Chapter 9

Transportation Problem

"We want these assets to be productive. We buy them. We own them. To say we care only about the short term is wrong. What I care about is seeing these assets in the best hands."

– Carl Icahn

PREVIEW

The structure of transportation problem involves a large number of shipping routes from several supply centres to several demand centres. The objective is to determine the number of units of an item (commodity or product) that should be shipped from an origin to a destination in order to satisfy the required quantity of goods or services at each destination centre.

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- recognize and formulate a transportation problem involving a large number of shipping routes.
- drive initial feasible solution using several methods.
- drive optimal solution by using Modified Distribution Method.
- handle the problem of degenerate and unbalanced transportation problem.
- examine multiple optimal solutions, and prohibited routes in the transportation problem.
- construct the initial transportation table for a trans-shipment problem.
- solve a profit maximization transportation problem using suitable changes in the transportation algorithm.

CHAPTER OUTLINE

9.1 INTRODUCTION

One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply centres to several demand centres. A transportation problem when expressed in terms of an LP model can also be solved by the simplex method. However a transportation problem involves a large number of variables and constraints, solving it using simplex methods takes a long time. Two transportation algorithms, namely *Stepping Stone Method* and the *MODI* (modified distribution) *Method* have been developed for solving a transportation problem.

The structure of transportation problem involves a large number of shipping routes from several supply centres to several demand centres. Thus, objective is to determine shipping routes between supply centres and demand centres in order to satisfy the required quantity of goods or services at each destination centre, with available quantity of goods or services at each supply centre at the minimum transportation cost and/or time.

The transportation algorithms help to minimize the total cost of transporting a homogeneous commodity (product) from supply centres to demand centres. However, it can also be applied to the maximization of total value or utility.

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock (1941). It was further developed by T C Koopmans (1949) and G B Dantzig (1951). Several extensions of transportation models and methods have been subsequently developed.

The study of **transportation problem** helps to identify optimal transportation routes along with units of commodity to be shipped in order to minimize total transportation cost.

9.2 MATHEMATICAL MODEL OF TRANSPORTATION PROBLEM

Let us consider Example 9.1 to illustrate the mathematical model formulation of transportation problem of transporting a single commodity from three sources of supply to four demand destinations. The *sources of supply* are production facilities, warehouses, or supply centres, each having certain amount of commodity to supply. The *destinations* are consumption facilities, warehouses or demand centres each having certain amount of requirement (or demand) of the commodity.

Example 9.1 A company has three production facilities S_1, S_2 and S_3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses D_1, D_2, D_3 and D_4 with requirement of 5, 6, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D_1	D_2	D_3	D_4	*Supply (Availability)*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand (Requirement)	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Model formulation Let x_{ij} = number of units of the product to be transported from a production facility i (i = 1, 2, 3) to a warehouse j (j = 1, 2, 3, 4)

The transportation problem is stated as an LP model as follows:

$$\text{Minimize (total transportation cost) } Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$$

subject to the constraints

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 7 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 9 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 18 \end{aligned}\right\} \text{(Supply)}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 5 \\ x_{12} + x_{22} + x_{32} &= 8 \\ x_{13} + x_{23} + x_{33} &= 7 \\ x_{14} + x_{24} + x_{34} &= 14 \end{aligned}\right\} \text{(Demand)}$$

and $\quad x_{ij} \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, and 4.

In the above LP model, there are $m \times n = 3 \times 4 = 12$ decision variables, x_{ij} and $m + n = 7$ constraints, where m are the number of rows and n are the number of columns in a general transportation table.

9.2.1 General Mathematical Model of Transportation Problem

Let there be m sources of supply, $S_1, S_2, \ldots, S_m$ having a_i ($i = 1, 2, \ldots, m$) units of supply (or capacity), respectively to be transported to n destinations, $D_1, D_2, \ldots, D_n$ with b_j ($j = 1, 2, \ldots, n$) units of demand (or requirement), respectively. Let c_{ij} be the cost of shipping one unit of the commodity from source i to destination j. If x_{ij} represents number of units shipped from source i to destination j, the problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand conditions. Mathematically, the transportation problem, in general, may be stated as follows:

$$\text{Minimize (total cost) } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \tag{1}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m \text{ (supply constraints)} \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n \text{ (demand constraints)} \tag{3}$$

and $\quad x_{ij} \geq 0$ for all i and j. $\quad$ (4)

For easy presentation and solution, a transportation problem data is generally presented as shown in Table 9.1.

Transportation table is a convenient way to summarize data.

Existence of feasible solution A necessary and sufficient condition for a feasible solution to the transportation problems is:

$$\text{Total supply} = \text{Total demand}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{(also called } \textit{rim conditions}\text{)}$$

For proof, see Appendix at the end of this chapter.

Table 9.1 General Transportation Table.

From \ To	D_1	D_2	. . .	D_n	*Supply* a_i
S_1	c_{11} x_{11}	c_{12} x_{12}	. . .	c_{1n} x_{1n}	a_1
S_2	c_{21} x_{21}	c_{22} x_{22}	. . .	c_{2n} x_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	. . .	c_{mn} x_{mn}	a_m
Demand b_j	b_1	b_2	. . .	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

In this problem, there are $(m + n)$ constraints, one for each source of supply, and distinction and $m \times n$ variables. Since all $(m + n)$ constraints are equations, therefore, one of these equations is extra (redundant). The extra constraint (equation) can be derived from the other constraints (equations), without affecting the feasible solution. It follows that any feasible solution for a transportation problem must have exactly $(m + n - 1)$ non-negative basic variables (or allocations) x_{ij} satisfying the rim conditions.

Remarks 1. When the total supply is equal to the total demand, the problem is called a *balanced transportation problem,* otherwise it is called an *unbalanced transportation problem*. The unbalanced transportation problem can be made balanced by adding a dummy supply centre (row) or a dummy demand centre (column) as the need arises.

2. When the number of positive allocations (values of decision variables) at any stage of the feasible solution is less than the required number (rows + columns – 1), i.e. number of independent constraint equations, the solution is said to be *degenerate,* otherwise *non-degenerate*. For proof, see Appendix at the end of this chapter.
3. Cells in the transportation table having positive allocation, i.e., $x_{ij} > 0$ are called *occupied cells,* otherwise are known as *non-occupied (or empty) cells.*

When total demand equals total supply, the transportation problem is said to be balanced.

9.3 THE TRANSPORTATION ALGORITHM

The algorithm for solving a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and arrange the data in the matrix form The formulation of the transportation problem is similar to the LP problem formulation. In transportation problem, the objective function is the total transportation cost and the constraints are the amount of supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution In this chapter, following three different methods are discussed to obtain an initial solution:

- *North-West Corner Method,*
- *Least Cost Method,* and
- *Vogel's Approximation (or Penalty) Method.*

The initial solution obtained by any of the three methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called *rim conditions*).
(ii) The number of positive allocations must be equal to $m + n - 1$, where m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called *non-degenerate basic feasible solution,* otherwise, *degenerate solution.*

Step 3: Test the initial solution for optimality In this chapter, the Modified Distribution (MODI) method is discussed to test the optimality of the solution obtained in Step 2. If the current solution is optimal, then stop. Otherwise, determine a new improved solution.

Step 4: Updating the solution Repeat Step 3 until an optimal solution is reached.

9.4 METHODS OF FINDING INITIAL SOLUTION

There are several methods available to obtain an initial basic feasible solution. In this chapter, we shall discuss only following three methods:

9.4.1 North-West Corner Method (NWCM)

This method does not take into account the cost of transportation on any route of transportation. The method can be summarized as follows:

Step 1: Start with the cell at the upper left (north-west) corner of the transportation table (or matrix) and allocate commodity equal to the minimum of the rim values for the first row and first column, i.e. min (a_1, b_1).

Step 2: (a) If allocation made in Step 1 is equal to the supply available at first source (a_1, in first row), then move vertically down to the cell (2, 1), i.e., second row and first column. Apply Step 1 again, for next allocation.

(b) If allocation made in Step 1 is equal to the demand of the first destination (b_1 in first column), then move horizontally to the cell (1, 2), i.e., first row and second column. Apply Step 1 again for next allocation.

(c) If $a_1 = b_1$, allocate $x_{11} = a_1$ or b_1 and move diagonally to the cell (2, 2).

Step 3: Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table.

Remark If during the process of making allocation at a particular cell, the supply equals demand, then the next allocation of magnitude zero can be made in a cell either in the next row or column. This condition is known as degeneracy.

