BMC 101 Calculus

Unit I

Real Numbers: Totallity of rationals and irrational numbers

Rational Numbers: are of the form $\frac{p}{q}$, $p,q \in \mathbb{Z}$ or non-terminating & reccuring e.g. $\frac{2}{5}, \frac{1}{3} = 0.333\ldots$

Irration numbers: Non-terminating and non-recurring e.g. $\sqrt{2} = 1.41421356\ldots$

System of real numbers are constructed from rational numbers in axiomatic manner

Completeness Axiom: This axiom distinguishes between $\mathbb{Q}$ and $\mathbb{R}$, hence justifies existence of irrational number.

Statement: Every non-empty subset of $\mathbb{R}$ which is bounded above has supremum in $\mathbb{R}$.

Before understanding this we need to understand few other concepts/terms.

Dedikind-Cantor Axiom: To every real number corresponds a unique point on directed line and to every point on directed line corresponds a unique real number. Hence, these can be used interchangeably

The following results are obvious:

1. If $a, b \in \mathbb{R}$ and $a < b$ then a lies to left of b
2. -ive number lies to left of zero & +ive number to right

3. If $a < c < b$, then point c lies between a & b

Intervals: set which contains all real numbers between any two given points

$(a,b) = \{x \in \mathbb{R} : a < x < b)$ ↳ open

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ ↳ closed

$(a,b] = \{x \in \mathbb{R} : a < x \le b\}$ $[a,b) = \{x \in \mathbb{R} : a \le x < b\}$

semi-open

Note: length of each of the above intervals $= (b-a)$.

(a, ∞), $[a, \infty)$, $(-\infty, a)$, $(-\infty, a]$ are infinite intervals

Absolute value: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

(1) $|x| \geq 0$

(2) $|-x| = |x|$

(3) $|xy| = |x|\,|y|$

(4) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$

(5) $|x+y| \le |x| + |y|$

(6) $|x-y| \geq ||x| - |y||$

(7) $|x| < a \Rightarrow -a < x < a \cong x \in (-a, a)$

Note: (5) & (6) are triangle inequality.

Bounded sets: A subset $S \subset \mathbb{R}$ is said to be bounded above if $\exists\, M$ such that $x \le M$, $\forall x \in S$.

A subset $S \subset \mathbb{R}$ is said to be bounded below if $\exists\, m$ s.t. $x \geq m$ $\forall x \in S$.

A set is said to be bounded iff both lower and upper bound exist for set S.

Example: (1) $\mathbb{N} \to$ bounded below by 1 & unbounded above

(2) $\mathbb{Z}^- \to$ upper bound 0 & unbdd below.

(3) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \to$ bounded above by 1 & below by 0

(4) $\mathbb{Q}, \mathbb{Z}, \mathbb{R} \to$ unbdd from above as well as below

(5) Every finite set is bounded.

Ex Which of the following sets are bounded below, above neither or both.

(a) $\{x : x = (-2)^n, n \in \mathbb{N}\} \to$ unbounded.

(b) $\left\{ x : x = \frac{4(-1)^n}{n}, n \in \mathbb{N} \right\} \to$ bounded below by -4 & bounded above by 2

(c) $\left\{ x : x = \frac{1}{2^n}, n \in \mathbb{N} \right\} \to$ bounded above by $\frac{1}{2}$ & bounded below by 0.

Ex Prove that $\{|x| : x \in S\}$ is bounded iff S is bounded

soln S is bounded $\Rightarrow \exists\, m$ & M such that $m \leq x \leq M$ $\forall x \in S$. In particular, take $m = -M$. Then $-M \leq x \leq M \Rightarrow |x| \leq M \Rightarrow \{|x| : x \in S\}$ is bounded //

Conversely Let $\{|x| : x \in S\}$ is bounded.

$\Rightarrow \exists\, M$ such that $|x| \leq M$ $\forall x \in S$.

$\Rightarrow -M \leq x \leq M$ $\forall x \in S$.

$\Rightarrow$ Upper and lower bound exists for S.

$\Rightarrow S$ is bounded //

Supremum (or Least upper bound (lub)): If $S \subset \mathbb{R}$ is bounded above then it will have infinitely many upper bounds. Then

Sup S = lub S = least among all the upperbounds of S.

So, if $\alpha = \text{Sup} S$ then

(1) α is upperbound of S (2) $\alpha \leq M$ for every upperbound M of S.

In a similar way one can define <u>Infimum</u> as

$\beta = \text{Inf} S$ then

(1) β is lower bound of S (2) $\beta \geq m$ for every lower bound m of S.

Infimum is also called ~~greattest~~ greatest lower bound (glb).

<u>Ex</u> Find Supremum & Infimum of the following, if they exist:

(1) $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$ $\text{Sup} S = 1$ $\text{Inf} S = 0$.

