3 Time Response Analysis of Control Systems

3.1 Introduction

The first step in the analysis of a control system is, describing the system in terms of a mathematical model. In chapter 2 we have seen how any given system is modelled by defining its transfer function. The next step would be, to obtain its response, both transient and steadystate, to a specific input. The input can be a time varying function which may be described by known *mathematical functions* or it may be a *random signal*. Moreover these input signals may not be known *apriori*. Thus it is customary to subject the control system to some standard input test signals which strain the system very severely. These standard input signals are : an impulse, a step, a ramp and a parabolic input. Analysis and design of control systems are carried out, defining certain performance measures for the system, using these standard test signals.

It is also pertinent to mention that any arbitary time function can be expressed in terms of linear combinations of these test signals and hence, if the system is linear, the output of the system can be obtained easily by using supersition principle. Further, convolution integral can also be used to determine the response of a linear system for any given input, if the response is known *for a step or an impulse input*.

3.2 Standard Test Signals

3.2.1 Impulse Signal

An impulse signal is shown in Fig. 3.1.

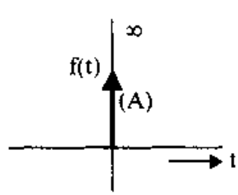

Fig. 3.1 An Impulse signal.

The impulse function is zero for all $t \neq 0$ and it is infinity at $t = 0$. It rises to infinity at $t = 0^-$ and comes back to zero at $t = 0^+$ enclosing a finite area. If this area is A it is called as an impulse function of strength A. If A = 1 it is called a unit impulse function. Thus an impulse signal is denoted by $f(t) = A\ \delta(t)$.

3.2.2 Step Signal

A step signal is shown in Fig. 3.2.

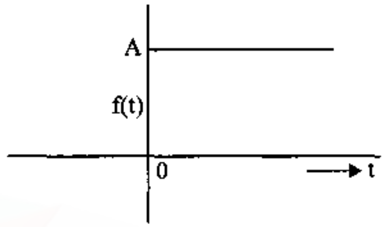

Fig. 3.2 A Step Signal.

It is zero for $t < 0$ and suddenly rises to a value A at $t = 0$ and remains at this value for $t > 0$: It is denoted by $f(t) = Au(t)$. If A = 1, it is called a *unit step function.*

3.2.3 Ramp signal

A ramp signal is shown in Fig. 3.3.

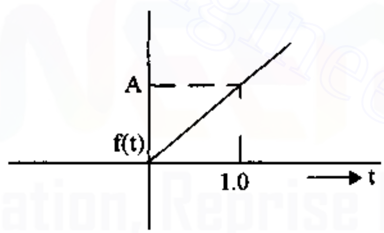

Fig. 3.3 A Ramp Signal.

It is zero for $t < 0$ and uniformly increases with a slope equal to A. It is denoted by $f(t) = At$. If the slope is unity, then it is called a *unit ramp signal.*

3.2.4 Parabolic signal

A parabolic signal is shown in Fig. 3.4.

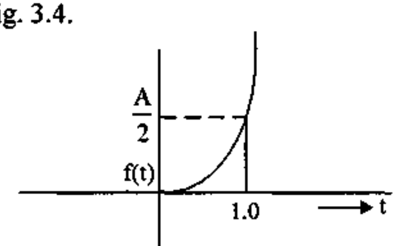

Fig. 3.4 A unit parabolic signal.

A parabolic signal is denoted by $f(t) = \frac{At^2}{2}$. If A is equal to unity then it is known as a *unit parabolic signal.*

It can be easily verified that the step function is obtained by integrating the impulse function from o to ∞; a ramp function is obtained by integrating the step function and finally the prabolic function is obtained by integrating the ramp function. Similarly ramp function, step function and impulse function can be to obtained by successive differentiations of the parabolic function.

Such a set of functions which are derived from one another are known as *singularity functions*. If the response of a linear system is known for any one of these input signals, the response to any other signal, out of these singularity functions, can be obtained by either differentiation or integration of the known *response*.

3.3 Representation of Systems

The input output description of the system is mathematically represented either as a differential equation or a transfer function.

The differential equation representation is known as a time domain representation and the transfer function is said to be a frequency domain representaiton. We will be considering the transfer function representation for all our analysis and design of control systems.

The open loop transfer function of a system is represented in the following two forms.

1. Pole-zero form

$$G(s) = K_1 \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \qquad \ldots..(3.1)$$

Zeros occur at $s = -z_1, -z_2, ---, -z_m$

Poles occur at $s = -p_1, -p_2, ---, -p_m$

The poles and zeros may be simple or repeated. Poles and zeros may occur at the origin. In the case where some of the poles occur at the origin, the transfer function may be written as

$$G(s) = \frac{K_1(s+z_1)(s+z_2)\ldots(s+z_m)}{s^r(s+p_{r+1})(s+p_{r+2})\ldots(s+p_n)} \qquad \ldots..(3.2)$$

The poles at the origin are given by the term $\frac{1}{s^r}$. The term $\frac{1}{s}$ indicates an integration in the system and hence $\frac{1}{s^r}$ indicates the number of integrations present in the system. Poles at origin influence the steadystate performance of the system as will be explained later in this chapter. Hence the systems are classified according to the number of poles at the origion.

If $r = 0$, the system has no pole at the origin and hence is known as a type – 0 system. If $r = 1$, there is one pole at the origin and the system is known as a type - 1 system. Similarly if $r = 2$, the system is known as type - 2 system. Thus it is clear that the type of a system is given by the number of poles it has at the origin.

2. Time Constant Form

The open loop transfer function of a system may also be written as,

$$G(s) = \frac{K(\tau_{z_1}s+1)(\tau_{z_2}s+1)...(\tau_{z_m}s+1)}{(\tau_{p_1}s+1)(\tau_{p_2}s+1)...(\tau_{p_n}s+1)} \quad .....(3.3)$$

The poles and zeros are related to the respective time constants by the relation

$$z_i = \frac{1}{\tau_{zi}} \qquad \text{for } i = 1, 2, ...m$$

$$p_j = \frac{1}{\tau_{p_j}} \qquad \text{for } j = 1, 2, ...n$$

The gain constans K_1 and K are related by

$$K = K_1 \frac{\prod_{i=1}^{m} z_i}{\prod_{j=1}^{n} p_j}$$

The two forms described above are equivalent and are used whereever convenience demands the use of a particular form.

In either of the forms, the degree of the denomination polynomial of G(s) is known as the order of the system. The complexity of the system is indicated by the order of the system. In general, systems of order greater than 2, are difficult to analyse and hence, it is a practice to approximate higher order systems by second order systems, for the purpose of analysis.

Let us now find the response of first order and second order systems to the test signals discussed in the previous section.

The impulse test signal is difficult to produce in a laboratory. But the response of a system to an impulse has great significance in studying the behavior of the system. The response to a unit impulse is known as *impulse response of the system*. This is also known as the *natural response of the system*.

For a unit impulse function, R(s) = 1

and $\qquad C(s) = T(s).1$

and $\qquad c(t) = \mathcal{L}^{-1}[T(s)]$

The Laplace inverse of T(s) is the impulse response of the system and is usually denoted by $h(t)$.

$\therefore \qquad \mathcal{L}^{-1}[T(s)] = h(t)$

If we know the impulse response of any system, we can easily calculate the response to any other arbitrary input $v(t)$ by using convolution integral, namely

$$c(t) = \int_0^t h(\tau)\, v(t-\tau)\, d\tau$$

Since the impulse function is difficult to generate in a laboratory at is seldon used as a test signal. Therefore, we will concentrate on other three inputs, namely, unit step, unit velocity and unit acceleration inputs and find the response of first order and second order systems to these inputs.

3.4 First Order System

3.4.1 Response to a Unit Step Input

Consider a feedback system with $G(s) = \dfrac{1}{\tau s}$ as show in Fig. 3.5.

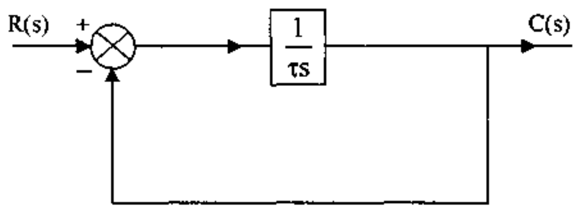

Fig. 3.5 A first order feedback system.