Example 9.2 Use North-West Corner Method (NWCM) to find an initial basic feasible solution to the transportation problem using data of Example 9.1

Solution The cell (S_1, D_1) is the north-west corner cell in the given transportation table. The rim values for row S_1 and column D_1 are compared. The smaller of the two, i.e. 5, is assigned as the first allocation; otherwise it will violate the feasibility condition. This means that 5 units of a commodity are to be transported from source S_1 to destination D_1. However, this allocation leaves a supply of 7 – 5 = 2 units of commodity at S_1.

Move horizontally and allocate as much as possible to cell (S_1, D_2). The rim value for row S_1 is 2 and for column D_2 is 8. The smaller of the two, i.e. 2, is placed in the cell. Proceeding to row S_2, since the demand of D_1 is fullfilled. The unfulfilled demand of D_2 is now 8 – 2 = 6 units. This can be fulfilled by S_2 with capacity of 9 units. So 6 units are allocated to cell (S_2, D_2). The demand of D_2 is now satisfied and a balance of 9 – 6 = 3 units remains with S_2.

Table 9.2
Initial Solution using NWCM

	D_1	D_2	D_3	D_4	*Supply*
S_1	19 5	30 2	50	10	7
S_2	70	30 6	40 3	60	9
S_3	40	8	70 4	20 14	18
Demand	5	8	7	14	34

Continue to move horizontally and vertically in the same manner to make desired allocations. Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal to $m + n - 1 = 3 + 4 - 1 = 6$. If yes, then solution is non-degenerate feasible solution. Otherwise degenerate solution.

The total transportation cost of the initial solution is obtained by multiplying the quantity x_{ij} in the occupied cells with the corresponding unit cost c_{ij} and adding all the values together. Thus, the total transportation cost of this solution is

Total cost = 5 × 19 + 2 × 30 + 6 × 30 + 3 × 40 + 4 × 70 + 14 × 20 = Rs 1,015

9.4.2 Least Cost Method (LCM)

Since the main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step 2: After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

Example 9.3 Use Least Cost Method (LCM) to find initial basic feasible solution to the transportation problem using data of Example 9.1.

Solution The cell with lowest unit cost (i.e., 8) is (S_3, D_2). The maximum units which can be allocated to this cell is 8. This meets the complete demand of D_2 and leave l0 units with S_3, as shown in Table 9.3.

In the reduced table without column D_2, the next smallest unit transportation cost, is 10 in cell (S_1, D_4). The maximum which can be allocated to this cell is 7. This exhausts the capacity of S_1 and leaves 7 units with D_4 as unsatisfied demand. This is shown in Table 9.3.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19	30	50	10 (7)	7
S_2	70	30	40	60	9
S_3	40	8 (8)	70	20	18
Demand	5	8	7	14	34

Table 9.3

In Table 9.3, the next smallest cost is 20 in cell (S_3, D_4). The maximum units that can be allocated to this cell is 7 units. This satisfies the entire demand of D_4 and leaves 3 units with S_3, as the remaining supply, shown in Table 9.4.

In Table 9.4, the next smallest unit cost cell is not unique. That is, there are two cells – (S_2, D_3) and (S_3, D_1) – that have the same unit transportation cost of 40. Allocate 7 units in cell (S_2, D_3) first because it can accommodate more units as compared to cell (S_3, D_1). Then allocate 3 units (only supply left with S_3) to cell (S_3, D_1). The remaining demand of 2 units of D_1 is fulfilled from S_2. Since supply and demand at each supply centre and demand centre is exhausted, the initial solution is arrived at, and is shown in Table 9.4.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19	30	50	10 (7)	7
S_2	70 (2)	30	40 (7)	60	9
S_3	40 (3)	8 (8)	70	20 (7)	18
Demand	5	8	7	14	34

Table 9.4

The total transportation cost of the initial solution by LCM is calculated as given below:

$$\text{Total cost} = 7 \times 10 + 2 \times 70 + 7 \times 40 + 3 \times 40 + 8 \times 8 + 7 \times 20 = \text{Rs } 814$$

The total transportation cost obtained by LCM is less than the cost obtained by NWCM.

9.4.3 Vogel's Approximation Method (VAM)

Vogel's approximation (penalty or regret) is preferred over NWCR and LCM methods. In this method, an allocation is made on the basis of the opportunity (or penalty or extra) cost that would have been incurred if the allocation in certain cells with minimum unit transportation cost were missed. Hence, allocations are made in such a way that the penalty cost is minimized. An initial solution obtained by using this method is nearer to an optimal solution or is the optimal solution itself. The steps of VAM are as follows:

Step 1: Calculate the penalties for each row (column) by taking the difference between the smallest and next smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if decision-maker fails to allocate to the cell with the minimum unit transportation cost.

Step 2: Select the row or column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row or column and satisfies the rim conditions. If there is a tie in the values of penalties, it can be broken by selecting the cell where the maximum allocation can be made.

Step 3: Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

Step 4: Repeat Steps 1 to 3 until the available supply at various sources and demand at various destinations is satisfied.

Example 9.4 Use Vagel's Approximation Method (VAM) to find the initial basic feasible solution to the transportation problem using the data of Example 9.1.

Solution The differences (penalty costs) for each row and column have been calculated as shown in Table 9.5. In the first round, the maximum penalty, 22 occurs in column D_2. Thus the cell (S_3, D_2) having the least transportation cost is chosen for allocation. The maximum possible allocation in this cell is 8 units and it satisfies demand in column D_2. Adjust the supply of S_3 from 18 to 10 (18 – 8 = 10).

Table 9.5 Initial Solution Using VAM

	D_1	D_2	D_3	D_4	*Supply*	*Row differences*
S_1	19 (5)	30	50	10 (2)	7	9 9 40 40
S_2	70	30	40 (7)	60 (2)	9	10 20 20 20
S_3	40	8 (8)	70	20 (10)	18	12 20 50 –
Demand	5	8	7	14	34	
Column differences	21 21 – –	22 – – –	10 10 10 10	10 10 10 50		

The new row and column penalties are calculated except column D_2 because D_2's demand has been satisfied. In the second round, the largest penalty, 21 appears at column D_1. Thus the cell (S_1, D_1) having the least transportation cost is chosen for allocating 5 units as shown in Table 9.5. After adjusting the supply and demand in the table, we move to the third round of penalty calculations.

In the third round, the maximum penalty 50 appears at row S_3. The maximum possible allocation of 10 units is made in cell (S_3, D_4) that has the least transportation cost of 20 as shown in Table 9.5.

The process is continued with new allocations till a complete solution is obtained. The initial solution using VAM is shown in Table 9.5. The total transportation cost associated with this method is:

$$\text{Total cost} = 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 = \text{Rs } 779$$

Example 9.5 A dairy firm has three plants located in a state. The daily milk production at each plant is as follows:

Plant 1 : 6 million litres, Plant 2 : 1 million litres, and Plant 3 : 10 million litres

Each day, the firm must fulfil the needs of its four distribution centres. The minimum requirement of each centre is as follows:

Distribution centre 1 : 7 million litres, Distribution centre 2 : 5 million litres,
Distribution centre 3 : 3 million litres, and Distribution centre 4 : 2 million litres

Cost (in hundreds of rupees) of shipping one million litre from each plant to each distribution centre is given in the following table:

		Distribution Centre			
		D_1	D_2	D_3	D_4
	P_1	2	3	11	7
Plant	P_2	1	0	6	1
	P_3	5	8	15	9

Find the initial basic feasible solution for given problem by using following methods:

(a) North-west corner rule
(b) Least cost method
(c) Vogel's approximation method

Solution (a) ***North-West Corner Rule***

		Distribution Centre				
		D_1	D_2	D_3	D_4	*Supply*
	P_1	2 (6)	3	11	7	$6 = a_1$
Plant	P_2	1 (1)	0	6	1	$1 = a_2$
	P_3	5	8 (5)	15 (3)	9 (2)	$10 = a_3$
Demand		$7 = b_1$	$5 = b_2$	$3 = b_3$	$2 = b_4$	

Table 9.6 Initial Solution by NWCR

(i) Comparing a_1 and b_1, since $a_1 < b_1$; allocate $x_{11} = 6$. This exhausts the supply at P_1 and leaves 1 unit as unsatisfied demand at D_1.

(ii) Move to cell (P_2, D_1). Compare a_2 and b_1 (i.e. 1 and 1). Since $a_2 = b_1$, allocate $x_{21} = 1$.

(iii) Move to cell (P_3, D_2). Since supply at P_3, is equal to the demand at D_2, D_3 and D_4, therefore, allocate $x_{32} = 5$, $x_{33} = 3$ and $x_{34} = 2$.

