Duality in LPP: To every problem in LPP there corresponds another LPP called dual. The original is called primal

Importance:

(1) If primal contains large no. of constraints & smaller no. of variables, the computations will be more. These can be reduced by solving the corresponding dual problem

(2) Calculation of dual checks accuracy of primal solution

(3) Duality concept helps in sensitivity analysis (which will be dealt with lala)

(4) Importance of dual variables from cost or economic point of view proves useful in making future decision.

Formulation of Dual Problem

Primal

max $z = \sum c_i x_i$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, 2, -- m$

$x_j \geq 0 . \quad j = 1, 2, -- n$

Dual

min $w = \sum b_i y_i$

subject to

$\sum_{i=1}^{m} a_{ji} y_i \geq c_j$

$j = 1, 2, -- n$

$y_i \geq 0 \quad i = 1, 2 -- m$

Primal	Dual
(1) No. of variables	No. of constraints
(2) No. of constraints	No. of variables
(3) Cost coff.	Resources (constraint constants)
(4) Constraint constants (Resources)	cost coff
(5) ith variable is unrestricted	ith constraint is equality
(6) jth constraint is equality	jth variable is unrestricted
(7) A	A^T

Ex: Write the dual of following LPP!

max $z = x_1 + 2x_2 + x_3$ → Primal

subject to

$$\left.\begin{aligned} 2x_1 + x_2 - x_3 &\le 2 \\ 2x_1 - x_2 + 5x_3 &\le 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}\right\} \quad \text{—(I)}$$

Dual min $w = 2y_1 + 6y_2$

subject to

$$\left.\begin{aligned} 2y_1 + 2y_2 &\geq 1 \\ y_1 - y_2 &\geq 2 \\ -y_1 + 5y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}\right\} \quad \text{—(II)}$$

Note: If (II) is primal then (I) will be dual.

Ex Given min $z = 3x_1 - 2x_2 + 4x_3$

subject to $3x_1 + 5x_2 + 4x_3 \geq 7$

$7x_1 - 2x_2 - x_3 \leq 10 \Leftrightarrow -7x_1 + 2x_2 + x_3 \geq -10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

find dual.

Soln Dual max $w = 7y_1 - 10y_2 + 3y_3 + 2y_4$

subject to

$3y_1 - 7y_2 + y_3 + 4y_4 \leq 3$

$5y_1 + 2y_2 - 2y_3 + 7y_4 \leq -2 \Leftrightarrow -5y_1 - 2y_2 + 2y_3 - 7y_4 \geq 2$.

$4y_1 + y_2 + 5y_3 - 2y_4 \leq 4$

$y_1, y_2, y_3, y_4 \geq 0$

Ex min $z = 2x_2 + 5x_3$

subject to

$2x_1 + x_2 + 6x_3 \leq 6 \Leftrightarrow -2x_1 - x_2 - 6x_3 \geq -6$

$x_1 + x_2 \geq 2$

$x_1 - x_2 + 3x_3 = 4 \Leftrightarrow x_1 - x_2 + 3x_3 \leq 4$

$-x_1 + x_2 - 3x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

$2x + 3y = 5$

$\Leftrightarrow$

$2x + 3y \leq 5$

& $2x + 3y \geq 5$

$\Leftrightarrow 2x + 3y \leq 5$

& $-2x - 3y \leq -5$.

$\therefore$ Dual

max $w = -6y_1 + 2y_2 + 4y_3 - 4y_4$

subject to

$-2y_1 + y_2 + y_3 - y_4 \geq 0$

$-y_1 + y_2 - y_3 + y_4 \geq 2$

$-6y_1 + 3y_3 - 3y_4 \geq 5$

$y_1, y_2, y_3, y_4 \geq 0$

max $w = -6y_1 + 2y_2 + 4(y_3 - y_4)$

Subject to

$-2y_1 + y_2 + (y_3 - y_4) \geq 0$ Let $y_3 - y_4 = y_5$.

