Sate variable Method of Analysis:

Topics covered:

1. What is a state model
2. Comparison with TF model
3. Methods of obtaining state-space models of dynamic systems
4. Transfer function from SS model
5. Solution of state equation
   a. State Transition Matrix
   b. Solution of Homogeneous (free) system equation
   c. Solution of non-homogeneous (forced) system equations
   d. Properties of State Transition Matrix
6. Controllability and Observability of a Control system

1. State Model (or Sate variable or State space model)) :

   State: The state of a dynamic system is the minimum set of variables called state variables, which if known at time t_0, along with the input, the system behaviour at $t > t_0$ can be determined.

A dynamic system represented by an nth order differential equations can be split into n first order equations, which can represent a state model. The model consists of the chosen state variables $x_1, x_2, \ldots x_n$, the input u, and an output y (for a single-input single output system).

The state model representation of a system is

$$\dot{\mathbf{X}} = A\,\mathbf{X} + \mathbf{B}\,u$$

$$y = C\,\mathbf{X} + \mathbf{D}\,u,$$ where

X is $(x_1, x_2, \ldots x_n)$ - a (nx1) matrix of state variables called State matrix,
A is a (nxn) matrix called Coefficient matrix,
B is a (nx1) matrix called Driving matrix,
C is a (1xn) matrix called Output matrix ,
and D is (nx1) matrix called Transmission matrix.
U is input and y is the output.

The state variables are the minimum set of variables.
The choice of the variables in the first order equations may be different, which leads to a different state model of the same system.

For example: consider a transfer function

$C(s)/R(s) = k/(s^2 + as + b)$,

i.e., $\ddot{c} + a\dot{c} + b\,c = k.\,r$ ... (1)

Let $x_1 = c$ and $x_2 = \dot{x}_1 = \dot{c}$

Eqn. 1 becomes, $\dot{x}_2 = -\,a\,x_2 - b\,x_1 + kr$

Above equations are written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} u$$

$$y \quad [\,1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $$\dot{X} = A\,X + B\,u$$

$$y = C\,X + D\,u$$

is the state model of the given second order system.

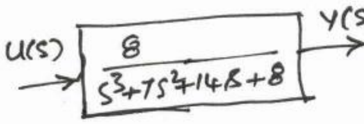
2. Comparison of Classical (TF) method and State variable method

	Transfer function	State variable (State –Space)
1	The classical Transfer function is a frequency domain approach	Time-domain approach
2	Transfer function is unique	State model is not unique.
3	Only input, output and error signals are important and are physical variables and measurable	Chosen state variables need not represent physical or measurable variables
4	Design methods are trial and error methods and are approximate	Not trial and error. Systems can be designed optimally.
5	Can handle only linear and time – invariant systems	Also handles non-linear and time-varying systems
6	Generally limited to single input-single output (SISO) systems. Multi input –Multi output systems (MIMO) are difficult	Can handle SISO and MIMO systems with equal ease.
7	Initials conditions (ICs) are neglected.	Ics can be considered.
8	Internal variables cannot be fed back	Internal variables can be fed back
9	Requires Laplace transforms for continuous signals and Z-transforms for discrete signals.	Treats both similarly.
10	TF models can be handled by manual methods	Computational ntensive and often needs use of computers.

State models can be obtained by different techniques.

Method 1 :

Example $\dfrac{y(t)}{u(t)} = \dfrac{8}{D^3+7D^2+14D+8}$



or $\dfrac{Y(s)}{U(s)} = \dfrac{8}{s^3+7s^2+14s+8}$

There are no s terms in the numerator. Hence we may write the governing differential equation as:

$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = 8u \quad \text{——} \; (1)$$

Letting
$$x_1 = y \qquad \text{or } \dot{x}_1 = x_2 \quad \text{——} \; (2)$$
$$x_2 = \dot{x}_1 = \dot{y}$$
$$x_3 = \ddot{y} = \dot{x}_2 \qquad \dot{x}_2 = x_3 \quad \text{——} \; (3)$$

Eqn.(1) becomes,
$$\dddot{y} = -8y - 14\dot{y} - 7\ddot{y} + 8u \quad \text{——} \; (4)$$