(2) $S = \{1, 2, 3, 4, 5\}$ $\text{Sup} S = 5$ $\text{Inf} S = 1$

(3) $S = \{-1, 0, 2, 1, -2, 3\}$ $\text{Sup} S = 3$ $\text{Inf} S = -2$.

(4) $\mathbb{N} \rightarrow$ Sup $\mathbb{N}$ does not exist, $\text{Inf} \mathbb{N} = 0$.

<u>Theorem</u>: If Sup S exists where $S \neq \varphi$, and $S \subseteq R$, then it is unique.

<u>Pf</u>: Let α, β be two distinct supremum of S

$\Rightarrow \alpha \leq \beta$ ($\because \beta = \text{Sup} S$) & $\beta \leq \alpha$ ($\because \alpha = \text{Sup} S$)

$\Rightarrow \alpha = \beta$

Hence, supremum of a set if it exists is unique.

<u>Theorem</u>: If Inf S exists where, $S \neq \varphi$ and $S \subseteq \mathbb{R}$, then it is unique

<u>Proof</u>: Do it.

Ex Find supremum & infimum if they exists.

(1) $S = \{x : x = \frac{n}{n+1}, n \in \mathbb{N}\}$ Sup S = 1
Inf S = $\frac{1}{2}$

(2) $S = \{x \in \mathbb{R} : -2x < 3\}$ Sup S = does not exist
Inf S = -1.5

(3) $S = [1, 2] \cup [3, 4)$ Sup S = 4
Inf S = 1

(4) $\{\sin \frac{n\pi}{2} : n \in \mathbb{N}\}$ Sup S = 1
Inf S = -1.

Ex Prove that $\sqrt{2}$ is irrational.

Proof: Suppose not.
$\Rightarrow \exists\, p, q \in \mathbb{N}, q \neq 0$ such that $\sqrt{2} = \frac{p}{q}$, and p & q are in smallest form (i.e. they have no factor in common)
$\Rightarrow p^2 = 2q^2 \Rightarrow p$ is even. Let $p = 2k$. $k \in \mathbb{N}$.
$\therefore (2k)^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q$ is even

contradiction as p & q have a factor of 2 common here. ~~and~~
Hence, the assumption was wrong.
$\Rightarrow \sqrt{2}$ is a irrational number.

Ex ~~Prot~~ Prove $\sqrt{3}, \sqrt{8}$ are irrationals.

Order Axioms:
(1) Law of trichotomy: If $a, b \in \mathbb{R}$, then one and only one of the following is true.
(i) $a > b$ (ii) $a = b$ (iii) $a < b$
(2) Transitivity law: If $a, b, c \in \mathbb{R}$ then $a > b$ & $b > c \Rightarrow a > c$
(3) Monotone property for addition & multiplication: ~~If a,~~
If $a, b, c \in \mathbb{R}$ then $a > b$

$\Rightarrow a+c > b+c$ and $ac > bc$ provided $c > 0$.

Note: $\mathbb{R}$ is order complete (as real numbers satisfy order axioms)

Completeness property of $\mathbb{R}$:

Completeness axiom: Every non-empty subset S of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

Alternative form: Define $T = \{y : y = -x, x \in S\}$

Every non-empty set T of $\mathbb{R}$ which is bounded below has infimum in $\mathbb{R}$.

But the same is not true for set of rational numbers $\mathbb{Q}$.

But set of rational numbers are not order-complete.

Theorem: Set of rational numbers are not order-complete.

Proof: Let $S = \{x : x \in \mathbb{Q}, x > 0 \text{ and } x^2 < 2\}$

clearly $S \subseteq \mathbb{Q}$, $S \neq \phi$ as $1 \in S$ and S is bounded above by 2 (as for all $x \in S$ $x^2 < 2$)

$\Rightarrow$ ~~Let~~ (Completeness axiom)

$\Rightarrow$ Sup S exists in $\mathbb{R}$.

Let $K = \text{Sup } S$. ~~Suppose~~ Assume that $K \in \mathbb{Q}$

From law of trichotomy only one of the following holds

(1) $K^2 < 2$ (2) $K^2 = 2$ (3) $K^2 > 2$

Consider positive rational number $y = \frac{4+3K}{3+2K}$. Then

$$K - y = \frac{2(K^2-2)}{3+2K} \quad \text{--- (1)}$$

$$\text{and } y^2 - 2 = \frac{(K^2-2)}{(3+2K)^2} \quad \text{--- (2)}.$$

Case I: $k^2 < 2$

$\Rightarrow k - y < 0$ (from ①) $\Rightarrow k < y$] $\Rightarrow y \in S$ and $y > \sup S$

and $y^2 - 2 < 0 \Rightarrow y^2 < 2 \Rightarrow y \in S$] which is impossible

$\therefore k^2 \nless 2$

Case II: $k^2 = 2 \Rightarrow k = \pm\sqrt{2} \notin \mathbb{Q}$ $\therefore$ Not possible

Case III: $k^2 > 2$

$\Rightarrow k - y > 0$ or $k > y$ (from ①)] $\Rightarrow y < \sup S$. Not possible

& $y^2 - 2 > 0$ or $y^2 > 2$ (from ②) $\Rightarrow y$ is upper bound of S.]