$-y_1 + y_2 - (y_3 - y_4) \geq 2$

$-6y_1 + 3(y_3 - y_4) \geq 5$

then max $w = -6y_1 + 2y_2 + y_5$

subject to

$-2y_1 + y_2 + y_5 \geq 0$ $2y_1 - y_2 - y_5 \leq 0$

$-y_1 + y_2 - y_5 \geq 2$ or $y_1 - y_2 + y_5 \leq -2$

$-6y_1 + 3y_5 \geq 5$

$y_1, y_2, \geq 0$ $y_5 \rightarrow$ unrestricted

'or' max $w = -6y_1 + 2y_2 + y_3$

subject to

$-2y_1 + y_2 + y_3 \geq 0$

$-y_1 + y_2 - y_3 \geq 2$

$-6y_1 + 3y_3 \geq 5$

$y_1, y_2 \geq 0$, $y_3 \rightarrow$ unrestricted.

Ex Find dual of max $z = 2x_1 - 6x_2 + x_3$

subject to

$x_1 - 2x_2 + x_3 \leq 0$

$x_1 - x_2 - x_3 \leq 2$

$-6x_2 + 3x_3 \leq 5$

$x_1, x_2 \geq 0$ $x_3 \rightarrow$ unrestricted.

Soln $x_3 \rightarrow$ unrestricted $\therefore$ let $x_3 = x_3' - x_3''$

$\therefore$ max $z = 2x_1 - 6x_2 + (x_3' - x_3'')$

subject to

$x_1 - 2x_2 + (x_3' - x_3'') \leq 0$

$x_1 - x_2 - (x_3' - x_3'') \leq 2$

$-6x_2 + 3(x_3' - x_3'') \leq 5$

$x_1, x_2, x_3', x_3'' \geq 0$.

Dual $\min w = 2y_2 + 5y_3$

subject to

$y_1 + y_2 \geq 2$

$-2y_1 - y_2 - 6y_3 \geq -6 \Rightarrow 2y_1 + y_2 + 6y_3 \leq 6$

$y_1 - y_2 + 3y_3 \geq 1 \qquad y_1 - y_2 + 3y_3 \geq 1$

$-y_1 + y_2 - 3y_3 \geq -1 \Rightarrow y_1 - y_2 + 3y_3 \leq 1$

$y_1, y_2, y_3, y_4 \geq 0 \qquad \Rightarrow y_1 - y_2 + 3y_3 = 1$

$\therefore \min w = 2y_2 + 5y_3$

subject to

$y_1 + y_2 \geq 2$

$2y_1 + y_2 + 6y_3 \leq 6$

$y_1 - y_2 + 3y_3 = 1$

$y_3, y_1, y_2 \geq 0$, ~~y~~

Ex Write dual of $\min z = 4x_1 + 5x_2 - 3x_3$

subject to $x_1 + x_2 + x_3 = 22$

$3x_1 + 5x_2 - 2x_3 \leq 65$

$x_1 + 7x_2 + 4x_3 \geq 120$

$x_1, x_2 \geq 0$, $x_3 \rightarrow$ unrestricted.

Soln $\min z = 4x_1 + 5x_2 - 3(x_3' - x_3'')$

subject to

$x_1 + x_2 + (x_3' - x_3'') \geq 22$

$-x_1 - x_2 - (x_3' - x_3'') \geq -22$

$-3x_1 - 5x_2 + 2(x_3' - x_3'') \geq -65$

$x_1 + 7x_2 + 4(x_3' - x_3'') \geq 120$

$x_1, x_2, x_3', x_3'' \geq 0$.