The closed loop transfer function of the system is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{1}{\tau s + 1} \qquad .....(3.4)$$

For a unit step input $R(s) = \dfrac{1}{s}$ and the output is given by

$$C(s) = \frac{1}{s(\tau s + 1)} \qquad .....(3.5)$$

Inverse Laplace transformation yields

$$c(t) = 1 - e^{-t/\tau} \qquad .....(3.6)$$

The plot of c(t) Vs t is shown in Fig. 3.6.

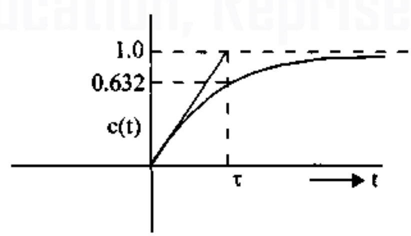

Fig. 3.6 Unit step response of a first order system.

The response is an exponentially increasing function and it approaches a value of unity as $t \to \infty$. At $t = \tau$ the response reaches a value,

$$c(\tau) = 1 - e^{-1} = 0.632$$

which is 63.2 percent of the steady value. This time, τ, is known as the *time constant of the system.* One of the characteristics which we would like to know about the system is its speed of response or how fast the response is approaching the final value. The time constant τ is indicative of this measure and the speed of response is inversely proportional to the time constant of the system.

Another important characteristic of the system is the error between the desired value and the actual value under steady state conditions. This quantity is known as the steady state error of the system and is denoted by e_{ss}.

The error E(s) for a unity feedback system is given by

$$E(s) = R(s) - C(s)$$

$$= R(s) - \frac{G(s)R(s)}{1+G(s)}$$

$$= \frac{R(s)}{1+G(s)} \qquad .....(3.7)$$

For the system under consideration $G(s) = \frac{1}{\tau s}$, $R(s) = \frac{1}{s}$ and therefore

$$E(s) = \frac{\tau}{\tau s + 1}$$

$$\therefore \quad e(t) = e^{-t/\tau}$$

As $t \to \infty$ $e(t) \to 0$. Thus the output of the first order system approaches the reference input, which is the desired output, without any error. In other words, we say a first order system tracks the step input without any steadystate error.

3.4.2 Response to a Unit Ramp Input or Unit Velocity Input

The response of the system in Fig. 3.4 for a unit ramp input, for which,

$$R(s) = \frac{1}{s^2},$$

is given by,

$$C(s) = \frac{1}{s^2(\tau s + 1)} \qquad .....(3.9)$$

The time response is obtained by taking inverse Laplace transform of eqn. (3.9).

$$c(t) = t - \tau(1 - e^{-t/\tau}) \qquad .....(3.10)$$

If eqn. (3.10) is differentiated we get

$$\frac{dc(t)}{dt} = 1 - e^{-t/\tau} \qquad .....(3.11)$$

Eqn. (3.11) is seen to be identical to eqn. (3.6) which is the response of the system to a step input. Thus no additional information about the speed of response is obtained by considering a ramp input. But let us see the effect on the steadystate error. As before,

$$E(s) = \frac{1}{s^2} \cdot \frac{\tau s}{\tau s + 1} = \frac{\tau}{s(\tau s + 1)}$$

$$\therefore \quad e(t) = \tau(1 - e^{-t/\tau})$$

and $$e_{ss} = \underset{t \to \infty}{\text{Lt}} \; e(t) = \tau \qquad .....(3.12)$$

Thus the steady state error is equal to the time constant of the system. The first order system, therefore, can not track the ramp input without a finite steady state error. If the time constant is reduced not only the speed of response increases but also the steady state error for ramp input decreases. Hence the ramp input is important to the extent that it produces a finite steady state error. Instead of finding the entire response, it is sufficient to estimate the steady state value by using the final value theorem. Thus

$$e_{ss} = \underset{s \to 0}{\text{Lt}} \; s\,E(s)$$

$$= \underset{s \to 0}{\text{Lt}} \; \frac{\tau s}{s(\tau s + 1)}$$

$$= \tau$$

which is same as given by eqn. (3.12)

The response of a first order system for unit ramp input is plotted in Fig. 3.7.

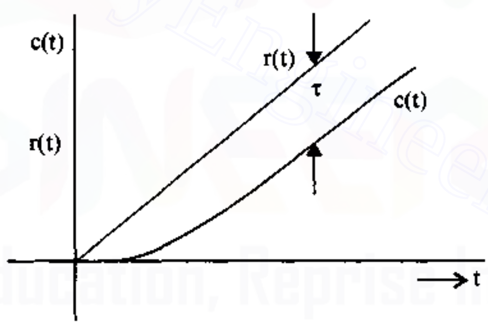

Fig. 3.7 Unit ramp response of a first order system.

3.4.3 Response to a Unit Parabolic or Acceleration Input

The response of a first order system to a unit parabolic input, for which

$$R(s) = \frac{1}{s^3} \text{ is given by,}$$

$$C(s) = \frac{1}{s^3(\tau s + 1)}$$

$$c(t) = \tau^2 - \tau\, t + \frac{t^2}{2} - \tau^2\, e^{-\frac{1}{\tau} t} \qquad .....(3.13)$$

Differentiating eqn. (3.13), we get,

$$\frac{dc(t)}{dt} = -\tau + t + \tau\, e^{-\frac{1}{\tau}t}$$

$$= t - \tau\left(1 - e^{-\frac{1}{\tau}t}\right) \qquad .....(3.14)$$

Eqn. (3.14) is seen to be same as eqn. (3.10), which is the response of the first order system to unit velocity input. Thus subjecting the first order system to a unit parabolic input does not give any additional information regarding transient behaviour of the system. But, the steady state error, for a prabolic input is given by,

$$e(t) = r(t) - c(t)$$

$$= \frac{t^2}{2} - \tau^2 + \tau t - \frac{t^2}{2} + \tau^2 e^{-\frac{1}{\tau}t}$$

$$e_{ss} = \underset{t \to \infty}{Lt}\ e(t) = \infty$$

Thus a first ordr system has infinite state error for a prabolic input. The steady state error can be easily obtained by using the final value theorem as :

$$e_{ss} = \underset{s \to 0}{Lt}\ s\,E(s) = \underset{s \to 0}{Lt}\ \frac{R(s)}{\tau + 1}$$

$$= \underset{s \to 0}{Lt}\ \frac{s.1}{s^3(\tau s + 1)} = \infty$$

Summarizing the analysis of first order system, we can say that the step input yields the desired information about the speed of transient response. It is observed that the speed of response is inversely proportional to the time constant τ of the system. The ramp and parabolic inputs do not give any additional information regarding the speed of response. However, the steady state errors are different for these three different inputs. For a step input, the steadystate error e_{ss} is zero, for a velocity input there is a finite error equal to the time constant τ of the system and for an acceleration input the steadystate error is infinity.

It is clear from the discussion above, that it is sufficient to study the behaviour of any system to a unit step input for understanding its transient response and use the velocity input and acceleration input for understanding the steadystate behaviour of the system.

3.5 Second Order System

3.5.1 Response to a Unit Step Input

Consider a Type 1, second order system as shown in Fig. 3.8. Since G(s) has one pole at the origin, it is a type one system.

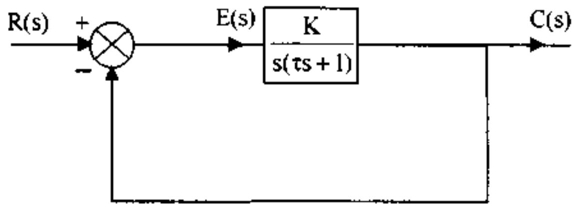

Fig. 3.8 Second Order System.