Eqns. (2), (3) & (4) are written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u \quad \text{——} \; (5)$$

and $y = x_1$ as
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{——} \; (6)$$

Eqns. (5) & (6) represent the state-variable model of the given systems in the form

$$\dot{X} = AX + Bu$$
$$y = CX \qquad \text{where}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}; \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$
$$\& \; D = 0$$

Method 2: This is applied when the denominator of TF is in factored form of 1st order terms, and numerator is a constant (i.e. without s terms), as in

$$\frac{Y(s)}{U(s)} = \frac{8}{(s+1)(s+2)(s+3)}$$

$$= \frac{8/3}{s+1} - \frac{4}{(s+2)} + \frac{4/3}{(s+4)}$$

We may write

$$Y(s) = \frac{8/3}{s+1}U(s) - \frac{4}{(s+2)}U(s) + \frac{4/3}{(s+4)}U(s)$$

or $$Y(s) = X_1(s) + X_2(s) + X_3(s) \qquad \text{— (1)}$$

in which $X_1(s) = \frac{8/3}{s+1}U(s)$

or $sX_1(s) + X_1(s) = 8/3\ U(s)$

or in time-domain,

$$\dot{x}_1(t) + x_1(t) = 8/3\ u(t)$$

or $$\dot{x}_1 = -x_1 + \frac{8}{3}u \qquad \text{— (2)}$$

Similarly $X_2(s) = -\frac{4}{s+2}U(s)$

or $$\dot{x}_2 = -2x_2 - 4u \qquad \text{— (3)}$$

and $X_3(s) = \frac{4/3}{s+4}U(s)$

or $$\dot{x}_3 = -4x_3 + \frac{4}{3}u \qquad \text{— (4)}$$

Eqns. (2), (3), (4) and eqn. (1) may be written as

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} &= \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}8/3\\ -4\\ 4/3\end{bmatrix}u \\
\text{and} \quad y &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}
\end{aligned}\right\} \text{ (P)}$$

which represent the SS model of the given system.

However, if the denominator of the T.F. is given in polynomial form as in the example of Method 1, we can get a different state model; i.e.

$$\frac{Y(s)}{U(s)} = \frac{8}{(s+1)(s+2)(s+3)} = \frac{8}{s^3+6s^2+11s+6}$$

$$\rightarrow \dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 8u$$

This is handled as in Method 1, where

$$y = x_1 \; ; \quad \dot{y} = \dot{x}_1 = x_2 \; ; \quad \ddot{y} = \dot{x}_2 = x_3$$

$$\text{and } \dddot{y} = -6y - 11\dot{y} - 6\ddot{y} + 8u = \dot{x}_3$$

Leading to

$$\left.\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}\right\} - (Q)$$

state-space models (P) & (Q) are the models of the same system $\frac{Y(s)}{U(s)} = \frac{8}{(s+1)(s+2)(s+3)}$

Note: We can obtain the transfer function of a given state model as

$$\frac{Y(s)}{U(s)} = C\,(sI-A)^{-1}B + D \qquad \text{— (R)}$$

By substituting the appropriate matrices in form (P) or (Q) into eqn. (R), the same T.F. is obtained. i.e. the same transfer function can be represented by different state models.

<u>Method 3</u> : (When numerator has s (or D) terms of order $< n$). Example

$$\frac{Y(s)}{U(s)} = \frac{s^2+2s+1}{s^3+7s^2+14s+8}$$

We may write this as

$$\frac{Y(s)}{F(s)} \cdot \frac{F(s)}{U(s)} = (s^2+2s+1) * \frac{1}{(s^3+7s^2+14s+8)}$$

$$\frac{Y(s)}{F(s)} = s^2+2s+1$$

$$\text{or } y(t) = \ddot{f}(t)+2\dot{f}(t)+f(t)$$

Let $x_1 = f(t)$; $x_2 = \dot{f}(t) = \dot{x}_1$ — (1)

— (2)

$x_3 = \dot{x}_2 = \ddot{f}(t)$ — (3)

