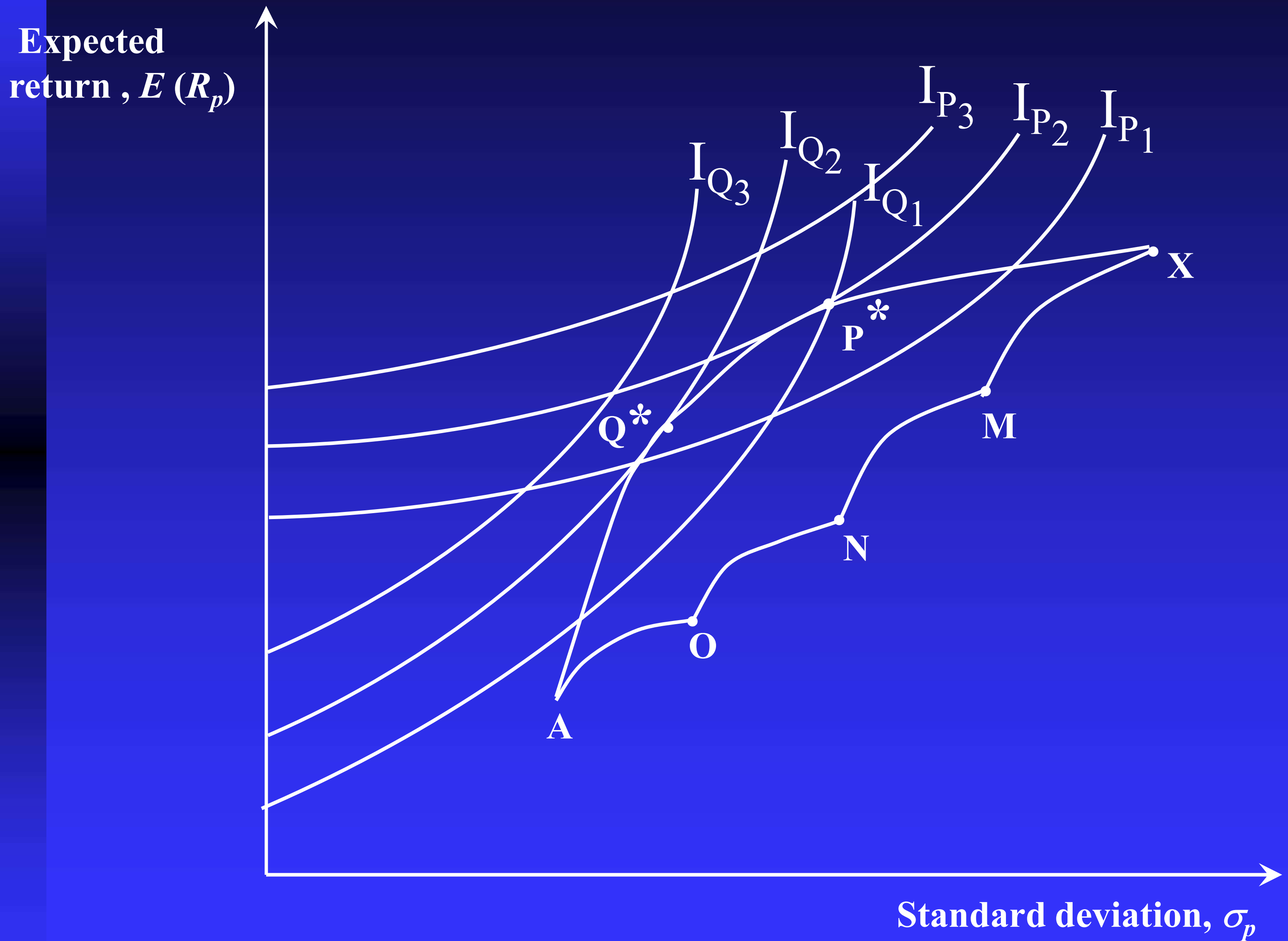
FEASIBLE FRONTIER UNDER VARIOUS DEGREES OF COEFFICIENT OF CORRELATION



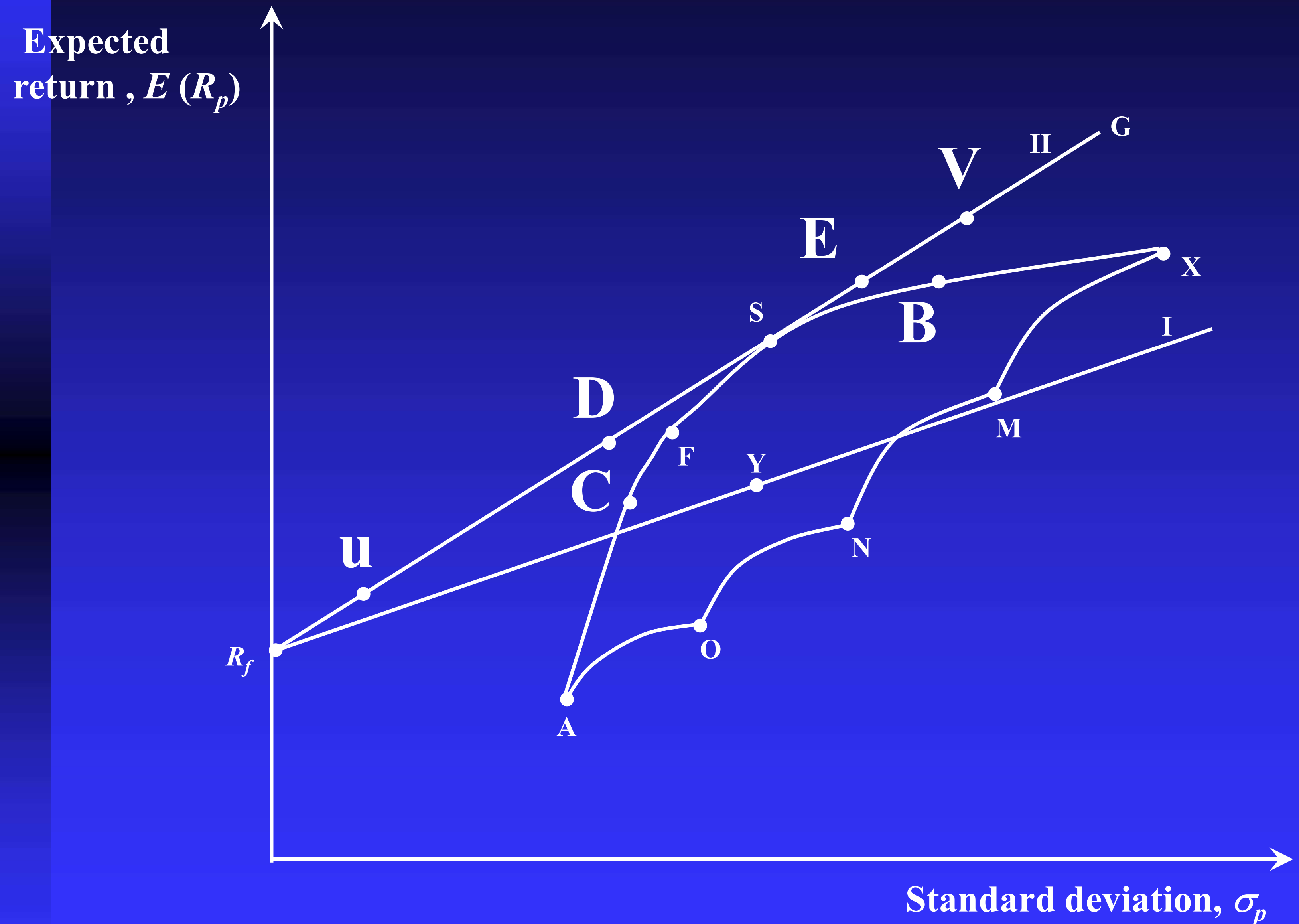
EFFICIENT FRONTIER FOR THE n-SECURITY CASE



OPTIMAL PORTFOLIO



RISKLESS LENDING AND BORROWING OPPORTUNITY



Thus, with the opportunity of lending and borrowing, the efficient frontier changes. It is no longer AFX. Rather, it becomes R_f SG as it domniates AFX.

SEPARATION THEOREM

- **Since R_f SG dominates AFX, every investor would do well to choose some combination of R_f and S. A conservative investor may choose a point like U, whereas an aggressive investor may choose a point like V.**

- **Thus, the task of portfolio selection can be <u>separated</u> into two steps:**

  1. **Identification of S, the optimal portfolio of risky securities.**
  2. **Choice of a combination of R_f and S, depending on one's risk attitude.**

This is the import of the celebrated <u>separation theorem</u>

SINGLE INDEX MODEL

INFORMATION - INTENSITY OF THE MARKOWITZ MODEL

n SECURITIES

n VARIANCE TERMS & $n(n-1)/2$ COVARIANCE TERMS

SHARPE'S MODEL

$$R_{it} = a_i + b_i R_{Mt} + e_{it}$$

$$E(R_i) = a_i + b_i E(R_M)$$

$$\text{VAR}(R_i) = b_i^2 [\text{VAR}(R_M)] + \text{VAR}(e_i)$$

$$\text{COV}(R_i, R_j) = b_i b_j \text{VAR}(R_M)$$

MARKOWITZ MODEL	SINGLE INDEX MODEL
$n(n+3)/2$	$3n+2$
$E(R_i)$ & VAR (R_i) FOR EACH SECURITY $n(n-1)/2$ COVARIANCE TERMS	b_i, b_j VAR (e_i) FOR EACH SECURITY & $E(R_M)$ & VAR (R_M)