It may be noted that the number of allocated cells (also called *basic cells*) are 5 which is one less than the required number $m + n - 1$ $(3 + 4 - 1 = 6)$. Thus, this solution is the degenerate solution. The transportation cost associated with this solution is:

Total cost = Rs $(2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,600

(b) ***Least Cost Method***

Table 9.7 Initial Solution by LCM

		Distribution Centre				
		D_1	D_2	D_3	D_4	*Supply*
	P_1	2 (6)	3	11	7	6
Plant	P_2	1	0 (1)	6	1	1
	P_3	5 (1)	8 (4)	15 (3)	9 (2)	10
Demand		7	5	3	2	

(i) The lowest unit cost in Table 9.7 is 0 in cell (P_2, D_2), therefore the maximum possible allocation that can be made is 1 unit. Since this allocation exhausts the supply at plant P_2, therefore row 2 is crossed off.

(ii) The next lowest unit cost is 2 in cell (P_1, D_1). The maximum possible allocation that can be made is 6 units. This exhausts the supply at plant P_1, therefore, row P_1 is crossed off.

(iii) Since the total supply at plant P_3 is now equal to the unsatisfied demand at all the four distribution centres, therefore, the maximum possible allocations satisfying the supply and demand conditions, are made in cells (P_3, D_1), (P_3, D_2), (P_3, D_3) and (P_3, D_4).

The number of allocated cells in this case are six, which is equal to the required number $m + n - 1$ ($3 + 4 - 1 = 6$). Thus, this solution is non-degenerate. The transportation cost associated with this solution is

Total cost = Rs $(2 \times 6 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,200

(c) ***Vogel's Approximation Method:*** First calculating penalties as per rules and then allocations are made in accordance of penalties as shown in Table 9.8.

Table 9.8 Initial Solution by VAM

		D_1	D_2	D_3	D_4	*Supply*	*Row penalty*		
	P_1	2 (1)	3 (5)	11	7	6	1	1	5
Plant	P_2	1	0	6	1 (1)	1	0	–	–
	P_3	5 (6)	8	15 (3)	9 (1)	10	3	3	4
Demand		7	5	3	2				
Column penalty		1	3	5	6				
		3	5	4	2				
		3	–	4	2				
		Distribution Centre							

The number of allocated cells in Table 9.8 are six, which is equal to the required number $m + n - 1$ ($3 + 4 - 1 = 6$), therefore, this solution is non-degenerate. The transportation cost associated with this solution is:

Total cost = Rs $(2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1) \times 100$ = Rs 10,200

Remark: Total transportation cost found by VAM is lower than the costs of transportation determined by the NWCR and LCM methods. Therefore, it is of advantage to use this method in order to reduce computational time required to obtain optimum solution.

CONCEPTUAL QUESTIONS A

1. Show that all the bases for a transportation problem are triangular.
2. With reference to a transportation problem define the following terms:
   (i) Feasible solution (ii) Basic feasible solution
   (iii) Optimal solution
   (iv) Non-degenerate basic feasible solution
3. Given a mathematical formulation of the transportation problem and the simplex methods, what are the differences in the nature of problems that can be solved by using these methods?
4. Prove that there are only $m + n - 1$ independent equations in a transportation problem, m and n being the number of origins and destination, and only one equation can be dropped as being redundant. (For proof see Appendix).
5. Describe the transportation problem with its general mathematical formulation.
6. Show that a transportation problem is a special type of LP problem. In what areas of management can the transportation model be effectively used? Discuss.
7. What are the characteristics of transportation problem of linear programming?
8. What is meant by the triangular form of a system of linear equations? When does a system of linear equations have a triangular basis? (See Appendix for proof.)
9. What is meant by non-degenerate basic feasible solution of a transportation problem?
10. Explain in brief three, methods of initial feasible solution for transportation problem.
11. Explain the various steps involved in solving transportation problem using (i) Least cost method, and (ii) Vogel's approximation method.
12. Explain the (i) North-West Corner method, (ii) Least-Cost method, and (iii) Vogel's Approximation method, for obtaining an initial basic feasible solution of a transportation problem.
13. State the transportation problem. Describe clearly the steps involved in solving it.
14. Is the transportation model an example of decision-making under certainty or under uncertainty? Why?
15. Why does Vogel's approximation method provide a good initial feasible solution? Can the North-West Corner method ever be able to provide an initial solution with a cost as low as this?

SELF PRACTICE PROBLEMS A

1. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCR, (b) LCM and (c) VAM.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	S_1	21	16	15	3	11
	S_2	17	18	14	23	13
	S_3	32	27	18	41	19
	Demand	6	6	8	23	

2. Determine an initial basic feasible solution to the following transportation problem by using (a) the least cost method, and (b) Vogel's approximation method.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	S_1	1	2	1	4	30
	S_2	3	3	2	1	30
	S_3	4	2	5	9	40
	Demand	20	40	30	10	

3. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCM, (b) LCM, and (c) VAM.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	*A*	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
	Demand	200	225	275	250	

4. Determine an initial basic feasible solution to the following transportation problem by using the North-West corner rule, where O_i and D_j represent ith origin and jth destination, respectively.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	O_1	6	4	1	5	14
	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
	Demand	6	10	15	4	

HINTS AND ANSWERS

1. $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 12$; Total cost = 686.
2. (a) and (b): $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{33} = 20$, $x_{24} = 10$, $x_{32} = 20$; Total cost = 180.
3. (a) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{33} = 150$, $x_{34} = 250$; Total cost = 12,200.
   (b) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{33} = 150$, $x_{34} = 250$; Total cost = 12,200.
   (c) $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$; Total cost = 12,075.
4. $x_{11} = 6$; $x_{12} = 8$; $x_{22} = 2$; $x_{23} = 14$; $x_{33} = 1$; $x_{34} = 4$; Total cost = Rs 128.

9.5 TEST FOR OPTIMALITY

Once an initial solution is obtained, the next step is to check its optimality in terms of feasibility of the solution and total minimum transportation cost.

The test of optimality begins by calculating an opportunity cost associated with each unoccupied cell (represents unused route) in the transportation table. An unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes (allocations). *This value indicates the per unit cost reduction that can be achieved by making appropriate allocation in the unoccupied cell.* This cell is also known as an *incoming cell* (*or variable*). The outgoing cell (or variable) from the current solution is the occupied cell (basic variable) where allocation will become zero as allocation is made in the unoccupied cell with the largest negative opportunity cost. Such an exchange reduces the total transportation cost. The process is continued until there is no negative opportunity cost. That is, the current solution is an optimal solution.

The negative opportunity cost indicates the per unit cost reduction that can be achieved by raising the shipment allocation in the unoccupied cell from its present level of zero.

The *Modified-distribution* (MODI) method (also called *u-v method* or *method of multipliers*) is used to calculate opportunity cost associated with each unoccupied cell and then improving the current solution leading to an optimal solution. The steps of MODI method based on the concept of duality are summarized in section 9.5.3.

9.5.1 Dual of Transportation Model

For a given basic feasible solution if we associate numbers (also called *dual variables* or *multipliers*) u_i and v_j with row i (i = 1, 2, . . ., m) and column j (j = 1, 2, . . ., n) of the transportation table, respectively, then u_i and v_j must satisfy the equation

$$u_i + v_j = c_{ij}\,, \quad \text{for each occupied cell } (i, j)$$

These equations yield $m + n - 1$ equations in $m + n$ unknown dual variables. The values of these variables can be determined by arbitrarily assigning a zero value to any one of these variables. The value of the remaining $m + n - 2$ variables can then be obtained algebraically by using the above equation for the occupied cells. The opportunity cost of each unoccupied cell (called *non-basic variable* or *unused route*) is calculated by using following equation that involves u_i and v_j values.:

$$d_{rs} = c_{rs} - (u_r + v_s)\,, \quad \text{for each unoccupied cell } (r, s)$$

This equation also indicates the per unit reduction in the total transportation cost for the route (r, s). To prove these two results, consider the general transportation model:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i\,, \quad i = 1, 2, \ldots, m \quad \text{(Supply)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,, \quad j = 1, 2, \ldots, n \quad \text{(Demand)}$$

and $\quad x_{ij} \geq 0 \quad$ for all i and j

Modi method helps in comparing the relative advantage of alternative allocations for all the unoccupied cells simultaneously.

Since all of the constraints are equalities, write each equality constraint equivalent to two inequalities as follows:

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &\geq a_i\,, \quad i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} -(x_{ij}) &\geq -a_i \end{aligned}\right\} \text{(Supply constraints)}$$

$$\left.\begin{aligned} \sum_{i=1}^{m} x_{ij} &\geq b_j\,, \quad j = 1, 2, \ldots, m \\ \sum_{i=1}^{m} -(x_{ij}) &\geq -b_j \end{aligned}\right\} \text{(Demand constraints)}$$

Let u_i^+ and u_i^- be the dual variables, one for each supply constraint i. Similarly v_j^+, v_j^- be the dual variables one for each demand constraint j. Then, the dual of the transportation model can be written as:

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (u_i^+ - u_i^-)\, a_i + \sum_{j=1}^{n} (v_j^+ - v_j^-)\, b_j$$

subject to the constraints

$$(u_i^+ - u_i^-) + (v_j^+ - v_j^-) \le c_{ij}$$

and $\quad u_i^+, u_i^-, v_j^+, v_j^- \ge 0,\quad$ for all i and j.