The closed loop transfer function is give by,

$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s^2 + s + K} \qquad .....(3.15)$$

The transient response of any system depends on the poles of the transfer function T(s). The roots of the denominator polynomial in *s* of *T*(*s*) are the poles of the transfer function. Thus the denominator polynomial of T(s), given by

$$D(s) = \tau s^2 + s + K$$

is known as the *characteristic polynomial* of the system and D(s) = 0 is known as the *characteristic* equation of the system. Eqn. (3.15) is normally put in standard from, given by,

$$T(s) = \frac{K/\tau}{s^2 + \frac{1}{\tau}s + K/\tau}$$

$$= \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \qquad .....(3.16)$$

Where,

$$\omega_n = \sqrt{\frac{K}{\tau}} = \text{natural frequency}$$

$$\delta = \frac{1}{2\sqrt{K\tau}} = \text{damping factor}$$

The poles of T(s), or, the roots of the characteristic equation

$$s^2 + 2\,\delta\,\omega_n\,s + \omega_n^2 = 0$$

are given by,

$$s_{1,2} = \frac{-2\delta\omega_n \pm \sqrt{4\delta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\delta\,\omega_n \pm j\,\omega_n\,\sqrt{1-\delta^2} \qquad \text{(assuming } \delta < 1\text{)}$$

$$= -\delta\,\omega_n \pm j\,\omega_d$$

Where $\omega_d = \omega_n\sqrt{1-\delta^2}$ is known as the *damped natural frequency* of the system. If $\delta > 1$, the two roots s_1, s_2 are real and we have an over damped system. If $\delta = 1$, the system is known as a *critically damped system*. The more common case of $\delta < 1$ is known as the *under damped system*.

If ω_n is held constant and δ is changed from 0 to ∞, the locus of the roots is shown in Fig. 3.9. The magnitude of s_1 or s_2 is ω_n and is independent of δ. Hence the locus is a semicircle with radius ω_n until $\delta = 1$. At $\delta = 0$, the roots are purely imaginary and are given by $s_{1,2} = \pm j\omega_n$. For $\delta = 1$, the roots are purely real, negative and equal to $-\omega_n$. As δ increases beyond unity, the roots are real and negative and one root approached the origin and the other approaches infinity as shown in Fig. 3.9.

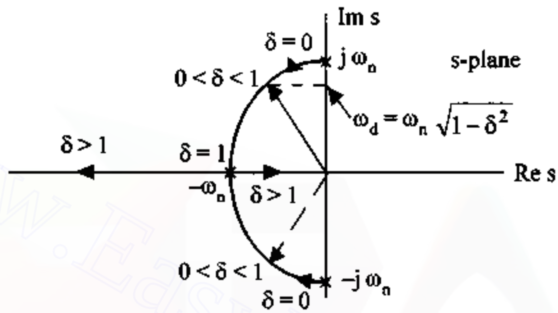

Fig. 3.9 Locus of the roots of the characteristic equation.

For a unit step input $R(s) = \frac{1}{s}$ and eqn. 3.16 can be written as

$$C(s) = T(s).\,R(s) = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\cdot\frac{1}{s} \qquad .....(3.17)$$

Splitting eqn. (3.17) in to partial fractions, assuming δ to be less than 1, we have

$$C(s) = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Evaluating K_1, K_2 and K_3 by the usual procedure, we have,

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$= \frac{1}{s} - \frac{s + \delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n\sqrt{1-\delta^2}}{\sqrt{1-\delta^2}\,(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$.....(3.18)$$

Taking inverse Laplae transform of eqn. (3.18), we have

$$c(t) = 1 - e^{-\delta\omega_n t}\left[\cos\omega_n\sqrt{1-\delta^2}\,t + \frac{\delta}{\sqrt{1-\delta^2}}\sin\omega_n\sqrt{1-\delta^2}\,t\right] \qquad .....(3.19)$$

Eqn. (3.19) can be put in a more convenient from as,

$$c(t) = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \operatorname{Sin}(\omega_d t + \phi)$$

Where $\omega_d = \omega_n \sqrt{1-\delta^2}$

and $\tan\phi = \frac{\sqrt{1-\delta^2}}{\delta}$ .....(3.20)

This response is plotted in Fig. 3.10. The response is oscillatory and as $t \to \infty$, it approaches unity.

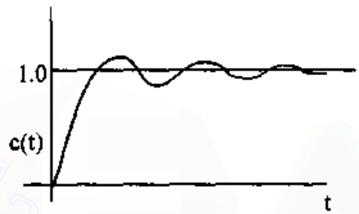

Fig. 3.10 Step response of an underdamped second order system.

If $\delta = 1$, the two roots of the characteristic equations are $s_1 = s_2 = -\omega_n$ and the response is given by

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \cdot \frac{1}{s}$$

and $c(t) = 1 - e^{-\omega_n t} - t\,\omega_n\, e^{-\omega_n t}$ .....(3.21)

This is plotted in Fig. 3.11.

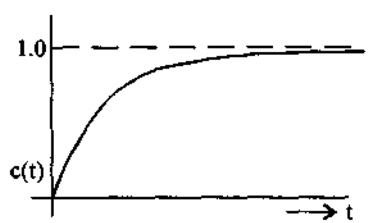

Fig. 3.11 Response of a critically damped second order system.

As the damping is increased from a value less than unity, the oscillations decrease and when the damping factor equals unity the oscillations just disappear. If δ is increaed beyond unity, the roots of the characteristic equation are real and negative and hence, the response approaches unity in an exponential way. This response is known as overdamped response and is shown in Fig. 3.12.

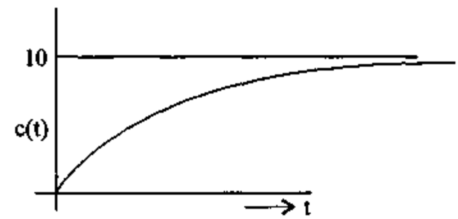

Fig. 3.12 Step response of an overdamped second order system.

$$c\,(t) = K_1\, e^{-s_1 t} + K_2\, e^{-s_2 t} \qquad .....(3.22)$$

Where s_1 and s_2 are given by,

$$s_{1,2} = -\,\delta\,\omega_n \pm \omega_n\,\sqrt{\delta^2 - 1}$$

and K_1 and K_2 are constants.

3.5.2 Response to a Unit Ramp Input

For a unit ramp input,

$$R\,(s) = \frac{1}{s^2}$$

and the output is given by,

$$C\,(s) = \frac{\omega_n^{\,2}}{s^2(s^2 + 2\delta\omega_n s + \omega_n^{\,2})}$$

Taking inverse Laplace transform, we get the time response $c\,(t)$ as,

$$c\,(t) = t - \frac{2\delta}{\omega_n} + \frac{e^{-\delta\omega_n t}}{\omega_n\sqrt{1-\delta^2}}\ \text{Sin}\left(\omega_n\sqrt{1-\delta^2}\ t + \phi\right)\ \text{for}\ \delta < 1 \qquad .....(3.23)$$

The time response for a unit ramp input is plotted in Fig. 3.13.

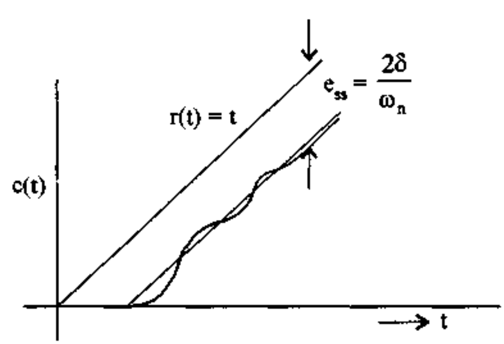

Fig. 3.13 Unit ramp response of a second order system.

The response reveals two aspects of the system.

1. The transient response is of the same form as that of a unit step response. No new information is obtained regarding speed of response or oscillations in the system.

2. It has a steadystate error $e_{ss} = \dfrac{2\delta}{\omega_n}$, unlike the step response, where the steady state error was zero. Thus, no new information is gained by obtaining the transient response of the system for a ramp input. The steadystate error could be easily calculated using final value theorem instead of laboriously solving for the entire reponse. For the given system, the error E (s) is given by

$$E(s) = R(s) - C(s)$$

$$= \frac{1}{s^2} - \frac{\omega_n^2}{s^2(s^2 + 2\delta\omega_n s + \omega_n^2)} = \frac{s^2 + 2\delta\omega_n s + \omega_n^2 - \omega_n^2}{s^2(s^2 + 2\delta\omega_n s + \omega_n^2)}$$

and from the final value theorem,

$$e_{ss} = \underset{s \to 0}{\text{Lt}}\ s\,E(s) = \underset{s \to 0}{\text{Lt}}\ \frac{s^2(s + 2\delta\omega_n)}{s^2(s^2 + 2\delta\omega_n s + \omega_n^2)}$$

$$= \frac{2\delta\omega_n}{\omega_n^2} = \frac{2\delta}{\omega_n} \qquad \text{.....(3.24)}$$

In a similar manner, the unit parabolic input does not yield any fresh information about the transient response. The steadystate error can be obtained using final value theorem in this case also. For the given system, for a unit acceleration input,

$$e_{ss} = \infty \qquad \text{.....(3.25)}$$

3.5.3 Time Domain Specifications of a Second Order System

The performance of a system is usually evaluated in terms of the following qualities.