Then $y(t) = x_3 + 2x_2 + x_1$

Now $$\frac{F(s)}{U(s)} = \frac{1}{s^3+7s^2+14s+8}$$

$$\dddot{f} + 7\ddot{f} + 14\dot{f} + 8f = u$$

or $\dddot{f} = -7\ddot{f} - 14\dot{f} - 8f + u$

or $\dot{x}_3 = -7x_3 - 14x_2 - 8x_1 + u$ — (4)

Eqns. (1), (2) and (4) and eqn. (3) are written as

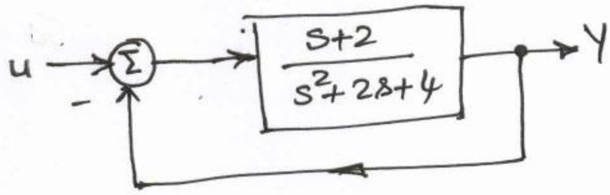
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

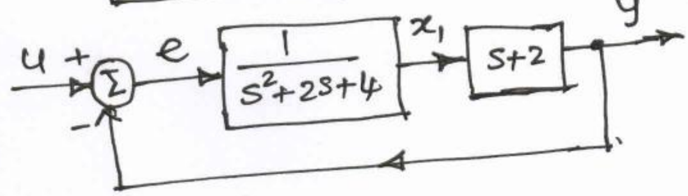
$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which is the s-s representation of the given system.

From Block diagrams: We may also obtain SS models from block diagrams. Consider a system shown.

Note: state variables $x_1, x_2 \ldots$ etc. have to be chosen as outputs of integrator blocks with $\frac{1}{s}$ or $\frac{1}{s^2 + \ldots}$ etc terms.





From the modified block diagram, $\frac{Y(s)}{X_1(s)} = s+2 \rightarrow y = \dot{x}_1 + 2x_1$
$= x_2 + 2x_1 \quad (\dot{x}_1 = x_2)$

Also $\frac{X_1(s)}{E(s)} = \frac{1}{s^2+2s+4}$, or in time-domain

$$\ddot{x}_1 + 2\dot{x}_1 + x_1 = e = u - y$$

or $$\dot{x}_2 = -2x_2 - x_1 + u - (x_2 + 2x_1)$$
$$= -3x_1 - 3x_2 + u$$

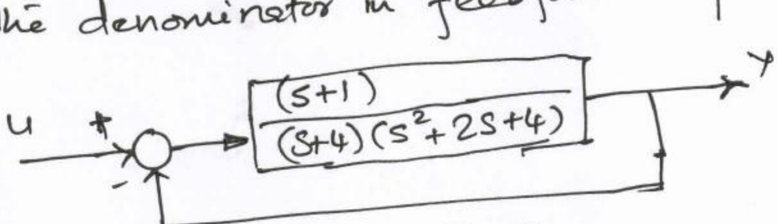
from which

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

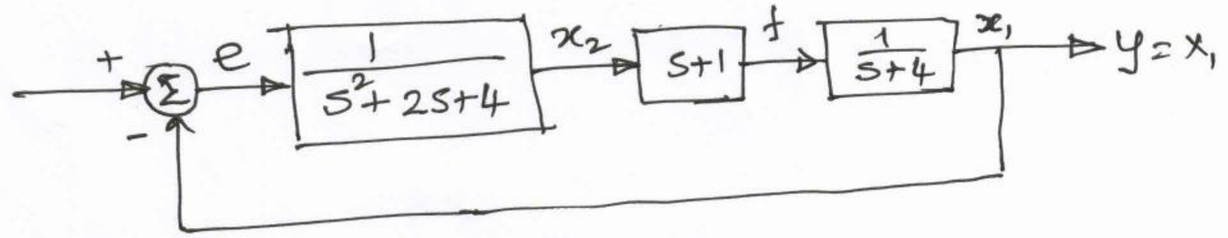
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the state variable representation.

Ex2. Similarly, if the denominator in feedforward part has factors,



it is represented in the form



Note that the output of $(s+1)$ is given a dummy variable, while output of $1/s$ or $\frac{1}{s^2}$ terms is assigned a state variable.