The variables u_i^+ and u_i^- that appear in the objective function, may take positive, negative or zero values. Thus, either of these will appear in the optimal basic feasible solution because one is the negative of the other. The same argument may be given for v_j^+ and v_j^-. Thus, let

$$u_i = u_i^+ - u_i^-,\quad i = 1, 2, \ldots, m$$
$$v_j = v_j^+ - v_j^-,\ j = 1, 2, \ldots, n`$$

The values of u_i and v_j will then be unrestricted in sign. Hence, the dual of the transportation model can now be written as

$$\text{Maximize } Z^* = \sum_{i=1}^{m} u_i\, a_i + \sum_{j=1}^{n} v_j\, b_j$$

subject to the constraints

$$u_i + v_j \le c_{ij}$$

and $\quad u_i, v_j$ unrestricted in sign for all i and j.

The relationship $(c_{ij} - u_i - v_j)\, x_{ij} = 0$ is known as *complementary slackness* for a transportation problem and indicates that

(a) if $x_{ij} > 0$ and solution is feasible, then $c_{ij} - u_i - v_j = 0$ or $c_{ij} = u_i + v_j$, for each occupied cell,
(b) if $x_{ij} = 0$ and $c_{ij} > u_i + v_j$, then it is not desirable to have $x_{ij} > 0$ in the solution mix because it would cost more to transport on a route (i, j),
(c) if $c_{ij} \le u_i + v_j$ for some $x_{ij} = 0$, then x_{ij} can be brought into the solution mix.

9.5.2 Economic Interpretation of u_i's and v_j's

The u_i values measures the comparative advantage of additional unit of supply or shadow price (or value) of available supply at centre i. This may also be termed as *location rent.* Similarly, the v_j values measures the comparative advantage of an additional unit of commodity demanded at demand centre j. This may also be termed as *market price.*

The **dual variables** u_is and v_js represent the shadow price (value of the commodity) for the supply centres and demand centres, respectively.

Illustration The concept of duality in transportation problem is applied on Example 9.1 in the following manner:

Reproducing transportation data of Example 9.1 for ready reference in Table 9.9. In Table 9.9, there are $m = 3$ rows and $n = 4$ columns. Let u_1, u_2 and u_3 be dual variables corresponding to each of the supply constraint in that order. Similarly, v_1, v_2, v_3 and v_4 be dual variables corresponding to each of demand constraint in that order. The dual problem then becomes

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19	30	50	10	7	u_1
S_2	70	30	40	60	9	u_2
S_3	40	8	70	20	18	u_3
Demand	5	8	7	14	34	
v_j	v_1	v_2	v_3	v_4		

Table 9.9

Maximize $Z = (7u_1 + 9u_2 + 18u_3) + (5v_1 + 8v_2 + 7v_3 + 14v_4)$

subject to the constraints

(i) $u_1 + v_1 \le 19$, (ii) $u_1 + v_2 \le 30$, (iii) $u_1 + v_3 \le 50$, (iv) $u_1 + v_4 \le 10$,
(v) $u_2 + v_1 \le 70$, (vi) $u_2 + v_2 \le 30$, (vii) $u_2 + v_3 \le 40$, (viii) $u_2 + v_4 \le 60$,
(ix) $u_3 + v_1 \le 40$, (x) $u_3 + v_2 \le 8$, (xi) $u_3 + v_3 \le 70$, (xii) $u_3 + v_4 \le 20$,

and $\quad u_i, v_j$ unrestricted in sign for all i and j.

Interpretation Consider the dual constraint $u_1 + v_1 \leq 19$ or $v_1 \leq 19 - u_1$. This represents the delivered market value of the commodity at destination D_1 which should be less than or equal to the unit cost of transportation from S_1 to D_1 minus the per unit value of commodity at D_1. A similar interpretation can also be given for other constraints.

The optimal value of dual variables can be obtained either by simplex method or by reading values of these variables from the optimal solution of transportation problem. It may be noted that the total transportation cost at optimal solution would be the same as obtained by putting values of u_i's and v_j's from optimal solution of transportation problem in the dual objective function:

$$\text{Maximize } Z = \sum_{i=1}^{3} a_i\, u_i + \sum_{j=1}^{4} b_j\, v_j$$

9.5.3 Steps of MODI Method (Transportation Algorithm)

The steps to evaluate unoccupied cells are as follows:

Step 1: For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate u_i and v_j for rows and columns. The initial solution can be obtained by any of the three methods discussed earlier.

To start with, any one of u_is or v_js is assigned the value zero. It is better to assign zero to a particular u_i or v_j where there are maximum number of allocations in a row or column respectively, as this will reduce the considerably arithmetic work. The value of u_is and v_js for other rows and columns is calculated by using the relationship.

$$c_{ij} = u_i + v_j, \quad \text{for all occupied cells } (i, j).$$

Step 2: For unoccupied cells, calculate the opportunity cost by using the relationship

$$d_{ij} = c_{ij} - (u_i + v_j), \quad \text{for all } i \text{ and } j.$$

Step 3: Examine sign of each d_{ij}

(i) If $d_{ij} > 0$, then the current basic feasible solution is optimal.

(ii) If $d_{ij} = 0$, then the current basic feasible solution will remain unaffected but an alternative solution exists.

(iii) If one or more $d_{ij} < 0$, then an improved solution can be obtained by entering an unoccupied cell (i, j) into the solution mix (basis). An unoccupied cell having the largest negative value of d_{ij} is chosen for entering into the solution mix (new transportation schedule).

Changing the shipping route involves adding to cells on the closed path with plus signs and subtracting from cells with negative signs.

Step 4: Construct a closed-path (or loop) for the unoccupied cell with largest negative value of d_{ij}. Start the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner with a minus sign (–) and continue down the column (or row) to an occupied cell. Then mark the corner with plus sign (+) and minus sign (–) alternatively. Close the path back to the selected unoccupied cell.

Step 5: Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell, add it to occupied cells marked with plus signs, and subtract it from the occupied cells marked with minus signs.

Step 6: Obtain a new improved solution by allocating units to the unoccupied cell according to Step 5 and calculate the new total transportation cost.

Step 7: Test optimality of the revised solution. The procedure terminates when all $d_{ij} \geq 0$ for unoccupied cells.

Remarks 1. The closed-loop (path) starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except an end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell.

It is immaterial whether the loop is traced in a clockwise or anti-clockwise direction and whether it starts up, down, right or left (but never diagonally). However, for a given solution only one loop can be constructed for each unoccupied cell.

2. There can only be one plus (+) sign and only one minus (–) sign in any given row or column.
3. The closed path indicates changes involved in reallocating the shipments.