1. How fast it is able to respond to the input,
2. How fast it is reaching the desired output,
3. What is the error between the desired output and the actual output, once the transients die down and steady state is achieved,
4. Does it oscillate around the desired value,

and 5. Is the output continuously increasing with time or is it bounded.

The last aspect is concerned with the stability of the system and we would require the system to be stable. This aspect will be considered later. The first four questions will be answered in terms of time domain specifications of the system based on its response to a unit step input. These are the specifications to be given for the design of a controller for a given system.

In section 3.5, we have obtained the response of a type 1 second order system to a unit step input. The step response of a typical underdamped second order system is plotted in Fig. 3.14.

It is observed that, for an underdamped system, there are two complex conjugate poles. Usually, even if a system is of higher order, the two complex conjugate poles nearest to the $j\omega$ – axis (called dominant poles) are considered and the system is approximated by a second order system. Thus, in designing any system, certain design specifications are given based on the typical underdamped step response shown as Fig. 3.14.

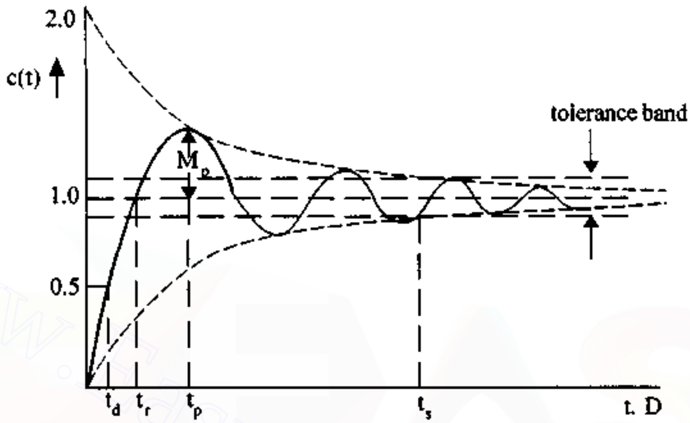

Fig. 3.14 Time domain specifications of a second order system.

The design specifications are :

1. *Delay time* t_d *:* It is the time required for the response to reach 50% of the steady state value for the first time
2. *Rise time* t_r *:* It is the time required for the response to reach 100% of the steady state value for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steadystate value.
3. *Peak time* t_p *:* It is the time required for the response to reach the maximum or Peak value of the response.
4. *Peak overshoot* M_p *:* It is defined as the difference between the peak value of the response and the steady state value. It is usually expressed in percent of the steady state value. If the time for the peak is t_p, percent peak overshoot is given by,

   $$\text{Percent peak overshoot } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100. \qquad .....(3.26)$$

   For systems of type 1 and higher, the steady state value $c(\infty)$ is equal to unity, the same as the input.
5. *Settling time* t_s *:* It is the time required for the response to reach and remain within a specified tolerance limits (usually $\pm$ 2% or $\pm$ 5%) around the steady state value.
6. *Steady state error* e_{ss} *:* It is the error betwen the desired output and the actual output as $t \to \infty$ or under steadystate conditions. The desired output is given by the reference input $r(t)$ and

   therefore, $e_{ss} = \underset{t \to \infty}{\text{Lt}} [r(t) - c(t)]$

From the above specifications it can be easily seen that the time response of a system for a unit step input is almost fixed once these specifications are given. But it is to be observed that all the above specifications are not independent of each other and hence they have to be specified in such a way that they are consistent with others.

Let us now obtain the expressions for some of the above design specifications in terms of the damping factor δ and natural frequency ω_n.

1. ***Rise time (t_r)***

   If we consider an underdamped system, from the definition of the rise time, it is the time required for the response to reach 100% of its steadystate value for the first time. Hence from eqn. (3.20).

$$C(t_r) = 1 = 1 - \frac{e^{-\delta\omega_n t_r}}{\sqrt{1-\delta^2}} \operatorname{Sin}\left(\omega_n\sqrt{1-\delta^2}\, t_r + \phi\right)$$

Or

$$\frac{e^{-\delta\omega_n t_r}}{\sqrt{1-\delta^2}} \operatorname{Sin}\left(\omega_n\sqrt{1-\delta^2}\, t_r + \phi\right) = 0$$

Since $\frac{e^{-\delta\omega_n t_r}}{\sqrt{1-\delta^2}}$ cannot be equal to zero,

$$\operatorname{Sin}(\omega_d t_r + \phi) = 0$$

$$\therefore \quad \omega_d t_r + \phi = \pi$$

and

$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\delta^2}}{\delta}}{\omega_n\sqrt{1-\delta^2}} \qquad .....(3.27)$$

2. ***Peak time (t_p)***

   At the peak time, t_p, the response attains its maximum value and this can be obtained by differentiating $c(t)$ and equating it to zero. Thus,

$$\frac{dc(t)}{dt} = \frac{\delta\omega_n}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{Sin}(\omega_d t + \phi) - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \cos(\omega_d t + \phi).\, \omega_d = 0$$

Simplifying we have,

$$\delta \operatorname{Sin}(\omega_d t + \phi) - \sqrt{1-\delta^2} \cos(\omega_d t + \phi) = 0$$

This can be written as,

$$\cos\phi \sin(\omega_d t + \phi) - \sin\phi \cos(\omega_d t + \phi) = 0$$

where $\tan\phi = \dfrac{\sqrt{1-\delta^2}}{\delta}$

$\therefore$ $\sin(\omega_d t + \phi - \phi) = \sin \omega_d t = 0$

or $\omega_d t = n\pi$ for n = 0, 1, 2, ...

Here

n = 0 Corresponds to its minimum value at t = 0

n = 1 Corresponds to its first peak value at $t = t_p$

n = 2 Corresponds to its first undershoot

n = 3 Corresponds to its second overshoot and so on

Hence for n = 1

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} \quad .....(3.28)$$

Thus, we see that the peak time depends on both ω_n and δ. If we consider the product of ω_n and t_p, which may be called as *normalised peak time*, we can plot the variation of this normalised peak time with the damping factor δ. This is shown in Fig. 3.15.

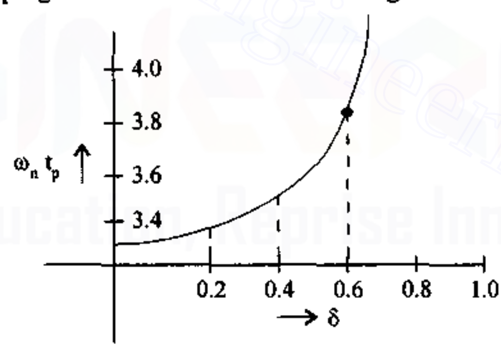

Fig. 3.15 Normalised peak time $\omega_n t_p$ Vs δ for a second order system.

3. ***Peak overshoot (M_p)***

The peak overshoot is defined as

$$M_p = c(t_p) - 1$$

$$= 1 - \frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \sin(\omega_d t_p + \phi) - 1$$

$$= -\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \sin\left(\omega_d \cdot \frac{\pi}{\omega_d} + \phi\right)$$

$$M_p = \frac{e^{-\frac{\delta\omega_n\pi}{\omega_n\sqrt{1-\delta^2}}}}{\sqrt{1-\delta^2}} \text{ Sin } \phi \quad (\because \text{ Sin } (\pi+\phi) = -\text{Sin } \phi)$$

$$= e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} \qquad \left(\because \text{Sin } \phi = \sqrt{1-\delta^2}\right)$$

Hence, peak overshoot, expressed as a percentage of steady state value, is given by,

$$M_p = 100\ e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}}\ \% \qquad .....(3.29)$$

It may be observed that peak overshoot M_p is a function of the damping factor δ only. Its variation with damping factor is shown in Fig. 3.16.