We now write, $y = x_1$; $\dot{x}_1 = x_2$; $\dot{x}_2 = x_3$

$$\frac{f}{x_2} = s+1 \quad \text{or} \quad f = \dot{x}_2 + x_2 = x_3 + x_2$$

Also
$$\frac{x_2}{e} = \frac{1}{s^2+2s+4}$$

$$\rightarrow \ddot{x}_2 + 2\dot{x}_2 + 4x_2 = e = u - y = u - x_1$$

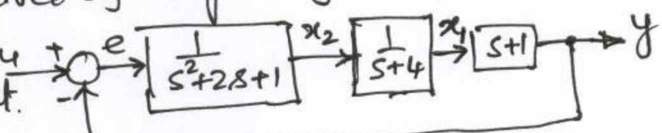
$$\text{or} \quad \dot{x}_3 = -2x_3 - 4x_2 + u - x_1$$

Thus we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- The above problem may also be solved by reorganising the block diagram. Noting x_1 the state variable is taken as output of $(s+1)$ block and as output.



Letting $\dot{x}_1 = x_2$; $\dot{x}_2 = x_3$

From $\frac{x_1}{x_2} = \frac{1}{s+4}$; $\frac{y}{x_1} = s+1$; $\frac{x_2}{e} = \frac{x_2}{u-y} = \frac{1}{s^2+2s+1}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\& \quad y = \begin{bmatrix} -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

becomes the alternative state model for the same system.

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4. TRANSFER FUNCTION from State Model.

To find $\frac{Y(s)}{U(s)}$, the T.F of the system, for which state model is

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Laplace form is,

$$s\,X(s) - X(0) = A\,X(s) + B\,U(s) \quad \text{--- (1)}$$
$$Y(s) = C\,X(s) + D\,U(s) \quad \text{--- (2)}$$

Eqn.(1) is $\quad s\,X(s) - A\,X(s) = X(0) + B\,U(s)$

or $\quad (sI - A)\,X(s) = X(0) + B\,U(s)$

or $\quad X(s) = [sI - A]^{-1}\left(X(0) + B\,U(s)\right)$

From eqn.(2), $\quad Y(s) = C\,(sI-A)^{-1}\left(X(0) + B\,U(s)\right) + D\,U(s)$

Letting initial conditions $X(0) = 0$ (i.e. $x_1(0) = 0$; $x_2(0) = 0, \ldots$ etc.)

$$Y(s) = C\,(sI-A)^{-1}B\,U(s) + D\,U(s)$$

or $\boxed{\dfrac{Y(s)}{U(s)} = C\,(sI-A)^{-1}B + D}$ is the transfer function.

Example: Find T.F. for the state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \;;\; y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C\,(sI-A)^{-1}B + D$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 0$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\frac{1}{s(s+2)+3}\begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$= \frac{1}{s^2+2s+3}\begin{bmatrix} s+2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{2}{s^2+2s+3}$$

Hence T.F $\quad \dfrac{Y(s)}{U(s)} = \dfrac{2}{s^2+2s+3}$

for the given state-space model.

5. Solution of State Equation

Solution means – finding $x(t)$ i.e. $x_1(t), x_2(t)$.. etc. given $x(t_0)$ (or $x(0)$ when $t_0=0$) for a given state model.

5.a We know that for a differential equation (homogeneous)

$$\dot{x} = ax, \quad \text{the solution is}$$

$$x(t) = K e^{at} \quad \text{where } k = x(0), \text{ so that}$$

$$x(t) = e^{at} x(0).$$

Similarly for the matrix equation,

$$\dot{X} = AX, \quad \text{the Solution is}$$

$$x(t) = e^{At} K$$

To find K, if $x(t_0)$ is the initial state of x at $t=t_0$, then

$$x(t_0) = e^{At_0} \cdot K \quad \text{or} \quad K = e^{-At_0} x(t_0)$$

$$\text{and} \quad x(t) = e^{At} K = e^{At} \cdot (e^{-At_0} \cdot x(t_0))$$

$$= e^{A(t-t_0)} x(t_0)$$

$$\text{If } t_0 = 0; \quad \underline{x(t) = e^{At} x(0)}$$

i.e. we can find the state $x(t)$ at time 't', knowing the state at some initial time; and multiplying it with e^{At}.

