Some Theorems on LPP

$$\boxed{\begin{array}{l}\text{maximize } z = CX \\ \text{Subject to } AX = b,\ X \geq 0\end{array}} \quad (*)$$

$A_{m\times n}$, $b_{m\times 1}$, $C_{1\times n}$, $X_B^T \in \mathbb{R}^m$, $C_B \in \mathbb{R}^m$

$b^T \in \mathbb{R}^m$ $\quad C \in \mathbb{R}^n$ $\quad (X_B)_{m\times 1}$

$B \to$ Basis matrix $\quad X = [X_B, 0] \quad X_B = B^{-1}b$.

$C_B \to$ cost coefficients of basis variables X_B.

$a_j \in A$ (i.e, a_j are columns of A)

$b_j \in B$ (" b_j " " " B)

<u>Theorem 2</u>: (<u>Replacement Basis Vector</u>): Let LPP (*) have a BFS. If we drop one of the basis vectors and introduce a non-basic vector in basic set, the new solution obtained is also BFS.

<u>Proof</u>: Let $\rho(A) = m$. Suppose there exists BFS X_B so that $BX_B = b$, $X_B \geq 0$ — (1)

where B is basis set for columns vectors of A. Hence for any vector $a_j \in A$, we have

$$a_j = y_{1j}b_1 + y_{2j}b_2 + \cdots + y_{mj}b_m$$

(as a_j can be written as l.c of elements of B (basis))

$$= [b_1\ b_2\ \cdots\ b_m]\begin{bmatrix} y_{1j} \\ \vdots \\ y_{mj}\end{bmatrix} = By_j$$

where $b_i \in B$ and y_{ij} are suitable scalars

Let b_r be basis vector for which $y_{rj} \neq 0$. Then

$$b_r = \frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i\neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}}\right) b_i \quad -(2)$$

From (1) & (2) we have

$$b = Bx_B = \sum_{i=1}^{m} x_{Bi} b_i = \sum_{\substack{i=1 \\ i\neq r}}^{m} x_{Bi} b_i + x_{Br}\left[\frac{a_j}{y_{rj}} - \sum_{\substack{i=1 \\ i\neq r}}^{m}\left(\frac{y_{ij}}{y_{rj}}\right) b_i\right]$$

$$= \sum_{\substack{i=1 \\ i\neq r}}^{m}\left[x_{Bi} - x_{Br}\left(\frac{y_{ij}}{y_{rj}}\right)\right] b_i + \left(\frac{x_{Br}}{y_{rj}}\right) a_j$$

Thus, new B.S $\hat{x}_B$ has components

$$\hat{x}_{Bi} = \begin{cases} x_{Bi} - x_{Br}\left(\frac{y_{ij}}{y_{rj}}\right) & i \neq r \\ \frac{x_{Br}}{y_{rj}} & i = r \end{cases}$$

To prove: $\hat{x}_B$ is Feasible. (ie, $\hat{x}_B \geq 0$)

Case I: $x_{Br} = 0 \Rightarrow \hat{x}_B = x_B$ and hence feasible (as x_B is BFS).

Case II: $x_{Br} \neq 0$. In this case we must have $y_{rj} > 0$ (*) (why!?)

Remaining y_{ij} $(i \neq r)$ are either zero or $\left[x_{Bi} - x_{Br}\left(\frac{y_{ij}}{y_{rj}}\right)\right] > 0$

$\Rightarrow \frac{x_{Bi}}{y_{ij}} > \frac{x_{Br}}{y_{rj}}$ for $y_{ij} > 0$ & $\frac{x_{Bi}}{y_{ij}} < \frac{x_{Br}}{y_{rj}}$ for $y_{ij} < 0$ $(i \neq r)$

Select index r $(y_{rj} \neq 0)$ in such a way that

$$\frac{x_{Br}}{y_{rj}} = \min_i \left\{\frac{x_{Bi}}{y_{ij}} : y_{ij} > 0, i \neq r\right\}.$$

Then $\hat{x}_B \geq 0$, hence feasible.

Therefore $\hat{x}_B$ is BFS //.

$\boxed{\text{Note}}$: $\hat{B} = (b_1, b_2, \ldots, b_{r-1}, a_j, b_{r+1}, \ldots b_m)$ $\therefore \hat{x}_B = \hat{B}^{-1} b$

(*) If $y_{rj} < 0$ then $\hat{x}_{Br} = \frac{x_{Br}}{y_{rj}} < 0$ hence not feasible)

Net evaluation. $a_j = \sum_{i=1}^{m} y_{ij} b_i$ (seen in above theorem)

Then the number $z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}$

is called evaluation corresponding to a_j and the number $c_j - z_j$ is called net evaluation corresponding to a_j

Theorem 3 (Improved BFS): Let X_B be a B.FS to LPP (*). Let $\hat{X}_B$ be another BFS obtained by admitting a non-basis column vector a_j in the basis, for which the net evaluation $c_j - z_j$ is positive. Then $\hat{X}_B$ is an improved BFS to the problem. i.e, $\hat{C}_B \hat{X}_B > C_B X_B$.