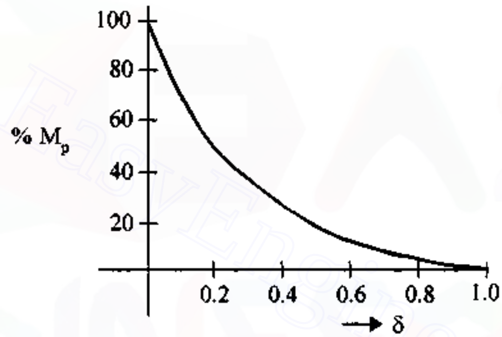

Fig. 3.16 Percent overshoot M_p Vs δ for a second order system.

4. ***Settling time*** (t_s)

The time varying term in the step response, $c(t)$, consists of a product of two terms; namely, an exponentially delaying term, $\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}$ and a sinusoidal term, $Sin\ (\omega_d t + \phi)$. It is clear that the response is a decaying sinusoid, the envelop of which is given by $\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}$. Thus, the response reaches and remains within a given band, around the steadystate value, when this envelop crosses the tolerance band. Once this envelop reaches this value, there is no possibility of subsequent oscillations to go beyond these tolerane limits. Thus for a 2% tolerance band,

$$\frac{e^{-\delta\omega_n t_s}}{\sqrt{1-\delta^2}} = 0.02$$

For low values of δ, $\delta^2 << 1$ and therefore $e^{-\delta\omega_n t_s} \simeq 0.02$

$$\therefore \qquad t_s \simeq \frac{4}{\delta\omega_n} = 4\ \tau \qquad .....(3.30)$$

where τ is the time constant of the exponential term.

Eqn. (3.30) shows that the settling time is a function of both δ and ω_n. Since damping factor is an important design specification, we would like to know the variation of the setting time with δ, with ω_n fixed. Or, in otherwords, we can define a normalised time $\omega_n t_s$, and find the variation of this quantity with respect to δ. The step response of a second order system is plotted in Fig. 3.17 for different values of δ, taking normalised time $\omega_n t$, on x-axis. The curves are magnified around the steady state value for clarity.

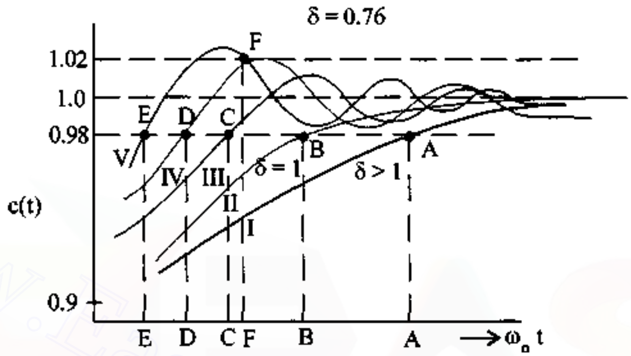

Fig. 3.17 C (t) plotted for different value of δ.

The settling time monotonically decreases as the damping is decreased from a value greater than one (over damped) to less than one (under damped). For 2% tolerance band, it decreases until the first peak of the response reaches the tolerance limit of 1.02 as shown by the curve IV in Fig. 3.17. Points A, B, C, and D marked on the graph give the values $\omega_n t_s$, for decreasing values of δ. The peak value of the response reaches 1.02 at a damping factor $\delta = 0.76$. The settling time for this value of δ is marked as point D on the curve. If δ is decreased further, since the response crosses the upper limit 1.02, the point E no longer represents the settling time. The settling time suddenly jumps to a value given by the point F on the curve. Thus there is a discontinuity at $\delta = 0.76$. If δ is decreased further the setting time increases until the first undershoot touches the lower limit of 0.98. Similarly, the third discontinuity occurs when the second peak touches the upper limit of 1.02 and so on. The variation of $\omega_n t_s$ with δ for a tolerance band of 2% is plotted in Fig. 3.18.

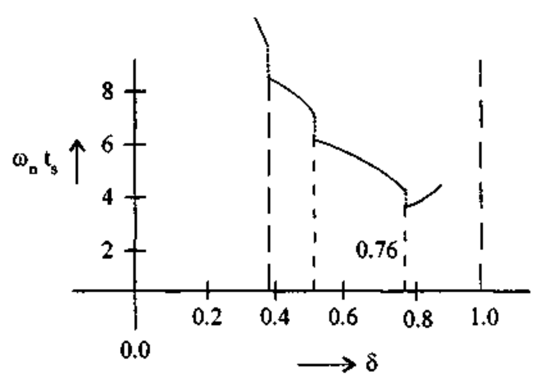

Fig. 3.18 Variation of normalised settling time $\omega_n t_s$ Vs δ.

From Fig. 3.18 it is observed that the least settling time is obtained for a damping factor of $\delta = 0.76$. Since settling time is a measure of how fast the system reaches a steady value, control systems are usually designed with a damping factor of around 0.7. Sometimes, the systems are designed to have even lesser damping factor because of the presence of certain nonlinearities which tend to produce an error under steadystate conditions. To reduce this steadystate error, normally the system gain K is increased, which in turn decreases the damping $\left(\because \delta = \frac{1}{2\sqrt{KT}}\right)$. However, for robotic control, the damping is made close to and slightly higher than unity. This is because the output of a robotic system should reach the desired value as fast as possible, but it should never overshoot it.

5. *Steady state error (e_{ss})*

For a type 1 system, considered for obtaining the design specifications of a second order control system, the steady state error for a step input is obviously zero. Thus

$$e_{ss} = \underset{t \to \infty}{Lt} \; 1 - c(t) = 0$$

The steady state error for a ramp input was obtained in eqn. (3.24) as $e_{ss} = \frac{2\delta}{\omega_n}$.

As the steadystate error, for various test signals, depends on the type of the system, it is dealt in the next section in detail.

3.6 Steady State Errors

One of the important design specifications for a control system is the steadystate error. The steady state output of any system should be as close to desired output as possible. If it deviates from this desired output, the performance of the system is not satisfactory under steadystate conditions. The steadystate error reflects the accuracy of the system. Among many reasons for these errors, the most important ones are the type of input, the type of the system and the nonlinearities present in the system. Since the actual input in a physical system is often a random signal, the steady state errors are obtained for the standard test signals, namely, step, ramp and parbaolic signals.

3.6.1 Error Constants

Let us consider a feedback control system shown in Fig. 3.19.

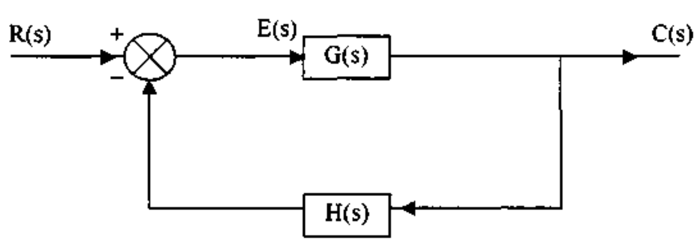

Fig. 3.19 Feedback Control System.

The error signal E (s) is given by

$$E(s) = R(s) - H(s)C(s) \quad .....(3.31)$$

But
$$C(s) = G(s)E(s) \quad .....(3.32)$$

From eqns. (3.31) and (3.32) we have

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Applying final value theorem, we can get the steady state error e_{ss} as,

$$e_{ss} = \underset{s \to 0}{Lt}\ s\,E(s) = \underset{s \to 0}{Lt}\ \frac{sR(s)}{1+G(s)H(s)} \quad .....(3.33)$$

Eqn. (3.33) shows that the steady state error is a function of the input R(s) and the open loop transfer function G(s). Let us consider various standard test signals and obtain the steadystate error for these inputs.