Hence e^{At} is called State Transition Matrix as it transfers $x(0)$ to $x(t)$.

$$\text{Just as} \quad e^{at} = 1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \dots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 2t \\ 0 & t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} t^2 & (2t)^2 \\ 0 & t^2 \end{bmatrix} + \dots$$

→ There are various methods of evaluating e^{At}. One of them is Laplace Transform approach.

5.b Solution for Homogeneous Equation :

We have the LT form of $\dot{x} = AX$ as

$$s\,X(s) - X(0) = A\,X(s)$$

from which $(sI - A)X(s) = X(0)$

$$X(s) = (sI-A)^{-1}X(0) \qquad \text{——— (1)}$$

or $x(t) = \mathcal{L}^{-1}[sI-A]^{-1}\,x(0)$

As $x(t) = e^{At}x(0)$

We have the State Transition Matrix $\underline{e^{At} = \mathcal{L}^{-1}(sI-A)^{-1}}$

Also eqn. (1) above may be written as

$$X(s) = \phi(s)\,X(0) \qquad \text{where } \phi(s) = (sI-A)^{-1}$$

and $x(t) = \phi(t)\,x(0)$ where $\phi(t) = \mathcal{L}^{-1}\phi(s)$

$\phi(t) = e^{At}$ is the STM.

The Laplace form of $\phi(t)$ is $\phi(s)$ and is called Resolvent Matrix. Hence one method of evaluating STM is $e^{At} = \mathcal{L}^{-1}(sI-A)^{-1}$

Example : Find $x(t)$, given $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t)$; for $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(B=0)$ (ie, No input u)

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} :$$

$$\phi(s) = (sI-A)^{-1} = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \right\}^{-1} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1}$$

$$= \frac{1}{s(s+4)+3} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

Given $X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x(t) = \phi(t)\,x(0) = \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+3}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\right]\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+3}\begin{bmatrix} s+4 \\ -3 \end{bmatrix}\right]$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+4}{s^2+4s+3} \\ \frac{-3}{s^2+4s+3} \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{3/2}{s+1} - \frac{1/2}{s+3} \\ \frac{3/2}{s+3} - \frac{3/2}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ 3/2\,(e^{-3t} - e^{-t}) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$ is the solution.

5.d Solution of State Equation with forcing function (u) (Non-homogeneous Equation)

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$\dot{x} - Ax = Bu$$ which on multiplying with e^{-At},

$$e^{-At}(\dot{x} - Ax) = e^{-At}Bu$$

$$\Rightarrow \frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu$$

Integrating between 0 to t,

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}B\,u(\tau)\,d\tau$$

Multiplying throughout by e^{At},

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau$$

or $$x(t) = \phi(t)\,x(0) + \phi(t)\int_0^t \phi(-\tau)\,B\,u(\tau)\,d\tau.$$

The forcing function effect is to add the second part to $\phi(t)\,x(0)$ which is the solution without 'u'.

Example: Find $x(t)$, for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u,$$ given

Initial condition $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\underline{u}$ is a unit step

We have $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$

$$\phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}$$

$$\phi(t) = e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

$$\therefore e^{At}x(0) = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is the solution when input $= 0$.

$$\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau = \int_0^t \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2(t-\tau)}) \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \int_0^t \frac{1}{2}(1-e^{-2(t-\tau)})\,d\tau \\ \int_0^t e^{-2(t-\tau)}\,d\tau \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} + \frac{t}{2} + \frac{e^{-2t}}{4} \\ \frac{1}{2} - \frac{e^{-2t}}{2} \end{bmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

or $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} + \frac{t}{2} + \frac{e^{-2t}}{4} \\ \frac{1}{2} - \frac{e^{-2t}}{2} \end{bmatrix}$$

$$x_1(t) = \frac{3}{4} + \frac{t}{2} + \frac{e^{-2t}}{4}$$

$$x_2(t) = \frac{1}{2}(1 - e^{-2t})$$

This is the solution i.e. time response equations of the state variables $x_1(t)$ and $x_2(t)$, when the input is a unit step.

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Another method of Evaluation of e^{At} (diagonalisation of A)

In the state model, if A is diagonal, as

$$A = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix},$$ then it can be readily seen that

$$e^{At} = \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^{-at} & 0 \\ 0 & e^{-bt} \end{bmatrix}$$

However, if A is not diagonal, but has distinct eigenvalues, (say λ_1 and λ_2), then we can obtain STM as

$$e^{At} = M\, e^{\lambda t} M^{-1}$$

where $e^{\lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ and

M is the modal matrix of eigenvectors of A.