Proof! Consider LPP(*).

Given X_B is BFS.

Let $z_0 = C_B X_B$. Let a_j be the column vector introduced in basis such that $c_j - z_j > 0$. Let b_r be vector removed from basis (outgoing variable) from the basis and let $\hat{X}_B$ be new BFS, then

$$\hat{x}_{Bi} = \begin{cases} x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}} & i \neq r \\[2ex] \dfrac{x_{Br}}{y_{rj}} & i = r \end{cases}$$ (Previous theorem) (thm 2)

Here, new value of objective function is

$$\hat{z} = \sum_{i=1}^{m} \hat{c}_{Bi}\hat{x}_{Bi} = \sum_{\substack{i=1 \\ i\neq r}}^{m} \hat{c}_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + \hat{c}_{Br}\left(\frac{x_{Br}}{y_{rj}}\right)$$

$$= \sum_{\substack{i=1 \\ i\neq r}}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + c_j \frac{x_{Br}}{y_{rj}}$$

$\because \hat{c}_{Bi} = c_{Bi}$ $i \neq r$

$\hat{c}_{Br} = c_j$ $i = r$

cost coff associated with entering column a_j

$$= z_0 - \sum_{i=1}^{m} c_{Bi} x_{Br}\frac{y_{ij}}{y_{rj}} + c_j \frac{x_{Br}}{y_{rj}}$$

$$= z_0 - \frac{x_{Br}}{y_{rj}}\sum_{i=1}^{m} c_{Bi} y_{ij} + c_j \frac{x_{Br}}{y_{rj}}$$

$= z_0 - z_j \frac{x_{Br}}{y_{rj}} + c_j \frac{x_{Br}}{y_{rj}}$ (defn of $z_j = \Sigma c_{Bi} y_{ij}$)

$= z_0 + \underbrace{(c_j - z_j)}_{>0} \underbrace{\frac{x_{Br}}{y_{rj}}}_{>0}$

$> z_0$

Hence new BFS $\hat{x}_B$ gives improved value of objective f^n z.

<u>Remark</u> (1) If given B.S. happens to be non-degenerate one, and their is increase in value of z_0, then $\frac{x_{Br}}{y_{rj}} > 0$

(2) If given feasible solution happens to be degenerate one, then an increase in value of z_0 depends on $\frac{x_{Br}}{y_{rj}}$. But in any case $\hat{z} \ngtr z_0$. ($\because c_j - z_j > 0$).

<u>Corollary</u>: If $c_j - z_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots m$, then another BFS is obtained which gives an unchanged value of objective f^n.

<u>Proof</u>: $\hat{z} = z_0 + (c_j - z_j) \frac{x_{Br}}{y_{rj}}$; $y_{rj} > 0$

$= z_0$ ($\because c_j - z_j = 0$)

Theorm 4: (Unbounded Solution): Let there exist a BFS to given LPP(*). If for atleast one j, for which $y_{ij} \leq 0$ $(i=1,2,\dots m)$ and $c_j - z_j$ is positive, then there does not exist any optimum solution to this LPP.

Proof: Let X_B be BFS to LPP(*), so $BX_B = b$ & $X_B \geq 0$ and value of objective function $z_0 = C_B X_B = \sum_{i=1}^{m} x_{B_i} c_{B_i}$

Now, $b = BX_B + \xi a_j - \xi a_j$ $\quad a_j \in A$, ξ is a scalar

$= \sum_{i=1}^{m} x_{B_i} b_i + \xi a_j - \xi \sum_{i=1}^{m} y_{ij} b_i$ ($\because$ a_j can be written as l.c. of columns of basis matrix B)

$= \sum_{i=1}^{m} (x_{B_i} - \xi y_{ij}) b_i + \xi a_j$

If $\xi > 0$, then $x_{B_i} - \xi y_{ij} \geq 0$ ($\because y_{ij} \leq 0$).

$\Rightarrow \exists$ a F.S. whose $(m+1)$ components may be strictly positive.

But in general, it may not be basic solution.