1. Unit step or position input.

For a unit step input, $R(s) = \frac{1}{s}$. Hence from eqn. (3.33)

$$e_s = \underset{s \to 0}{Lt}\ \frac{s.\frac{1}{s}}{1+G(s)H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{Lt}\ G(s)H(s)} \quad .....(3.34)$$

Let us define a useful term, *position error constant* K_p as,

$$K_p \triangleq \underset{s \to 0}{Lt}\ G(s)H(s) \quad .....(3.35)$$

In terms of the position error constant, e_{ss} can be written as,

$$e_{ss} = \frac{1}{1+K_p} \quad .....(3.36)$$

2. Unit ramp or velocity input.

For unit velocity input, $R(s) = \frac{1}{s^2}$ and hence,

$$e_{ss} = \underset{s \to 0}{Lt}\ \frac{s.\frac{1}{s}}{1+G(s)H(s)} = \underset{s \to 0}{Lt}\ \frac{1}{s+sG(s)H(s)}$$

$$= \frac{1}{\underset{s \to 0}{Lt}\ sG(s)H(s)} \quad .....(3.37)$$

Again, defining the *velocity error constant* K_v as,

$$K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s) \qquad .....(3.38)$$

$$\therefore \qquad e_{ss} = \frac{1}{K_v} \qquad .....(3.39)$$

3. Unit parabolic or acceleration input.

For unit acceleration input $R(s) = \frac{1}{s^3}$ and hence

$$e_{ss} = \underset{s \to 0}{Lt} \frac{s}{s^3[1+G(s)H(s)]} = \underset{s \to 0}{Lt} \frac{1}{s^2 + s^2G(s)H(s)}$$

$$= \frac{1}{\underset{s \to 0}{Lt}\ s^2G(s)H(s)} \qquad .....(3.40)$$

Defining the acceleration error constant K_a as,

$$K_a = \underset{s \to 0}{Lt}\ s^2\,G(s)\,H(s) \qquad .....(3.41)$$

$$\therefore \qquad e_{ss} = \frac{1}{K_a} \qquad .....(3.42)$$

For the special case of unity of feedback system, H (s) = 1 and eqns. (3.35) (3.38) or (3.41) are modified as,

$$K_p = \underset{s \to 0}{Lt}\ G(s) \qquad .....(3.43)$$

$$K_v = \underset{s \to 0}{Lt}\ sG(s) \qquad .....(3.44)$$

and

$$K_a = \underset{s \to 0}{Lt}\ s^2\,G(s) \qquad .....(3.45)$$

In design specifications, instead of specifying the steady state error, it is a common practice to specify the error constants which have a direct bearing on the steadystate error. As will be seen later in this section, if the open loop transfer function is specified in time constant form, as in eqn. (3.3), the error constant is equal to the gain of the open loop system.

3.6.2 Dependence of Steadystate Error on Type of the System

Let the loop transfer function G (s) H (s) or the open loop transfer function G (s) for a unity feedback system, be given is time constant form.

$$G(s) = \frac{K(T_{z1}s+1)(T_{z2}s+1)----}{S^r(T_{p1}s+1)(T_{p2}s+1)----} \qquad .....(3.46)$$

As $s \to 0$, the poles at the origin dominate the expression for $G(s)$. We had defined the type of a system, as the number of poles present at the origin. Hence the steady state error, which depends on $\underset{s \to 0}{\text{Lt}}\ G(s)$, $\underset{s \to 0}{\text{Lt}}\ s\,G(s)$ or $\underset{s \to 0}{\text{Lt}}\ s^2\,G(s)$, is dependent on the type of the system. Let us therefore obtain the steady state error for various standard test signals for type–0, type–1 and type–2 systems.

1. ***Type –0 system***

From eqn. (3.46) with $r = 0$, the error constants are given by

$$K_p = \underset{s \to 0}{\text{Lt}}\ G(s) \qquad = \underset{s \to 0}{\text{Lt}}\ \frac{K(\tau_{Z1}s+1)(\tau_{Z2}s+1)--}{(\tau_{p1}s+1)(\tau_{p2}s+1)--} = K$$

$$K_v = \underset{s \to 0}{\text{Lt}}\ s\,G(s) \qquad = \underset{s \to 0}{\text{Lt}}\ \frac{sK(\tau_{Z1}s+1)(\tau_{Z2}s+1)--}{(\tau_{p1}s+1)(\tau_{p2}s+1)--} = 0$$

Similarly
$$K_a = \underset{s \to 0}{\text{Lt}}\ s^2\,G(s) = 0 \qquad \text{.....(3.47)}$$

The steady state errors for unit step, velocity and acceleration inputs are respectively, from eqns. (3.34), (3.37) and (3.40),

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K} \text{ (step input)}$$

$$e_{ss} = \frac{1}{K_v} = \infty \text{ (velocity input)}$$

$$e_{ss} = \frac{1}{K_a} = \infty \text{ (acceleration input)}$$

2. ***Type 1 system***

For type 1 system, $r = 1$ in eqn. (3.46) and

$$K_p = \underset{s \to 0}{\text{Lt}}\ G(s) \quad = \text{Lt}\ \frac{K}{s} = \infty$$

$$K_v = \underset{s \to 0}{\text{Lt}}\ s\,G(s) = \underset{s \to 0}{\text{Lt}}\ s.\frac{K}{s} = K$$

and
$$K_a = \underset{s \to 0}{\text{Lt}}\ s\,G(s) = \underset{s \to 0}{\text{Lt}}\ s^2.\frac{K}{s} = 0$$

The steady state error for unit step, unit velocity and unit acceleration inputs are respectively,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0 \qquad \text{(position)}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K} \qquad \text{(velocity)}$$

and $$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty \qquad \text{(acceleration)}$$

3. ***Type 2-system***

For a type – 2 system $r = 2$ in eqn. (3.46) and

$$K_p = \underset{s \to 0}{Lt}\ G(s) \qquad = \underset{s \to 0}{Lt}\ \frac{K}{s^2} = \infty$$

$$K_v = \underset{s \to 0}{Lt}\ s\,G(s) \qquad = \underset{s \to 0}{Lt}\ \frac{sK}{s^2} = \infty$$

and $$K_a = \underset{s \to 0}{Lt}\ s^2\,G(s) \qquad = \underset{s \to 0}{Lt}\ \frac{s^2K}{s^2} = K$$

The steady state errors for the three test inputs are,

$$e_{ss} = \frac{1}{1+K_P} = \frac{1}{1+\infty} = 0 \quad \text{(position)}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0 \qquad \text{(velocity)}$$

and $$e_{ss} = \frac{1}{K_a} = \frac{1}{K} \qquad \text{(acceleration)}$$

Thus a type zero system has a finite steady state error for a unit step input and is equal to

$$e_{ss} = \frac{1}{1+K} = \frac{1}{1+K_p} \qquad .....(3.47)$$

Where K is the system gain in the time constant from. It is customary to specify the gain of a type zero system by K_p rather than K.

Similarly, a type –1 system has a finite steady state error for a velocity input only and is given by

$$e_{ss} = \frac{1}{K} = \frac{1}{K_v} \qquad .....(3.48)$$

Thus the gain of type –1 system in normally specified as K_v.

A type –2 system has a finite steady state error only for acceleration input and is given by

$$e_{ss} = \frac{1}{K} = \frac{1}{K_a} \qquad .....(3.49)$$

As before, the gain of type –2 system is specified as K_a rather than K.

The steady state errors, for various standard inputs for type – 0, type – 1 and type – 2 are summarized in Table. 3.1.

Table. 3.1 Steady state errors for various inputs and type of systems

Standard input	**Steadystate error e_{ss}**		
	Type - 0 $K_p = \underset{s \to 0}{Lt} G(s)$	**Type - 1** $K_v = \underset{s \to 0}{Lt} s\,G(s)$	**Type - 2** $K_a = \underset{s \to 0}{Lt} s^2 G(s)$
Unit step	$\frac{1}{1+K_P}$	0	0
Unit velocity	∞	$\frac{1}{K_v}$	0
Unit acceleration	∞	∞	$\frac{1}{K_a}$

If can be seen from Table. 3.1, as the type of the system and hence the number of integrations increases, more and more steady state errors become zero. Hence it may appear that it is better to design a system with more and more poles at the origin. But if the type of the system is higher than 2, the systems tend to be more unstable and the dynamic errors tend to be larger. The stability aspects are considered in chapter 4.

3.6.3 Generalized Error Coefficients - Error Series

The main disadvantage of defining the steadystate error in terms of error constants is that, only one of the constants is finite and non zero for a particular system, where as the other constants are either zero or infinity. If any error constant is zero, the steady state error is infinity, but we do not have any clue as to how the error is approaching infinity.