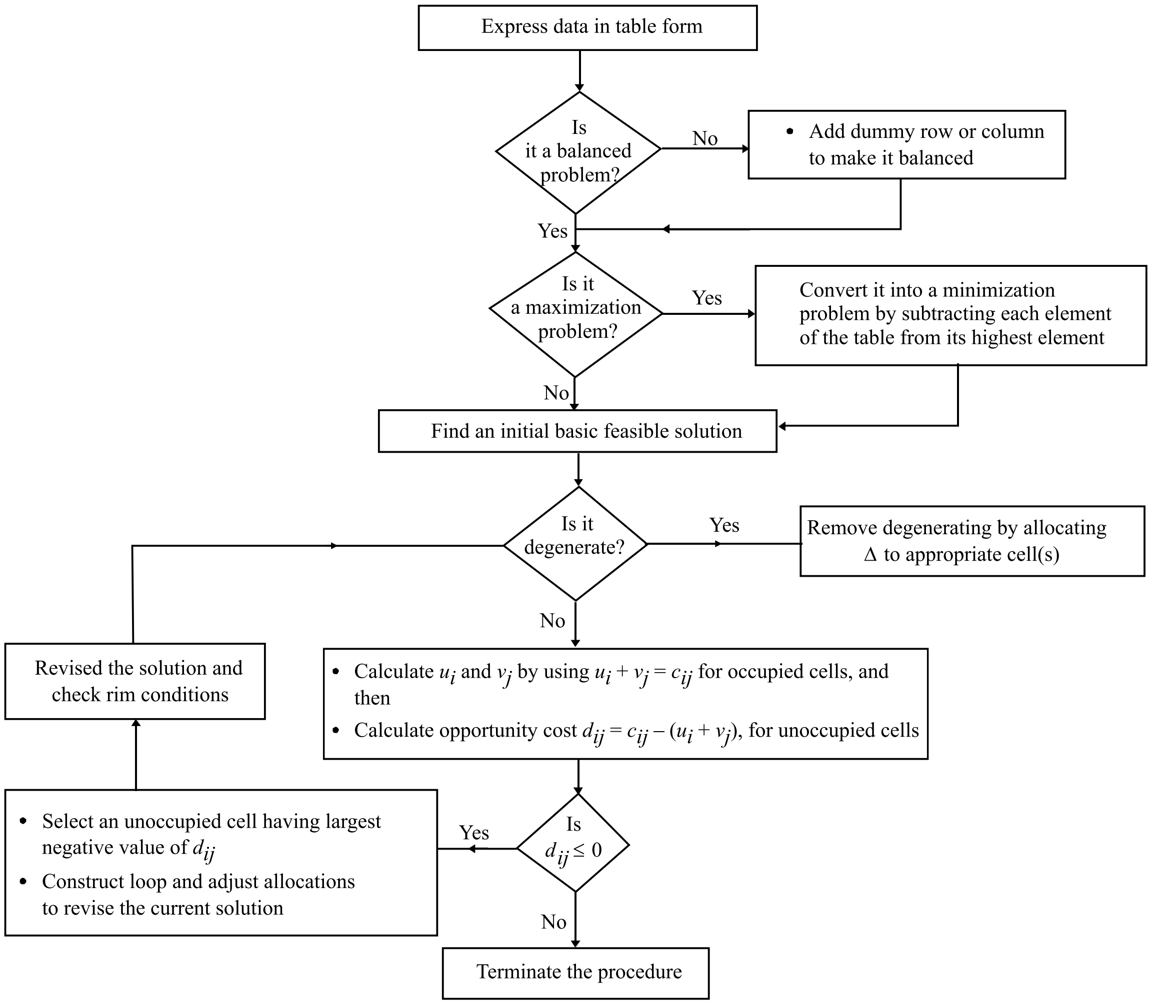


Fig. 9.1
Flow Chart of MODI Method

The steps of MODI method for solving a transportation problem are summarized in the flow chart shown in Fig. 9.1.

9.5.4 Close-Loop in Transportation Table and its Properties

An ordered set of at least four cells in a transportation table forms a **loop.**

Any basic feasible solution must contain $m + n - 1$ non-zero allocations provided.

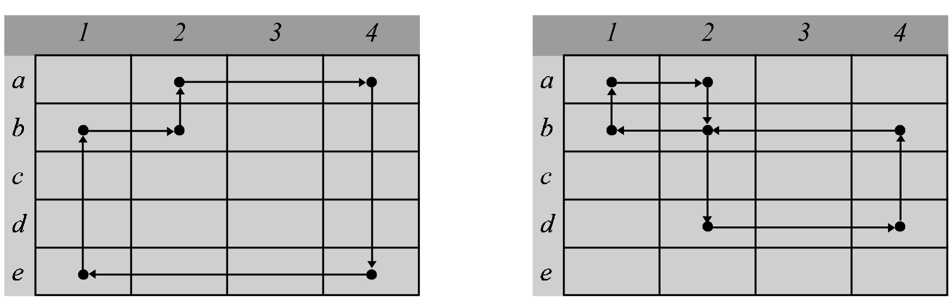
(i) any two adjacent cells of the ordered set lie either in the same row or in the same column, and

(ii) no three or more adjacent cells in the ordered set lie in the same row or column. The first cell of the set must follow the last in the set, i.e. each cell (except the last) must appear only once in the ordered set.

Consider the following two cases represented in Tables 9.10(a) and 9.10(b). In Table 9.10(a), if we join the positive allocations by horizontal and vertical lines, then a closed loop is obtained. The ordered set of cells forming a loop is:

$$L = \{(a, 2), (a, 4), (e, 4), (e, 1), (b, 1), (b, 2), (a, 2)\}$$

The loop in Table 9.10(b) is not allowed because it does not satisfy the conditions in the definition of a loop. That is, the cell $(b, 2)$ appears twice.



(a) (b)

Table 9.10

Remarks 1. Every loop has an even number of cells and has at least four cells.

2. The allocations are said to be in independent position if it is not possible to increase or decrease any individual allocation without changing the positions of these allocations, or if a closed loop cannot be formed through these allocations without violating the rim conditions.
3. Each row and column in the transportation table should have only one plus and minus sign. All cells that have a plus or a minus sign, except the starting unoccupied cell, must be occupied cells.
4. Closed loops may or may not be in the shape of a square.

Example 9.6 Apply MODI method to obtain optimal solution of transportation problem using the data of Example 9.1.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Solution Applying Vogel's approximation method to obtain an initial basic feasible solution. This solution is shown in Table 9.11 [for ready reference see Table 9.5].

1. In Table 9.11, since number of occupied cells are $m + n - 1 = 3 + 4 - 1 = 6$ (as required), therefore this initial solution is non-degenerate. Thus, an optimal solution can be obtained. The total transportation cost associated with this solution is Rs 779.

2. In order to calculate the values of u_is ($i = 1, 2, 3$) and v_js ($j = 1, 2, 3, 4$) for each occupied cell, assigning arbitrarily, $v_4 = 0$ in order to simplify calculations. Given $v_4 = 0$, u_1, u_2 and u_3 can be immediately computed by using the relation $c_{ij} = u_i + v_j$ for occupied cells, as shown in Table 9.11.

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19 (5)	30 +32	50 + 60	10 (2)	7	$u_1 = 10$
S_2	70 + 1	30 (+) – 18	40 (7)	60 (2) (–)	9	$u_2 = 60$
S_3	40 + 11	8 (–) (8)	70 + 70	20 (10) (+)	18	$u_3 = 20$
Demand	5	8	7	14	34	
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

Table 9.11
Initial Solution, VAM

$$c_{34} = u_3 + v_4 \quad \text{or} \quad 20 = u_3 + 0 \quad \text{or} \quad u_3 = 20$$
$$c_{24} = u_2 + v_4 \quad \text{or} \quad 60 = u_2 + 0 \quad \text{or} \quad u_2 = 60$$
$$c_{14} = u_1 + v_4 \quad \text{or} \quad 10 = u_1 + 0 \quad \text{or} \quad u_1 = 10$$

Given u_1, u_2, and u_3, value of v_1, v_2 and v_3 can also be calculated as shown below:

$$c_{11} = u_1 + v_1 \quad \text{or } 19 = 10 + v_1 \text{ or } v_1 = 9$$
$$c_{23} = u_2 + u_3 \quad \text{or } 40 = 60 + v_3 \text{ or } v_3 = -20$$
$$c_{32} = u_3 + v_2 \quad \text{or } 8 = 20 + v_2 \text{ or } v_2 = -12$$

3. The opportunity cost for each of the occupied cell is determined by using the relation $d_{ij} = c_{ij} - (u_i + v_j)$ and is shown below.

$$\begin{aligned}
d_{12} &= c_{12} - (u_1 + v_2) = 30 - (10 - 12) = 32\\
d_{13} &= c_{13} - (u_1 + v_3) = 50 - (10 - 20) = 60\\
d_{21} &= c_{21} - (u_2 + v_1) = 70 - (60 + 9) = 1\\
d_{22} &= c_{22} - (u_2 + v_2) = 30 - (60 - 12) = -18\\
d_{31} &= c_{31} - (u_3 + v_1) = 40 - (20 + 9) = 11\\
d_{33} &= c_{33} - (u_3 + v_3) = 70 - (20 - 20) = 70
\end{aligned}$$

4. According to the optimality criterion for cost minimizing transportation problem, the current solution is not optimal, since the opportunity costs of the unoccupied cells are not all zero or positive. The value of $d_{22} = -18$ in cell (S_2, D_2) is indicating that the total transportation cost can be reduced in the multiple of 18 by shifting an allocation to this cell.

5. A closed-loop (path) is traced along row S_2 to an occupied cell (S_3, D_2). A plus sign is placed in cell (S_2, D_2) and minus sign in cell (S_3, D_2). Now take a right-angle turn and locate an occupied cell in column D_4. An occupied cell (S_3, D_4) exists at row S_3, and a plus sign is placed in this cell.
   Continue this process and complete the closed path. The occupied cell (S_2, D_3) must be bypassed otherwise they will violate the rules of constructing closed path.

6. In order to maintain feasibility, examine the occupied cells with minus sign at the corners of closed loop, and select the one that has the smallest allocation. This determines the maximum number of units that can be shifted along the closed path. The minus signs are in cells (S_3, D_2) and (S_2, D_4). The cell (S_2, D_4) is selected because it has the smaller allocation, i.e. 2. The value of this allocation is then added to cell (S_2, D_2) and (S_3, D_4), which carry plus signs. The same value is subtracted from cells (S_2, D_4) and (S_3, D_2) because they carry minus signs.