$\Rightarrow y$ is upper bound of S has to be greater than or equal

as every upper bound of S has to be greater than or equal to $\sup S$.

Hence, $\nexists$ $k = \sup S$ and is a rational number.

$\Rightarrow$ Set of rational numbers is not complete

Archimedian Property (Theorem): If x and y are any two positive real numbers then $\exists$ a positive integer n s.t. $ny > x$

Proof: Suppose not. Then $ny \le x \ \forall n \in \mathbb{N}$.

$\Rightarrow S = \{ny : n \in \mathbb{N}\}$ is bounded above by 'x'.

$\Rightarrow \sup S$ exists in $\mathbb{R}$ (Completeness axiom)

Let $M = \sup S$, $M \in \mathbb{R}$.

$\therefore ny \le M \ \forall n \in \mathbb{N}$

$\Rightarrow (n+1)y \le M \ \forall n \in \mathbb{N}$ ($\because (n+1)$ is also natural number and above inequality holds $\forall n \in \mathbb{N}$)

$\Rightarrow ny \le M - y \ \forall n \in \mathbb{N}$.

$\Rightarrow M - y$ is upper bound of S & $M - y < M$ contradiction

Theorem 3: Between any two distinct real numbers there lies atleast one rational number & hence infinitely many real numbers.

Pf | Let x & y be two distinct real numbers such that $x < y \Rightarrow y - x > 0$. Consider $y-x$ & 1.

$\Rightarrow \exists\, n$ such that $n(y-x) > 1$ (Archimedian property)

$\Rightarrow ny > nx + 1$ — (1).

From theorem 1, $\exists$ unique integer m for which

$m - 1 \leq nx < m$ — (2)

$\Rightarrow m \leq nx + 1$ — (3)

From (1) & (3) $m < ny$ — (4)

From (2) & (4) $nx < m < ny$ or, $x < \frac{m}{n} < y$ or, $x < r < y$ where $r = \frac{m}{n} \in \mathbb{Q}$

$\Rightarrow \exists$ a rational number between two real numbers.

Continuing this with x and r & r and y we get $r_1, r_2 \in \mathbb{Q}$ and continuing this process we will have infinitely many rational numbers between x and y //

Rationals are dense in $\mathbb{R}$.

Note | Thm 3 tells that between any two distinct real numbers $\exists$ at least

Theorem 4 | Between any two distinct real numbers $\exists$ at least one irrational number and hence infinitely many irrationals.

Proof | Let $x \neq y$ be two real numbers s.t. $x < y$ and (α (x)) be any irrational number. Then

$x < y \Rightarrow x - \alpha < y - \alpha$

From thm 3 $\exists\, r \in \mathbb{Q}$ s.t. $x - \alpha < r < y - \alpha$.

or, $x < \alpha + r < y$ or, $x < ir < y$ where $ir = r + \alpha$ is irrational

Continuing in this way we will have infinitely many irrationals between two real numbers //

Note: Thm 4 tells that Irrationals are dense in $\mathbb{R}$.

Theorem 5: Between any two distinct real numbers there are infinitely many real numbers

Proof: From thm 3 & 4.

Hence, from this theorem we have Real number is dense.

Ex Let $a, b \in \mathbb{R}$. Show that if $a \leq b + \frac{1}{n}$ $\forall n \in \mathbb{N}$, then $a \leq b$.

Soln Assume $a \leq b + \frac{1}{n}$ $\forall n \in \mathbb{N}$ — (1) and $a > b$.

$\Rightarrow a - b > 0$ $\therefore$ From Archimedian property $\exists\, n_0 \in \mathbb{N}$ such that $n_0(a-b) \geq 1$ $\Rightarrow a \geq b + \frac{1}{n_0}$ contradiction to (1)

Hence, $a \leq b$ //.

Ex If for any $\epsilon > 0$, $|b-a| < \epsilon$, then $b = a$.

Soln $|b-a| < \epsilon \Rightarrow -\epsilon < b - a < \epsilon \Rightarrow a - \epsilon < b < a + \epsilon$

Consider $b < a + \epsilon$. From above example $b \leq a$ — (1)

Consider $a - \epsilon < b \Rightarrow a < b + \epsilon$.

$\Rightarrow a \leq b$ (from above example) — (2)

From (1) & (2) $a = b$ //