If the inputs are other than step, velocity or acceleration inputs, we can extend the concept of error constants to include inputs which can be represented by a polynomial. Many functions which are analytic can be represented by a polynomial in *t*. Let the error be given by,

$$E(s) = \frac{R(s)}{1+G(s)} \qquad .....(3.50)$$

Eqn. (3.50) may be written as

$$E(s) = Y(s).\, R(s) \qquad .....(3.51)$$

Where

$$Y(s) = \frac{1}{1+G(s)} \qquad .....(3.52)$$

Using Convolution theorem eqn. (3:51) can be written as

$$e(t) = \int_0^t y(\tau)\, r(t-\tau)\, d\tau \qquad .....(3.53)$$

Assuming that $r(t)$ has first n deriratives, $r(t-\tau)$ can be expanded into a Taylor series,

$$r(t-\tau) = r(t) - \tau r'(t) + \frac{\tau^2}{2!} r''(t) - \frac{\tau^3}{3!} r'''(t) ---- \qquad .....(3.54)$$

where the primes indicate time derivatives. Substituting eqn. (3.54) into eqn. (3.53), we have,

$$e(t) = \int_0^t y(\tau) \left[r(t) - \tau r'(t) + \frac{\tau^2}{2!} r''(t) - \frac{\tau^3}{3!} r'''(t) ---- \right] d\tau$$

$$= r(t) \int_0^t y(\tau)\, d\tau - r'(t) \int_0^t \tau y(\tau)\, d\tau + r''(t) \int_0^t \frac{\tau^2}{2!} y(\tau)\, d\tau + ... \qquad .....(3.55)$$

To obtain the steady state error, we take the limit $t \to \infty$ on both sides of eqn. (3.55)

$$e_{ss} = \underset{t\to\infty}{\text{Lt}}\, e(t) = \underset{t\to\infty}{\text{Lt}} \left[r(t)\int_0^t y(\tau)d\tau - r'(t)\int_0^t \tau y(\tau)d\tau + r''(t)\frac{\tau^2}{2!}y(\tau)d\tau..... \right] \qquad .....(3.56)$$

$$e_{ss} = r_{ss}(t) \int_0^\infty y(\tau)\, d\tau - r'_{ss}(t) \int_0^\infty \tau y(\tau)\, d\tau + r_{ss}''(t) \int_0^\infty \frac{\tau^2}{2!} y(\tau)\, d\tau + ... \qquad .....(3.57)$$

Where the suffix *ss* denotes steady state part of the function. It may be further observed that the integrals in eqn. (3.57) yield constant values. Hence eqn. (3.57) may be written as,

$$e_{ss} = C_O r_{ss}(t) + C_1 r'_{ss}(t) + \frac{C_2}{2!} r''_{ss}(t) + ... + \frac{C_n}{n!} r^{(n)}_{ss}(t) + \qquad ......(3.58)$$

Where,

$$C_0 = \int_0^\infty y(\tau)\, d\tau \qquad .....(3.59)$$

$$C_1 = -\int_0^\infty \tau y(\tau)\, d\tau \qquad .....(3.60)$$

$$C_n = (-1)^n \int_0^\infty \tau^n y(\tau)\, d\tau \qquad .....(3.61)$$

The coefficients C_0, C_1, C_2, ... C_n, ... are defined as generalized error coefficients. Eqn. (3.58) is known as generalised error series. It may be observed that the steady state error is obtained as a function of time, in terms of generalised error coefficients, the steady state part of the input and its derivatives. For a given transfer function G(s), the error coefficients can be easily evaluated as shown in the following.

Let $$y(t) = \mathcal{L}^{-1}\, Y(s)$$

$$\therefore \quad Y(s) = \int_0^\infty y(t)\, e^{-s\tau}\, d\tau \qquad \text{.....(3.62)}$$

$$\underset{s \to 0}{\text{Lt}}\, Y(s) = \underset{s \to 0}{\text{Lt}} \int_0^\infty y(\tau)\, e^{-s\tau}\, d\tau$$

$$= \int_0^\infty y(\tau) \underset{s \to 0}{\text{Lt}}\, e^{-s\tau}\, d\tau$$

$$= \int_0^\infty y(\tau)\, d\tau$$

$$= C_0 \qquad \text{.....(3.63)}$$

Taking the derivative of eqn. (3.62) with respect to *s*,

We have,

$$\frac{dY(s)}{ds} = \int_0^\infty y(\tau)(-\tau)\, e^{-s\tau}\, d\tau \qquad \text{.....(3.64)}$$

Now taking the limit of equation (3.64) as $s \to 0$, we have,

$$\underset{s \to 0}{\text{Lt}}\, \frac{dY(s)}{ds} = \int_0^\infty y(\tau)(-\tau) \underset{s \to 0}{\text{Lt}}\, e^{-s\tau}\, d\tau$$

$$= -\int_0^\infty \tau y(\tau)\, d\tau$$

$$= C_1 \qquad \text{.....(3.65)}$$

Similarly,

$$C_2 = \underset{s \to 0}{\text{Lt}}\, \frac{d^2Y(s)}{ds^2} \qquad \text{.....(3.66)}$$

$$C_3 = \underset{s \to 0}{\text{Lt}}\, \frac{d^3Y(s)}{ds^3} \qquad \text{.....(3.67)}$$

$$C_n = \underset{s \to 0}{\text{Lt}}\, \frac{d^nY(s)}{ds^n} \qquad \text{.....(3.68)}$$

Thus the constants can be evaluated using eqns. (3.63), (3.65) and (3.66) and so on and the time variation of the steadystate error can be obtained using eqn. (3.58).

The advantages of error series can be summarized as,

1. It provides a simple way of obtaining the nature of steadystate response to almost any arbitrary input.
2. We can obtain the complete steadystate response without actually solving the system differential equation.

Example 3.1

The angular position θ_C of a mass is controlled by a servo system through a reference signal θ_r. The moment of intertia of moving parts referred to the load shaft, J, is 150 kgm^2 and damping torque coefficient referred to the load shaft, B, is 4.5×10^3 Nwm / rad / sec. The torque developed by the motor at the load is 7.2×10^4 Nw–m per radian of error.

(a) Obtain the response of the system to a step input of 1 rad and determine the peak time, peak overshoot and frequency of transient oscillations. Also find the steadystate error for a constant angular velocity of 1 revolution / minute.

(b) If a steady torque of 1000 Nwm is applied at the load shaft, determine the steadystate error.

Solution :

The block diagram of the system may be written as shown in Fig. 3.20.

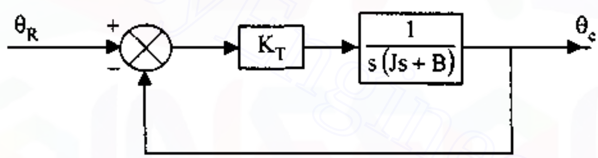

Fig. 3.20 Block diagram of the given system

From the block diagram, the forward path transfer function G (s) is given by,

$$G(s) = \frac{K_T}{s(Js + B)}$$

For the given values of K_T, J and B, we have

$$G(s) = \frac{7.2 \times 10^4}{s(150s + 4.5 \times 10^3)}$$

$$= \frac{16}{s(0.333s + 1)}$$

Thus $K_V = 16$

$\tau = 0.333$ sec.

and $\delta = \frac{1}{2\sqrt{K_V \tau}} = \frac{1}{2\sqrt{16 \times 0.0333}}$

$= 0.6847$

$$\omega_n = \sqrt{\frac{K_V}{\tau}} = \sqrt{\frac{16}{0.0333}}$$
$$= 21.91 \text{ rad/sec}$$

(a)
$$\theta(t) = 1 - \frac{e^{-\omega_n t}}{\sqrt{1-\delta^2}} \text{ Sin}\left(\omega_n\sqrt{1-\delta^2}\, t + \tan^{-1}\sqrt{\frac{1-\delta^2}{\delta}}\right)$$
$$= 1 - 1.372\, e^{-15\,t} \text{ Sin } (15.97\, t + 46.8^0)$$

Peak time, $t_P = \dfrac{\pi}{\omega_n\sqrt{1-\delta^2}} = \dfrac{\pi}{\omega_d}$

$$= \frac{\pi}{15.97} = 0.1967 \text{ sec}$$

Peak over shoot, $M_P = 100\, e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}}$

