Mutual Information →

It is defined as the amount of information transferred when x_i is transmitted and y_j is received.

It is represented by $I(x_i; y_j)$ and given by

$$I(x_i, y_j) = \log_2 \frac{P(x_i/y_j)}{P(x_i)} \text{ bits}$$

Where $P(x_i/y_j) \rightarrow$ conditional probability that was x_i transmitted and y_j is received.

$P(x_i) \rightarrow$ Probability of symbol x_i for transmission.

Average mutual Information →

It is defined as the amount of source information gained per received symbol.

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j)\, I(x_i; y_j)$$

$$\boxed{I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}}$$

Properties →

1-) The mutual Information of the channel is symmetric i.e.

$$I(X;Y) = I(Y;X)$$

Proof → We know that $P(x_i, y_j) = P(x_i/y_j)\, P(y_j)$ → ①

and $P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$ → (2)

where $P(x_i, y_j)$ → is joint probability that x_i transmitted and y_j received.

$P(x_i/y_j)$ → conditional probability that x_i is transmitted and y_j is received.

$P(y_j/x_i)$ → ,, ,, ,, y_j is ,, and x_i is received.

from eqn ① & ② we get

$$P(x_i/y_j)\, P(y_j) = P(y_j/x_i)\, P(x_i)$$

$$\frac{P(x_i/y_j)}{P(x_i)} = \frac{P(y_j/x_i)}{P(y_j)} \rightarrow ③$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} \rightarrow ④$$

$$I(Y;X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(y_j/x_i)}{P(y_j)} \rightarrow ⑤$$

$$I(Y;X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

$$\boxed{I(Y;X) = I(X;Y)}$$

Proved

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2-) The mutual information can be enpressed in terms of entrapies of channel input or output and conditional entrapies i.e

$$I(X;Y) = H(X) - H(X/Y) \rightarrow ①$$

$$I(X;X) = H(Y) - H(Y/X) \rightarrow ②$$

where $H(X/Y)$ & $H(Y/X)$ are conditional entrapies.

Proof $\rightarrow$ we farow that $\rightarrow$

$$H(X/Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)} \rightarrow ①$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{1}{P(x_i)}\right) + \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 P(x_i/y_j)$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{1}{P(x_i)}\right) - \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{1}{P(x_i/y_j)}\right)$$

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i)} - H(X/Y) \rightarrow (2)$$

we farhow that $\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$

$$I(X;Y) = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)} - H(X/Y) \rightarrow ③$$

$$\boxed{I(X;Y) = H(X) - H(X/Y)}$$

where $H(X) = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)}$

Similarly we can prove $I(Y;X) = H(Y) - H(Y/X)$

3-) mutual Information is always positive.

$$I(X;Y) \geq 0$$

Proof → we know that $I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$ → 1

we know that $\boxed{P(x_i, y_j) = P(x_i/y_j)\, P(y_j)}$

$$P(x_i/y_j) = \frac{P(x_i, y_j)}{P(y_j)}$$

put this value in eqn 1 we get

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log \frac{P(x_i, y_j)}{P(x_i)\, P(y_j)}$$

$$I(X;Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i)\, P(y_j)}{P(x_i, y_j)}$$

$$-I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i)\, P(y_j)}{P(x_i, y_j)}$$

Let $P(x_i)\, P(y_j) = q_k$

$P(x_i, y_j) = P_k$

$$-I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{q_k}{p_k} \rightarrow (2)$$

We know that $\sum_{k=1}^{m} p_k \log_2 \left(\frac{q_k}{p_k}\right) \leq 0$ { upper bound entropy

$$-I(X;Y) \leq 0$$

$$\boxed{I(X;Y) \geq 0}$$

Proved

4) The mutual information is related to the joint entropy $H(X,Y)$ by following relation:

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

Proof →

Proof: As we know,

$$I(x;y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{P(x_i|y_j)}{P(x_i)}\right) \quad \text{-①}$$

& $$P(x_i, y_j) = P(x_i|y_j)\, P(y_j)$$

$$P(x_i|y_j) = \frac{P(x_i, y_j)}{P(y_j)}$$

from ①

$$I(x;y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{P(x_i, y_j)}{P(x_i)P(y_j)}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\left\{\log_2 \frac{1}{P(x_i)} + \log_2 \frac{1}{P(y_j)} - \log_2 \frac{1}{P(x_i, y_j)}\right]$$

$$= \sum_{i=1}^{n} P(x_i) \log_2 \left(\frac{1}{P(x_i)}\right) + \sum_{j=1}^{m} P(y_j) \log_2 \left(\frac{1}{P(y_j)}\right)$$

$$- \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{1}{P(x_i, y_j)}\right)$$

$$\boxed{I(x;y) = H(x) + H(y) - H(x,y)}$$ Proved