7. The revised solution is shown in Table 9.12. The total transportation cost associated with this solution is:

   Total cost = 5 × 19 + 2 × 10 + 2 × 30 + 7 × 40 + 6 × 8 + 12 × 20 = Rs 743

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19 (5)	30 +32	50 +42	10 (2)	7	$u_1 = 0$
S_2	70 + 19	30 (2)	40 (7)	60 +14	9	$u_2 = 32$
S_3	40 + 11	8 (6)	70 + 52	20 (12)	18	$u_3 = 10$
Demand	5	8	7	14	34	
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

Table 9.12
Optimal Solution

8. Test the optimality of the revised solution once again in the same way as discussed in earlier steps. The values of u_is, v_js and d_{ij}s are shown in Table 9.12. Since each of $d_{ij}s$ is positive, therefore, the current basic feasible solution is optimal with a mi]nimum total transportation cost of Rs 743.

Example 9.7 A company has factories at F_1, F_2, and F_3 that supply products to warehouses at W_1, W_2 and W_3. The weekly capacities of the factories are 200, 160 and 90 units, respectively. The weekly warehouse requirements are 180, 120 and 150 units, respectively. The unit shipping costs (in rupees) are as follows:

Warehouse

		W_1	W_2	W_3	*Supply*
Factory	F_1	16	20	12	200
	F_2	14	8	18	160
	F_3	26	24	16	90
	Demand	180	120	150	450

Determine the optimal distribution for this company in order to minimize its total shipping cost.

Solution Initial basic feasible solution obtained by North-West Corner Rule is given in Table 9.13. Since, as required, this initial solution has $m + n - 1 = 3 + 3 - 1 = 5$ allocations, it is a non-degenerate solution. The optimality test can, therefore, be performed. The total transportation cost associated with this solution is:

Table 9.13
Initial Solution

	W_1	W_2	W_3	*Supply*
F_1	16 (180)	20 (20)	12	200
F_2	14	8 (100)	18 (60)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

$$\text{Total cost} = 16 \times 180 + 20 \times 20 + 8 \times 100 + 18 \times 60 + 16 \times 90 = \text{Rs } 6{,}800$$

Determine the values of u_is and v_js as usual, by arbitrarily assigning $u_1 = 0$. Given $u_1 = 0$, the values of others variables obtained by using the equation $c_{ij} = u_i + v_j$ for occupied cells, are shown in Table 9.14.

At each step a non-occupied cell with largest negative opportunity cost is selected to get maximum reduction in total transportation cost.

Table 9.14

	W_1	W_2	W_3	*Supply*	u_i
F_1	16 (180)	20 (–) (20)	12 (+) – 18	200	$u_1 = 0$
F_2	14 +10	8 (+) (100)	18 (60) (–)	160	$u_2 = -12$
F_3	26 +24	24 +18	16 (90)	90	$u_3 = -14$
Demand	180	120	150	450	
v_j	$v_1 = 16$	$v_2 = 20$	$v_3 = 30$		

$$
\begin{aligned}
c_{11} &= u_1 + v_1 &\text{or}\quad 16 &= 0 + v_1 &\text{or}\quad v_1 &= 16\\
c_{12} &= u_1 + v_2 &\text{or}\quad 20 &= 0 + v_2 &\text{or}\quad v_2 &= 20\\
c_{22} &= u_2 + v_2 &\text{or}\quad 8 &= u_2 + 20 &\text{or}\quad u_2 &= -12\\
c_{23} &= u_2 + v_3 &\text{or}\quad 18 &= -12 + v_3 &\text{or}\quad v_3 &= 30\\
c_{33} &= u_3 + v_3 &\text{or}\quad 16 &= u_3 + 30 &\text{or}\quad u_3 &= -14
\end{aligned}
$$

The opportunity cost for each of the unoccupied cells is determined by using the equation, $d_{ij} = c_{ij} - (u_i + v_j)$ as follows:

$$d_{13} = c_{13} - (u_1 + v_3) = 12 - (0 + 30) = -18$$
$$d_{21} = c_{21} - (u_2 + v_1) = 14 - (-12 + 16) = 10$$
$$d_{31} = c_{31} - (u_3 + v_1) = 26 - (-14 + 16) = 24$$
$$d_{32} = c_{32} - (u_3 + v_2) = 24 - (-14 + 20) = 18$$

The value of $d_{13} = -18$ in the cell (F_1, W_3) indicates that the total transportation cost can be reduced in a multiple of 18 by introducing this cell in the new transportation schedule. To see how many units of the commodity could be allocated to this cell (route), form a closed path as shown in Table 9.14.

The largest number of units of the commodity that should be allocated to the cell (F_1, W_3) is 20 units because it does not violate the supply and demand restrictions (minimum allocation among the occupied cells bearing negative sign at the corners of the loop). The new transportation schedule (solution) so obtained is shown in Table 9.15.

	W_1	W_2	W_3	*Supply*
F_1	16 (180)	20	12 (20)	200
F_2	14	8 (120)	18 (40)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

Table 9.15
Revised Schedule

The total transportation cost associated with this solution is

Total cost = 16 × 180 + 12 × 20 + 8 × 120 + 18 × 40 + 16 × 90 = Rs 6,240

To test the optimality of the new solution shown in Table 9.15, again calculate the opportunity cost of each unoccupied cell in the same manner as discussed earlier. The calculations for u_is, v_js and d_{ij}s are shown in Table 9.16.

$$c_{13} = u_1 + v_3 \quad \text{or} \quad 12 = u_1 + 0 \quad \text{or} \quad u_1 = 12$$
$$c_{23} = u_2 + v_3 \quad \text{or} \quad 18 = u_2 + 0 \quad \text{or} \quad u_2 = 18$$
$$c_{33} = u_3 + v_3 \quad \text{or} \quad 16 = u_3 + 0 \quad \text{or} \quad u_3 = 16$$
$$c_{11} = u_1 + v_1 \quad \text{or} \quad 16 = 12 + v_1 \quad \text{or} \quad v_1 = 4$$
$$c_{22} = u_2 + v_2 \quad \text{or} \quad 8 = 18 + v_2 \quad \text{or} \quad v_2 = -10$$

$$d_{12} = c_{12} - (u_1 + v_2) \quad \text{or} \quad 20 - (12 - 10) = 18$$
$$d_{21} = c_{21} - (u_2 + v_1) \quad \text{or} \quad 14 - (18 + 4) = -8$$
$$d_{31} = c_{31} - (u_3 + v_1) \quad \text{or} \quad 26 - (16 + 4) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) \quad \text{or} \quad 24 - (16 - 10) = 18$$

The value of $d_{21} = -8$ in the cell (F_2, W_1) indicates that the total cost of transportation can further be reduced in a multiple of 8 by introducing this cell in the new transportation schedule. The new solution is obtained by introducing 40 units of the commodity in the cell (F_2, W_1), as indicated in Table 9.16. The new solution is shown in Table 9.17.

	W_1	W_2	W_3	*Supply*	u_i
F_1	16 (–) 180	20 18	12 20 (+)	200	$u_1 = 12$
F_2	14 (+) –8	8 120	18 40 (–)	160	$u_2 = 18$
F_3	26 6	24	16 90	90	$u_3 = 16$
Demand	180	120	150		
v_j	$v_1 = 4$	$v_2 = -10$	$v_3 = 0$		

Table 9.16

	W_1	W_2	W_3	*Supply*
F_1	16 140	20	12 60	200
F_2	14 40	8 120	18	160
F_3	26	24	16 90	90
Demand	180	120	150	

Table 9.17
Revised Schedule

The total transportation cost associated with this solution is

Total cost = 16 × 140 + 12 × 60 + 14 × 40 + 8 × 120 + 16 × 90 = Rs 5,920

To test the optimality of the new solution shown in Table 9.17, calculate again the opportunity cost of each unoccupied cell in the same manner as discussed earlier. The calculations are shown in Table 9.18.

	W_1	W_2	W_3	*Supply*	u_i
F_1	16 140	20 +10	12 60	200	$u_1 = 16$
F_2	14 40	8 120	18 +8	160	$u_2 = 14$
F_3	26 +6	24 +10	16 90	90	$u_3 = 20$
Demand	180	120	150		
v_j	$v_1 = 0$	$v_2 = -6$	$v_3 = -4$		

Table 9.18

$$
\begin{aligned}
d_{12} &= c_{12} - (u_1 + v_2) \quad \text{or} \quad 20 - (16 - 6) = 10 \\
d_{23} &= c_{23} - (u_2 + v_3) \quad \text{or} \quad 18 - (14 - 4) = 8 \\
d_{31} &= c_{31} - (u_3 + v_1) \quad \text{or} \quad 26 - (20 + 0) = 6 \\
d_{32} &= c_{32} - (u_3 + v_2) \quad \text{or} \quad 24 - (20 - 6) = 10
\end{aligned}
$$

Since none of the unoccupied cells in Table 9.18 has a negative opportunity cost value, therefore, the total transportation cost cannot be reduced further. Thus, the solution shown in Table 9.18 is the optimal solution, giving the optimal transportation schedule with a total cost of Rs 5,920.