$= 5.23\%$

Frequency of transient oscillations, $\omega_d = 15.97$ rad/sec

Steady state error $\dot{\theta}_R = \dfrac{2\pi}{60}$ rad/sec

$K_v = 16$

$$e_{ss} = \frac{2\pi}{60 \times 16} = 6.54 \times 10^{-3} \text{ rad}$$

(b) When a load torque of 1000 Nwm is applied at the load shaft, using super position theorem, the error is nothing but the response due to this load torque acting as a step input with

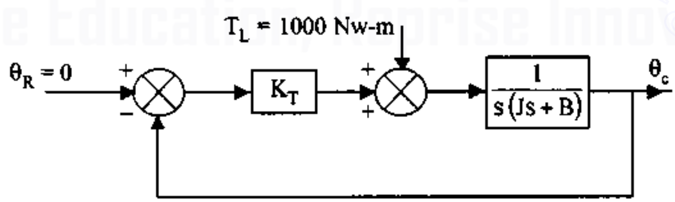

Fig. 3.21 Block diagram of the system with load torque applied

$\theta_R = 0$. The block diagram may be modified as shown in Fig. 3.21.

From Fig. 3.21, we have

$$\frac{\theta_C(s)}{T_L(s)} = \frac{\dfrac{1}{s(Js+B)}}{1 + \dfrac{K_T}{s(Js+B)}} = \frac{1}{Js^2 + Bs + K_T}$$

$$\theta_C(s) = \frac{1000}{s(150s^2 + 4.5 \times 10^3 s + 7.2 \times 10^4)}$$

Using final value theorem,

$$\theta_{Css} = \underset{s \to 0}{Lt}\ s\,\theta_C(s) = \frac{1000}{7.2\times10^4}$$

$$= 0.01389 \text{ rad}$$

$$= 0.796^0$$

Example 3.2

The open loop transfer function of a unity feedback system is given by,

$$G(s) = \frac{K}{s(\tau s+1)} \qquad K, \tau > 0$$

With a given value of K, the peak overshoot was found to be 80%. It is proposed to reduce the peak overshoot to 20% by decreasing the gain. Find the new value of K in terms of the old value.

Solution :

Let the gain be K_1 for a peak overshoot of 80%

$$\therefore \qquad e^{-\frac{\pi\delta_1}{\sqrt{1-\delta_1^2}}} = 0.8$$

$$\frac{\pi\delta_1}{\sqrt{1-\delta_1^2}} = ln\frac{1}{0.8} = 0.223$$

$$\pi^2\delta_1^2 = 0.223^2(1-\delta^2)$$

Solving for δ_1, we get

$$\delta_1 = 0.07$$

Let the new gain be K_2 for a peak overshoot of 20%

$$e^{-\frac{\pi\delta_2}{\sqrt{1-\delta_2^2}}} = 0.2$$

$$\frac{\pi\delta_2}{\sqrt{1-\delta_2^2}} = 1.61$$

Solving for δ_2,

$$\delta_2 = 0.456$$

But $$\delta = \frac{1}{2\sqrt{K\tau}}$$

$$\frac{\delta_1}{\delta_2} = \frac{1}{2\sqrt{K_1\tau}}.2\sqrt{K_2\tau} = \sqrt{\frac{K_2}{K_1}}$$

$$\frac{\delta_1^2}{\delta_2^2} = \frac{K_2}{K_1}$$

$\therefore$
$$K_2 = \frac{\delta_1^2}{\delta_2^2} . K_1 = 0.0236\ K_1$$

Example 3.3

Find the steadystate error for unit step, unit ramp and unit acceleration inputs for the following systems.

1. $\dfrac{10}{s(0.1s+1)(0.5s+1)}$ 2. $\dfrac{1000(s+1)}{(s+10)(s+50)}$ 3. $\dfrac{1000}{s^2(s+1)(s+20)}$

Solution :

1. $$G(s) = \frac{10}{s(0.1s+1)(0.5s+1)}$$

(a) Unit step input

$$K_p = \underset{s \to 0}{Lt}\ G(s) = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0$$

(b) Unit ramp input

$$K_v = \underset{s \to 0}{Lt}\ s\,G(s)$$

$$= \underset{s \to 0}{Lt}\ \frac{10}{(0.1s+1)(0.5s+1)} = 10$$

$$e_{ss} = \frac{1}{K_V} = \frac{1}{10} = 0.1$$

(c) Unit acceleration input

$$K_a = \underset{s \to 0}{Lt}\ s^2 G(s) = \underset{s \to 0}{Lt}\ \frac{10s}{(0.1s+1)(0.5s+1)} = 0$$

$$e_{ss} = \frac{1}{K_a} = \infty$$

2. $$G(s) = \frac{1000(s+1)}{(s+10)(s+50)}$$

The transfer function is given in pole zero form. Let us put this in time constant form.

$$G(s) = \frac{1000(s+1)}{500(0.1s+1)(0.02s+1)} = \frac{2(s+1)}{(0.1s+1)(0.02s+1)}$$

Since this is a type zero system we can directly obtain

$$K_p = 2, \qquad K_v = 0 \qquad K_a = 0$$

The steadystate errors are,

(a) Unit step input

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+2} = \frac{1}{3}$$

(b) Unit ramp input

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

(c) Unit acceleration input

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

3. $$G(s) = \frac{1000}{s^2(s+1)(s+20)}$$

Expressing G (s) in time constant form,

$$G(s) = \frac{1000}{20s^2(s+1)(0.05s+1)} = \frac{50}{s^2(s+1)(0.05s+1)}$$

The error constants for a type 2 system are

$$K_p = \infty \qquad K_v = \infty \qquad K_a = 50$$

The steadystate errors for,

(a) a unit step input

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

(b) a unit ramp input

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

(c) a unit acceleration input

$$e_{ss} = \frac{1}{K_a} = \frac{1}{50} = 0.02$$

Example 3.4

The open loop transfer function of a servo system is given by,

$$G(s) = \frac{10}{s(0.2s+1)}$$

Evaluate the error series for the input,

$$r(t) = 1 + 2t + \frac{3t^2}{2}$$

Solution :

$$G(s) = \frac{10}{s(0.2s+1)}$$

$$Y(s) = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10}{s(0.2s+1)}} = \frac{s(0.2s+1)}{0.2s^2+s+10}$$

The generalised error coefficients are given by,

$$C_0 = \underset{s\to 0}{\text{Lt}}\ Y(s)$$

$$= \underset{s\to 0}{\text{Lt}}\ \frac{s(0.2s+1)}{0.2s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{\text{Lt}}\ \frac{dY(s)}{ds}$$

$$C_1 = \underset{s\to 0}{\text{Lt}}\ \frac{(0.2s^2+s+10)(0.4s+1) - s(0.2s+1)(0.4s+1)}{(0.2s^2+s+10)^2}$$

$$= \underset{s\to 0}{\text{Lt}}\ \frac{10(0.4s+1)}{(0.2s^2+s+10)^2}$$

$$= \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{\text{Lt}}\ \frac{d^2Y(s)}{ds^2}$$

$$= \underset{s\to 0}{\text{Lt}}\ \frac{(0.2s^2+s+10)^2(4) - 10(0.4s+1)\left[2(0.2s^2+s+10)(0.4s+1)\right]}{(0.2s^2+s+10)^4}$$

$$= \frac{400 - 10(20)}{(10)^4} = 0.02$$

The input and its deriatives are,

$$r(t) = 1 + 2\,t + \frac{3t^2}{2}$$

$$r'(t) = 2 + \frac{6t}{2} = 2 + 3t$$

$$r''(t) = 3$$

$$r'''(t) = 0 = r^{iv}(t) = r^{v}(t)$$

∴ The error series is given by,

$$e_{ss}(t) = C_0\, r_{ss}(t) + C_1\, r_{ss}'(t) + \frac{C_2}{2!}\, r_{ss}''(t)$$

$$e_{ss}(t) = 0\left(1 + 2t + \frac{3t^2}{2}\right) + 0.1\,(2 + 3t) + \frac{0.02}{2}\,(3)$$

$$= 0.23 + 0.3t$$