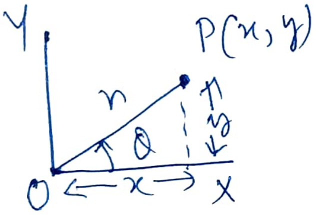
Polar Coordinates (r, θ)

Generally we use rectangular or cartesian coordinates to describe the motion in a straight line.

However rectangular coord. are not so useful to describe circular motion.
Since circular motion plays an important role in physics so it is worth to introduce another coordinate system which is well suited to describe the circular motion.



Our new coordinate system is known as polar coordinate system.

Let r be the distance of a point from the origin O and θ is the angle that r makes with the x-axis.

The relations between the polar coordinate (r, θ) and the cartesian coordinates (x, y) are given below

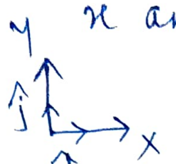
$$x = r\cos\theta, \quad y = r\sin\theta$$

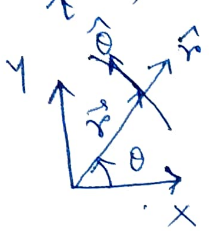
$$\therefore r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}(y/x).$$

The coordinates (r, θ) are known as plane polar coordinates.

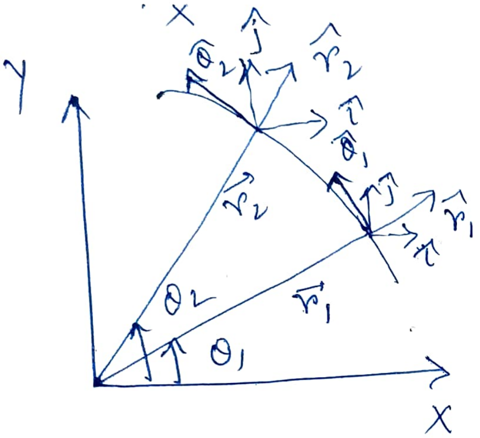
Next we will describe position, velocity and acceleration in plane polar coordinates.

In rectangular coordinate system we have introduced the unit vector $\hat{i}$ and $\hat{j}$ which point the direction of increasing x and y respectively.



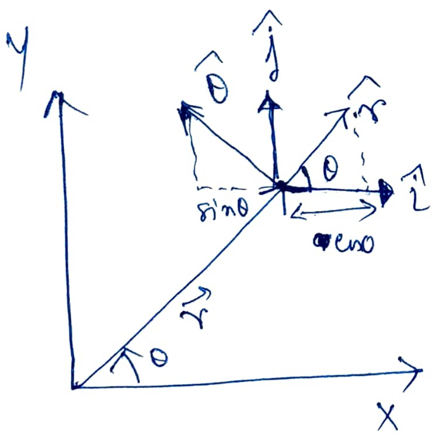


Similarly we now introduce the two new unit vectors $\hat{r}$ and $\hat{\theta}$ which points the direction of increasing r and increasing θ respectively.



There is an important difference between these base vectors and the Cartesian base vectors:
The direction of $\hat{r}$ and $\hat{\theta}$ vary with position whereas $\hat{i}$ and $\hat{j}$ have fixed directions. It is shown in the figure.

Although $\hat{r}$ and $\hat{\theta}$ vary with position note that they only depend on θ not on r.



From this figure we see

$$\hat{r} = |\hat{r}|\cos\theta\,\hat{i} + |\hat{r}|\sin\theta\,\hat{j}$$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

since $|\hat{r}| = 1$ unit vect.

and $$\hat{\theta} = -|\hat{\theta}|\sin\theta\,\hat{i} + |\hat{\theta}|\cos\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

since $|\hat{\theta}| = 1$ unit vect.

$$\therefore \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix}$$

So in polar coordinate system we write the position vector

$$\vec{r} = r\,\hat{r} \quad (1) \quad \text{where } r = |\vec{r}|$$

Velocity in Polar Cordinate System

Let us now turn our attention to describe velocity in polar coordinate system

We know

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j}) = \dot{x}\hat{i} + \dot{y}\hat{j}$$

where $\dot{x} = \frac{dx}{dt} \Rightarrow$ time derivative

$\dot{y} = \frac{dy}{dt} \Rightarrow$ time derivative.