Example 9.8 The following table provides all the necessary information on the availability of supply to each warehouse, the requirement of each market, and the unit transportation cost (in Rs) from each warehouse to each market.

Market

		P	*Q*	*R*	*S*	*Supply*
Warehouse	*A*	6	3	5	4	22
	B	5	9	2	7	15
	C	5	7	8	6	8
	Demand	7	12	17	9	45

The shipping clerk of the shipping agency has worked out the following schedule, based on his own experience: 12 units from A to Q, 1 unit from A to R, 9 units from A to S, 15 units from B to R, 7 units from C to P and 1 unit from C to R.

(a) Check and see if the clerk has the optimal schedule.

(b) Find the optimal schedule and minimum total transport cost.

(c) If the clerk is approached by a carrier of route C to Q, who offers to reduce his rate in the hope of getting some business, by how much should the rate be reduced before the clerk would offer him the business.

Solution (a) The shipping schedule determined by the clerk based on his experience is shown in Table 9.19. The total transportation cost associated with this solution is

$$\text{Total cost} = 3 \times 12 + 5 \times 1 + 4 \times 9 + 2 \times 15 + 5 \times 7 + 8 \times 1 = \text{Rs } 150$$

Since the number of occupied cells (i.e. 6) is equal to the required number of occupied cells (i.e. $m + n - 1$) in a feasible solution, therefore the solution is non-generate feasible solution. Now, to test the optimality of the solution given in Table 9.19, evaluate each unoccupied cell in terms of the opportunity cost associated with it. This is done in the usual manner and is shown in Table 9.20.

	P	*Q*	*R*	*S*	*Supply*
A	6	3 12	5 1	4 9	22
B	5	9	2 15	7	15
C	5 7	7	8 1	6	8
Demand	7	12	17	9	45

Table 9.19
Initial Solution

In Table 9.20, cell (C, S) has a negative opportunity cost (i.e. – 1). Thus, this solution is not the optimal solution and, therefore, the schedule prepared by the shipping clerk is not optimal.

(b) By forming a closed-loop to introduce the cell (C, S) into the new transportation schedule as shown in Table 9.20, we get a new solution that is shown in Table 9.21.

	P	*Q*	*R*	*S*	*Supply*	u_i
A	6 +4	3 12	5 (+) 1	4 9 (–)	22	$u_1 = 0$
B	5	9 + 9	2 15	7 + 6	15	$u_2 = -3$
C	5 7	7 + 1	8 (–) 1	6 (+) – 1	8	$u_3 = 3$
Demand	7	12	17	9	45	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

Table 9.20

While testing the optimality of the improved solution shown in Table 9.21, we found that the opportunity costs in all the unoccupied cells are positive. Thus the current solution is optimal and the optimal schedule is to transport 12 units from A to Q; 2 units from A to R; 8 units from A to S; 15 units from B to R; 7 units from C to P and 1 unit from *C* to *S*. The total minimum transportation cost associated with this solution is

Total cost = 3 × 12 + 5 × 2 + 4 × 8 + 2 × 15 + 5 × 7 + 6 × 1 = Rs 149

	P	*Q*	*R*	*S*	*Supply*	u_i
A	6 +3	3 12	5 2	4 8	22	$u_1 = 0$
B	5 +5	9 +9	2 15	7 +6	15	$u_2 = -3$
C	5 7	7 +2	8 +1	6 1	8	$u_3 = 2$
Demand	7	12	17	9	45	
v_j	$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

Table 9.21
Optimal Solution

(c) The total transportation cost will increase by Rs 2 (opportunity cost) if one unit of commodity is transported from C to Q. This means that the rate of the carrier on the route C to Q should be reduced by Rs 2, i.e. from Rs 7 to Rs 5 so as to get some business of one unit of commodity only.

In case all the 8 units available at C are shipped through the route (C, Q), then the solution presented in Table 9.21 may be read as shown in Table 9.22.

	P	*Q*	*R*	*S*	*Supply*
A	6 7	3 4	5 2	4 9	22
B	5	9	2 15	7	15
C	5	7 8	8	6	8
Demand	7	12	17	9	45

Table 9.22

The total cost of transportation associated with this solution is

Total cost = 6 × 7 + 3 × 4 + 5 × 2 + 4 × 9 + 2 × 15 + 7 × 8 = Rs 186.

Thus, the additional cost of Rs 37 (= 186 – 149) should be reduced from the transportation cost of 8 units from C to Q. Hence transportation cost per unit from C to Q should be at the most 7 – (37/8) = Rs 2.38.

Example 9.10 ABC Limited has three production shops that supply a product to five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The costs of transportation are given below:

Warehouse

		I	II	III	IV	V	Supply
	A	6	4	4	7	5	100
Shop	B	5	6	7	4	8	125
	C	3	4	6	3	4	175
	Demand	60	80	85	105	70	400

The cost of manufacturing the product at different production shops is

Shop	*Variable Cost*	*Fixed Cost*
A	14	7,000
B	16	4,000
C	15	5,000

Find the optimum quantity to be supplied from each shop to different warehouses at the minimum total cost. [*Delhi Univ., MBA, 2007*]

Solution In this case, the fixed cost data is of no use. The transportation cost matrix will include the given transportation cost plus the variable cost, as shown in Table 9.23.

	I	II	III	IV	V	Supply
A	6 + 14 = 20	4 + 14 = 18	4 + 14 = 18	7 + 14 = 21	5 + 14 = 19	100
B	5 + 16 = 21	6 + 16 = 22	7 + 16 = 23	4 + 16 = 20	8 + 16 = 24	125
C	3 + 15 = 18	4 + 15 = 19	6 + 15 = 21	3 + 15 = 18	4 + 15 = 19	175
Demand	60	80	85	105	70	400

Table 9.23

The optimal solution obtained by applying MODI method is shown in Table 9.24.

	I	II	III	IV	V	Supply	u_i
A	20 +3	18 (15)	18 (85)	21 +5	19 +1	100	u_1 = 18
B	21 0	22 (20)	23 +1	20 (105)	24 +2	125	u_2 = 22
C	18 (60)	19 (45)	21 +2	18 +1	19 (70)	175	u_3 = 19
Demand	60	80	85	105	70		
v_j	v_1 = –1	v_2 = 0	v_3 = 0	v_4 = –2	v_5 = 0		

Table 9.24
Optimal Solution

The transportation cost associated with the solution is

$$\text{Total cost} = 18 \times 15 + 18 \times 85 + 22 \times 20 + 20 \times 105 + 18 \times 60 + 19 \times 45 + 19 \times 70 = \text{Rs } 7{,}605$$

CONCEPTUAL QUESTIONS B

1. Describe the computational procedure of the optimality test in a transportation problem.
2. Indicate how you will test for optimality of initial feasible solution of a transportation problem.
3. Let S_i and D_j ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the supply and demand available, respectively, for a commodity at m godowns and n markets. Let c_{ij} be the cost of transporting one unit of the commodity from godown i to market j. Assuming that

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

Symbolically state the transportation problem. Establish that the optimal solution is not altered when the c_{ij}'s are replaced by c_{ij}^*s, where $c_{ij}^* = c_{ij} + u_i + v_j$, u_i ($i = 1, 2, \ldots, m$) and v_j ($j = 1, 2, \ldots, n$) are arbitrary real numbers.

SELF PRACTICE PROBLEMS B

1. Consider four bases of operation B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table:

Target (T_j)

Base (B_i)		T_1	T_2	T_3
	B_1	8	6	5
	B_2	6	6	6
	B_3	10	8	4
	B_4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement of sorties spread over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets. Explain each step in the process.

2. A company has four warehouses, a, b, c and d. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following amounts in stock:

Warehouse : a b c d
No. of units : 15 16 12 13

and the customers' requirements are

Customer : A B C
No. of units : 18 20 18

The table below shows the costs of transporting one unit from warehouse to the customer.

Warehouse

Customer		a	b	c	d
	A	8	9	6	3
	B	6	11	5	10
	C	3	8	7	9

Find the optimal transportation routes.

3. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units, respectively during this month. The firm had made a commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost of transporting from the three plants to the five customers is given below:

Customers

Plants		A	B	C	D	E
	I	4	1	3	4	4
	II	2	3	2	2	3
	III	3	5	2	4	4

4. The following table gives the cost of transporting material from supply points A, B, C and D to demand points E, F, G, H and I.