For these $(m+1)$ variables, objective function is

$\hat{z} = \sum_{i=1}^{m} c_{B_i} (x_{B_i} - \xi y_{ij}) + \xi c_j = \sum_{i=1}^{m} c_{B_i} x_{B_i} + \xi (c_j - \sum_{i=1}^{m} c_{B_i} y_{ij})$

$= z_0 + \xi (c_j - z_j) \longrightarrow \infty$ as $\xi \to \infty$

Hence, unbounded solution to given LPP(*)

Theorem 5: (Conditions of Optimality): A sufficient condition for a BFS of an LPP(*) to be an optimum (maximum) is that $c_j - z_j \leq 0$ for all j for which column vector $a_j \in A$ is not in basis B.

Proof: Let $\rho(A) = m$ and $B_{m \times m}$ be basis matrix. Let X_B be BFS to LPP(*), then $BX_B = b$, $X_B \geq 0$ & $z_0 = C_B X_B$.

Given: $c_j - z_j \leq 0$ $\forall$ j's s.t. $a_j \notin B$.

Let $a_j = b_j$ for all such j for which $a_j \in B$. Then

$y_j = B^{-1} b_j$ (since $y_j = B^{-1} a_j$) (see Thm 2)

$= e_j$, the unit vector

& $c_j - z_j = c_B e_j - c_j = c_{Bj} - c_j = 0$ ($\because a_j \in B \therefore c_{Bj} = c_j$)

Thus $c_j - z_j \le 0 \ \forall j$ for which $a_j \in A$.

Now, let X be F.S. Then $\sum_{j=1}^{n} (c_j - z_j) x_j \le 0$ $\because x_j \ge 0$

$\Rightarrow \sum_{j=1}^{n} z_j x_j \ge \sum_{j=1}^{n} c_j x_j$

$\Rightarrow \sum_{j=1}^{n} c_B y_j x_j \ge \sum_{j=1}^{n} c_j x_j$ ($\because z_j = c_B y_j$)

or, $\sum_{i=1}^{m} c_{Bi} \sum_{j=1}^{n} y_{ij} x_j \ge \sum_{j=1}^{n} c_j x_j$ ($\because c_B y_j = \sum_{i=1}^{m} c_{Bi} y_{ij}$) — (1) for all j for which $a_j \notin B$.

Now, as $x_B = B^{-1}(AX)$ ($\because AX = b$)

$= (B^{-1}A)X = YX$

or, $x_{Bi} = \sum_{j=1}^{n} y_{ij} x_j$ $i = 1, 2, \ldots m$ — (2)

$\therefore$ From (1) & (2)

$\sum_{i=1}^{m} c_{Bi} x_{Bi} \ge \sum_{j=1}^{n} c_j x_j$ or, $c_B x_B \ge cx$

or, $z_0 \ge z^*$

where z^* is value of objective fn that for feasible soln.

Hence z_0 is optimum ~~soln~~ for a BFS for which $c_j - z_j \le 0$ $\forall j$ such that $a_j \notin B$.

<u>Corollary</u>: A n.a.s.c. for a BFS to LPP(*) to be an optimum (maximum) is that $c_j - z_j \le 0$ for all j for which $a_j \notin B$.

<u>Pf</u>: For necessary: Thm 3 & 4 For sufficien: Thm 5

Theorem 6: Any convex combination of k different optimum solutions to a LPP is again an optimum solution to the problem.

Proof: Let $x_1, x_2, \dots x_k$ be k-different optimum solutions to LPP(*). Let z_0 be optimum value of z.

Obviously $z_0 = cx_1 = cx_2 = \dots = cx_k$

& $b = Ax_1 = Ax_2 = \dots = Ax_k$

where $x_i \geq 0 \quad 1 \leq i \leq k$.

Let x be any combination of $x_1, x_2, \dots x_k$ i.e,

$x = \sum_{i=1}^{k} \alpha_i x_i \qquad \alpha_i$'s are positive scalars s.t $\sum_{i=1}^{k} \alpha_i = 1$

$\therefore Ax = A\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i (Ax_i) = \cancel{\sum_{i=1}^{k}}$

$= \sum_{i=1}^{k} \alpha_i b = b \qquad \left(\because \sum_{i=1}^{k} \alpha_i = 1\right).$

Since $x_i \geq 0$ & $\alpha_i \geq 0 \ \forall i = 1, 2, \dots k$, therefore every component of x will be non-negative

Thus, x is a F.S. to LPP(*).

Again $cx = c\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i cx_i = \sum_{i=1}^{k} \alpha_i z_0 = z_0$.

Hence, convex combination of k different optimum solutions to a LPP is also optimum solution to LPP.