Example:

Let $\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x$

$[\lambda I - A] = 0 \Rightarrow \begin{bmatrix} \lambda & -1 \\ 3 & \lambda+4 \end{bmatrix} \Rightarrow$ Char. Eqn. $\lambda^2 + 4\lambda + 3 = 0$

with $\lambda_1 = -1$ and $\lambda_2 = -3$ are the eigenvalues.

Eigenvector $v_1 = \text{Adj}(\lambda I - A) = \begin{bmatrix} \lambda+4 & 1 \\ -3 & \lambda \end{bmatrix}$

$$\lambda_1 = -1 \Rightarrow \text{Adj}(\lambda I - A) = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -3 \Rightarrow \text{Adj}(\lambda I - A) = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Hence $M = [v_1 \;\; v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$ and $M^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Now
$$e^{At} = M e^{\lambda t} M^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}$$

or
$$\phi(t) = e^{At} = \frac{1}{2} \begin{bmatrix} 3e^{-t} - e^{-3t} & e^{-t} - e^{-3t} \\ 3e^{-3t} - e^{-t} & 3e^{-3t} - 3e^{-t} \end{bmatrix}$$

Properties of State Transition Matrix, $\phi(t) = e^{At}$.

Properties of STM can be readily obtained using exponential representation.

Property	Explanation
1. $\phi(0) = I$	$\rightarrow \phi(0) = e^{A0} = I$
2. $\phi^{-1}(t) = \phi(-t)$	$\rightarrow \phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$
3. $\phi^{k}(t) = \phi(kt)$	$\rightarrow \phi^{k}(t) = (e^{At})^{k} = e^{Akt} = \phi(kt)$
4. $\phi(t_1+t_2) = \phi(t_1).\phi(t_2)$	$\rightarrow \phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1}.e^{At_2} = \phi(t_1).\phi(t_2)$
5. $\phi(t_2-t_1)\phi(t_1-t_0) = \phi(t_2-t_0)$	$\rightarrow \phi(t_2-t_1)\phi(t_1-t_0) = e^{A(t_2-t_1)}.e^{A(t_1-t_0)} = e^{A(t_2-t_1+t_1-t_0)} = e^{A(t_2-t_0)} = \phi(t_2-t_0)$

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CONTROLLABILITY & OBSERVABILITY

Controllability: A system is said to be controllable at time t_0, if there exists an input $u(0,t)$ which transfers the system from initial state $x(t_0)$ to another state $x(t)$ in a finite interval of time.

The system is completely state controllable if and only if the vectors $B, AB, A^2B, \ldots, A^{n-1}B$ are linearly dependent or

the $n \times n$ matrix $[B : AB : A^2B : \ldots A^{n-1}B]$ is of rank 'n'.

Example: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Then $[B : AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, which is a non-singular matrix or of full rank.

Hence the system is controllable.

OBSERVABILITY: A system is said to be observable if the system in state $x(t_0)$ can be determined from the observation of output $y(t)$ over a finite interval of time from t_0 to t.

A system ($\dot{x} = Ax$; $y = cx$) is completely observable if and only if the $n \times n$ matrix

$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is of rank n or has linearly independent column vectors.

Example: $\dot{x} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$; $y = [1 \;\; 0]\, x$

$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has full rank $(=2)$ and is therefore observable.

Example 2 : $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$; $y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

We can see that $[B : AB] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$ has rank 2 and is Controllable.

$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$ is singular as the rows are linearly dependent

Hence the system is not observable.

Exercises :

Check that the system

(1) $\dot{x} = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & -1 \\ -3 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$ is Controllable & Observable.

(2) $A = \begin{bmatrix} -6 & 2 & -4 \\ -18 & 3 & -8 \\ -6 & 1 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$: $c = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$

is controllable but not observable.

(3) $A = \begin{bmatrix} -6 & -18 & -6 \\ 2 & 3 & 1 \\ -4 & -8 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$; $c = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}$

is not controllable & not observable.

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