In polar cordinat systm

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} r\hat{r}(\theta)$$

$$= \frac{dr}{dt}\hat{r}(\theta) + r\frac{d}{dt}\hat{r}(\theta)$$

$$= \dot{r}\hat{r}(\theta) + r\frac{d}{dt}\hat{r}(\theta)$$

We know $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$$\therefore \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}(\cos\theta\hat{i} + \sin\theta\hat{j})\dot{\theta} \quad \left[\text{where } \dot{\theta} = \frac{d\theta}{dt}\right]$$

$$= \underbrace{(-\sin\theta\hat{i} + \cos\theta\hat{j})}_{\hat{\theta}}\dot{\theta} = \dot{\theta}\hat{\theta}$$

$$\therefore \boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}} \Rightarrow \text{important result}$$

Similarly

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j})\dot{\theta}$$

$$= (-\cos\theta\hat{i} - \sin\theta\hat{j})\dot{\theta}$$

$$= -\underbrace{(\cos\theta\hat{i} + \sin\theta\hat{j})}_{\hat{r}}\dot{\theta}$$

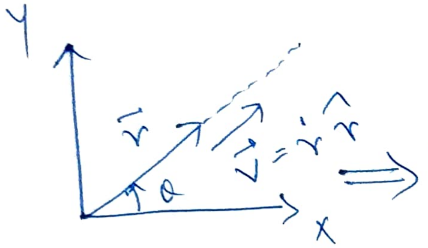
$$= -\hat{r}\dot{\theta}$$

$$\therefore \boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}}$$

important result

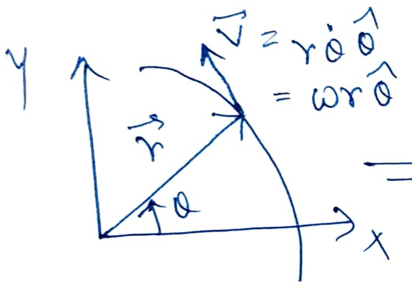
$\therefore$ The velocity $\vec{v} = \dot{r}\hat{r} + r\frac{d}{dt}\hat{r}(\theta)$

$$\boxed{\vec{v} = \underbrace{\dot{r}\hat{r}}_{\substack{\text{radial component}\\ \text{of velocity}}} + \underbrace{r\dot{\theta}\hat{\theta}}_{\substack{\text{tangential component of}\\ \text{velocity}}}}$$



$\Longrightarrow$ In this case θ is const. i.e. $\frac{d\theta}{dt} = 0 = \dot{\theta}$

$\therefore \vec{v} = \dot{r}\hat{r} \Rightarrow$ We have one dimensional motion in a fixed radial direction.



$\Longrightarrow$ In this case r = constant $\therefore \dot{r} = \frac{dr}{dt} = 0$

$\therefore$ The velocity $\vec{v} = r\dot{\theta}\hat{\theta} = \omega r\hat{\theta}$

since $|\vec{r}|$ is fixed motion lies on the arc of a circle.

For motion in general both r and θ change in time and velocity $\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$ where $v_r = \dot{r} = \frac{dr}{dt}$

$$= \underbrace{v_r\hat{r}}_{\text{radial part}} + \underbrace{v_\theta\hat{\theta}}_{\text{tangential part}}$$

$v_\theta = r\dot{\theta} = r\frac{d\theta}{dt}$

Acceleration in Polar coordinates

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$\dot{\theta}\hat{\theta}$

time derivative of $\dot{\theta}$

$$= \frac{d}{dt}(\dot{r})\hat{r} + \dot{r}\left(\frac{d}{dt}\hat{r}\right) + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r})$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

∴ Acceleration $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$ — (1)

$$\boxed{\vec{a} = a_r\hat{r} + a_\theta\hat{\theta}}$$

where $a_r = \ddot{r} - r\dot{\theta}^2$ is the radial part of acceleration

and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}\,\hat{\theta}$ is the tangential part of acceleration.

Now for uniform circular motion

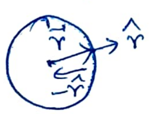
r = radius = constant ie $\dot{r} = 0$, $\ddot{r} = 0$

$v = \omega r = \dot{\theta} r$ = constant ⇒ ∴ $\ddot{\theta} = 0$ since $\omega = \frac{d\theta}{dt} = \dot{\theta} \Rightarrow$ constant

Then from eqn. (1)

$$\boxed{\vec{a} = -r\dot{\theta}^2\hat{r} = -\omega^2 r\,\hat{r}}$$ ⇒ centripetal acceleration

This centripetal acceleration acts towards the centre of the circle ie in $(-\hat{r})$ direction.



The term $2\dot{r}\dot{\theta}\hat{\theta}$ is the coriolis acceleration.

The coriolis acceln. $(2\dot{r}\dot{\theta}\hat{\theta})$ here is a real acceleration which is present when both r, θ change with time.