<u>Dual</u> max $w = 22(y_1 - 22y_2) - 65y_3 + 120y_4$

subject to

$(y_1 - y_2) - 3y_3 + y_4 \le 4$

$(y_1 - y_2) - 5y_3 + 7y_4 \le 5$

$(y_1 - y_2) + 2y_3 + 4y_4 \le -3$

$(-y_1 + y_2) - 2y_3 - 4y_4 \le 3$

$y_1, y_2, y_3, y_4 \ge 0$

Let $y_1 - y_2 = y$. Then

max $w = 22y - 65y_3 + 120y_4$

subject to

$y - 3y_3 + y_4 \le 4$

$y - 5y_3 + 7y_4 \le 5$

$\left.\begin{array}{l} y + 2y_3 + 4y_4 \le -3 \\ -y - 2y_3 - 4y_4 \le 3 \end{array}\right] \Leftrightarrow -y - 2y_3 - 4y_4 = 3.$

$y \to$ unrestricted $y_3, y_4 \ge 0$

$\therefore$ Dual is

max $w = 22y_1 - 65y_2 + 120y_3$.

subject to

$y_1 - 3y_2 + y_3 \le 4$

$y_1 - 5y_2 + 7y_4 \le 5$

$-y_1 - 2y_2 - 4y_3 = 3$

$y_2, y_3 \ge 0$ $y_1 \to$ unrestricted

Theorems on Duality:

Thm 1 Dual of dual is primal.

Proof: Consider primal problem: $\max z = cx$

subject to

$Ax \leq b$

$x \geq 0$

$A_{m\times n}$ $b^T \in \mathbb{R}^m$, $c, x^T \in \mathbb{R}^n$.

Dual $\min w = b^T y$

subject to

$(A^T) y \geq c^T$

$y \geq 0$.

Dual $\max v = (c^T)^T t$

subject to

$(A^T)^T t \leq (b^T)^T$

$t \geq 0$

(OR) $\max v = ct$

subject to

$At \leq b$

$t \geq 0$

which is same as primal with z replaced by v & t replaced by x.

Hence, dual of dual is primal.

Thm 2 Let x_0 be feasible solution to the primal problem $\max z = cx$ s.t. $Ax \leq b$, $x \geq 0$. where x^T & $c \in \mathbb{R}^n$, $b^T \in \mathbb{R}^m$ and $A_{m\times n}$ matrix. If y_0 be feasible solution to dual of the primal, namely, $\min w = b^T y$ s.t. $A^T y \geq c^T$, $y \geq 0$ where $y^T \in \mathbb{R}^m$

then $cx_0 \leq b^T y_0$.

Proof: Since x_0 & y_0 are feasible solutions to the primal and its dual respectively, we must have
$Ax_0 \leq b$, $x_0 \geq 0$ & $A^T y_0 \geq c^T$, $y_0 \geq 0$:

Thus, $c \leq y_0^T A$ or, $cx_0 \leq y_0^T A x_0 \leq y_0^T b$:
$\Rightarrow cx_0 \leq y_0^T b = b^T y_0$ //.

Thm 3: Let x_0 be feasible solution to the primal problem
$\max z = cx$ s.t $Ax \leq b$, $x \geq 0$
and y_0 be feasible solution to its dual
$\min w = b^T y$ s.t. $A^T y \geq c^T$, $y \geq 0$
where x^T and $c \in \mathbb{R}^n$, y^T & $b^T \in \mathbb{R}^m$ and $A_{m \times n}$ real matrix
If $cx_0 = b^T y_0$, then both x_0 and y_0 are optimum solutions to the primal and dual respectively.

Proof: Let x_0^* be any other feasible solution to primal problem, then
$$cx_0^* \leq b^T y_0 \quad \text{(from Thm 2)}$$
$$= cx_0 \quad \text{(given)}$$

$\Rightarrow cx_0^* \leq cx_0$
$\Rightarrow x_0$ is optimum solution to primal problem
($\because$ primal is maximization problem)

Similarly, if y_0^* is any other feasible solution to dual problem then $b^T y_0 \leq b^T y_0^*$ (prove it yourself)
$\Rightarrow y_0$ is optimum solution to the dual problem
($\because$ dual is minimization problem). //.