To

From		E	F	G	H	I
	A	8	10	12	17	15
	B	15	13	18	11	9
	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows:
A to E 90; A to F 10; B to F 150; C to F 10; C to G 50; C to I 120; D to H 210; D to I 70.

(a) Check if this allocation is optimum. If not, find an optimum schedule.
(b) If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum schedule?

5. A whole selling company has three warehouses from which the supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are:

Warehouse Number	*Supply (units)*	*Customer Number*	*Demand (units)*
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Total supply at the warehouses is equal to total demand from the customers. The table below gives the transportation costs, per unit, shipped from each warehouse to each customer.

Customer

		C_1	C_2	C_3	C_4
	W_1	3	6	8	5
Warehouse	W_2	6	1	2	5
	W_3	7	8	3	9

Determine what supplies should be dispatched from each of the warehouses to each customer so as to minimize the overall transportation cost.

6. A manufacturer has distribution centres at Agra, Allahabad and Kolkata. These centres have availability of 40, 20 and 40 units of his product, respectively. His retail outlets at A, B, C, D and E require 25, 10, 20, 30 and 15 units of the products, respectively. The transport cost (in rupees) per unit between each centre outlet is given below:

Distribution Centre	*Retail Outlets*				
	A	*B*	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Kolkata	40	60	95	35	30

Determine the optimal distribution so as to minimize the cost of transportation. [*Delhi Univ., MBA, 2002*]

7. A manufacturer has distribution centres located at Agra, Allahabad and Kolkata. These centres have available 40, 20 and 40 units of his product. His retail outlets at A, B, C, D and E requires 25, 10, 20, 30 and 15 units of the product, respectively. The shipping cost per unit (in rupees) between each centre and outlet is given in the following table.

Distribution Centre	*Retail Outlets*				
	A	*B*	*C*	*D*	*E*
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Kolkata	40	60	95	35	30

Determine the optimal shipping cost.

[*Delhi Univ. MBA, 2003*]

8. A steel company is concerned with the problem of distributing imported ore from three ports to four steel mills. The supplies of ore arriving at the ports are:

Port	*Tonnes per week*
a	20,000
b	38,000
c	16,000

The demand at the steel mills is as follows:

Steel mills	A	B	C	D
Tonnes per week	10,000	18,000	22,000	24,000

The transportation cost is Re 0.05 per tonne per km. The distance between the ports and the steel mills is as given below:

	A	*B*	*C*	*D*
a	50	60	100	50
b	80	40	70	50
c	90	70	30	50

Calculate a transportation plan that will minimize the distribution cost for the steel company. State the cost of this distribution plan.

9. A company has three factories at Amethi, Baghpat and Gwalior that have a production capacity of 5,000, 6,000, and 2,500 tonnes, respectively. Four distribution centres at Allahabad, Bombay, Kolkata and Delhi, require 6,000 tonnes, 4,000 tonnes, 2,000 tonnes and 1,500 tonnes, respectively, of the product. The transportation costs per tonne from different factories to different centres are given below:

Factories	*Distribution Centres*			
	Allahabad	*Bombay*	*Kolkata*	*Delhi*
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gwalior	2	5	4	5

Suggest an optimum transportation schedule and find the minimum cost of transportation.

10. A company has three plants and four warehouses. The supply and demand in units and the corresponding transportation costs are given. The table below has been taken from the solution procedure of a transportation problem:

Warehouses

		I	*II*	*III*	*IV*	*Supply*
	A	5	10	4 (10)	5	10
Plants	*B*	6 (20)	8	7	2 (5)	25
	C	4 (5)	2 (10)	5 (5)	7	20
	Demand	25	10	15	5	55

Answer the following questions, giving brief reasons for the same:

(a) Is this solution feasible?
(b) Is this solution degenerate?
(c) Is this solution optimum?
(d) Does this problem have more than one optimum solution? If so, show all of them.
(e) If the cost for the route B-III is reduced from Rs 7 to Rs 6 per unit, what will be the optimum solution?

11. A baking firm can produce a special bread in either of its two plants, the details of which are as follows:

Plant	*Production Capacity Loaves*	*Production Cost Rs/Loaf*
A	2,500	2.30
B	2,100	2.50

Four restaurant chains are willing to purchase this bread; their demand and the prices they are willing to pay are as follows:

Chain	*Maximum Demand Loaves*	*Price Offered Rs/Loaf*
1	1,800	3.90
2	2,300	3.70
3	550	4.00
4	1,750	3.60

The cost (in paise) of shipping a loaf from a plant to a restaurant chain is:

	Chain 1	*Chain 2*	*Chain 3*	*Chain 4*
Plant A	6	8	11	9
Plant B	12	6	8	5

Determine a delivery schedule for the baking firm that will maximize its profit from this bread.

Write the dual of this transportation problem and use it for checking the optimal solution to the given problem.

[*Delhi Univ., MBA, 2005*]

HINTS AND ANSWERS

1. The initial solution obtained by VAM is also the optimal solution: x_{11} = 50, x_{12} = 100, x_{21} = 150, x_{33} = 150, x_{42} = 100, x_{43} = 50. Maximum total tonnage = 3,300.
2. (A, b) = 5, (A, d) = 13, (B, b) = 8, (B, c) = 12, (C, a) = 15 and (C, b) = 3, Total cost = Rs 301.
3. (I, B) = 45, (I, F) = 15, (II, A) = 17, (II, D) = 18, (III, A) = 5, (III, C) = 20 and (III, E) = 15, Total cost = Rs 290.
4. (a) (A, F) = 100, (B, F) = 70, (B, I) = 80, (C, E) = 90, (C, G) = 50, (C, I) = 40, (D, H) = 210, (D, I) = 70, Total cost = Rs 6,600.
   (b) When transportation cost from A to G is reduced to 10, the optimal schedule given in (a) remains unchanged.
5. x_{11} = 15, x_{14} = 5, x_{22} = 19, x_{24} = 9, x_{33} = 13, x_{34} = 4, Total cost = Rs 209.
6. x_{11} = 5, x_{12} = 10, x_{13} = 20, x_{14} = 5, x_{21} = 20, x_{34} = 25, x_{35} = 15, Total cost = Rs 3,650.
7. x_{12} = 2, x_{13} = 4, x_{15} = 2, x_{21} = 4, x_{31} = 1, x_{34} = 6, x_{35} = 1, Total cost = Rs 720.
8. x_{11} = 10,000, x_{14} = 10,000, x_{22} = 18,000, x_{23} = 6,000, x_{24} = 14,000, x_{33} = 16,000, Total cost = Rs 1,66,000.
9. x_{11} = 3,500, x_{12} = 1,500, x_{22} = 2,500, x_{23} = 2,000, x_{24} = 1,500, x_{31} = 2,500, Total cost = Rs 39,500.
10. (a) The solution is feasible because it satisfies supply and demand constraints.
    (b) The solution is non-degenerate because the number of occupied cells are equal to the required number of $(m + n - 1)$ of occupied cells in the solution.
    (c) Solution is optimal.
    (d) The problem has alternative optimal solution because opportunity cost for cell (B, III) is zero; x_{11} = 10, x_{21} = 15, x_{23} = 5, x_{24} = 5, x_{31} = 10 and x_{32} = 10, Total cost = Rs 235.
    (e) If cell (B, III) has a unit cost of 6, the opportunity cost in this cell will be – 1 and, therefore, the given solution will not be optimal. The new solution obtained will be: x_{13} = 10, x_{21} = 15, x_{23} = 5, x_{24} = 5, x_{31} = 10, x_{32} = 10, Total cost = Rs 230.

9.6 VARIATIONS IN TRANSPORTATION PROBLEM

Some of the variations that often arise while solving a transportation problem are as follows.

9.6.1 Unbalanced Supply and Demand

For a feasible solution to exist, it is necessary that the total supply must equal the total demand. That is,

$$\text{Total supply} = \text{Total demand}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

A **dummy source** or destination is added to balance transportation problem where demand is not equal to supply.

But a situation may arise when the total available supply is not equal to the total demand [For proof see appendix]. The following two cases may arise:

(a) If the total supply exceeds the total demand, then an additional column (called a *dummy demand centre*) can be added to the transportation table in order to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these represent product items that are neither made nor sent.

(b) If the total demand exceeds the total supply, a dummy row (called a *dummy supply centre*) can be added to the transportation table to account for the excess demand quality. The unit transportation cost in such a case also, for the cells in the dummy row is set equal to zero.

Example 9.10 A company has received a contract to supply gravel to three new construction projects located in towns A, B and C. The construction engineers have estimated that the required amounts of gravel which will be needed at these construction projects are: