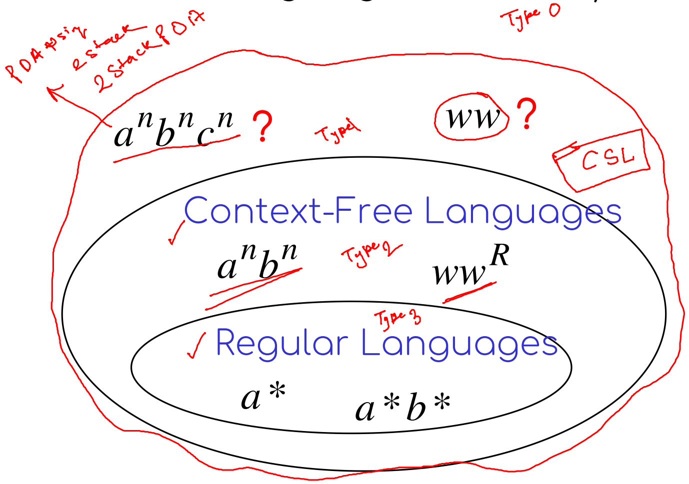
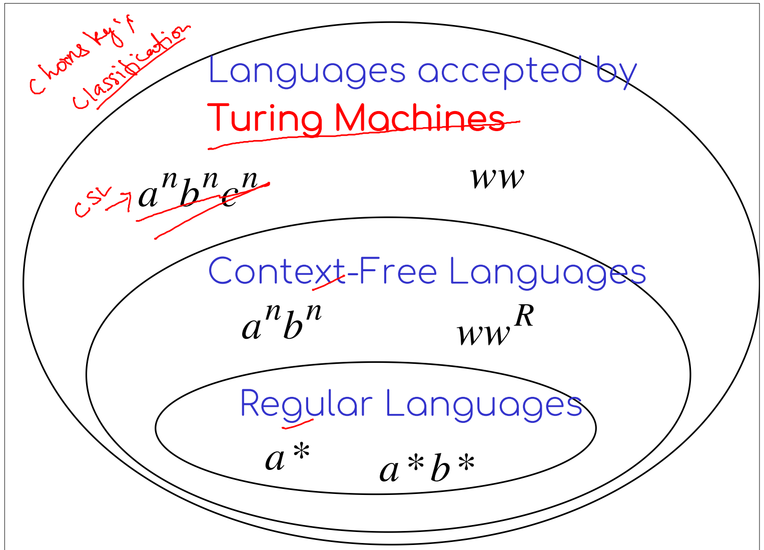
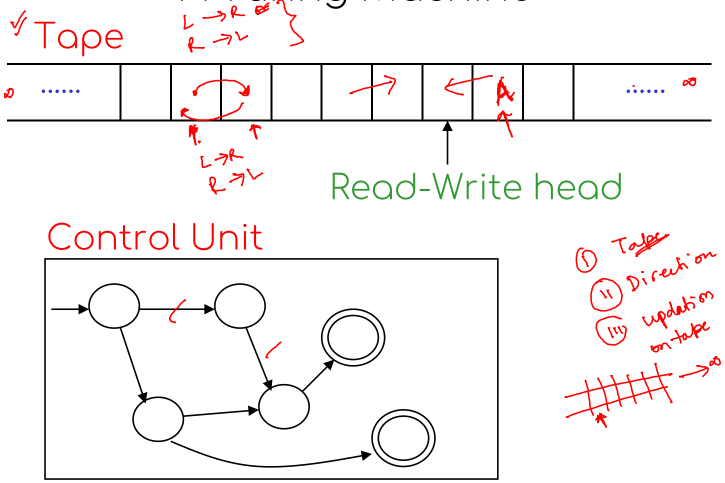
Turing Machines

The Language Hierarchy



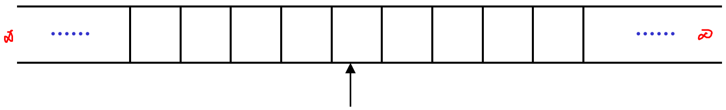


A Turing Machine



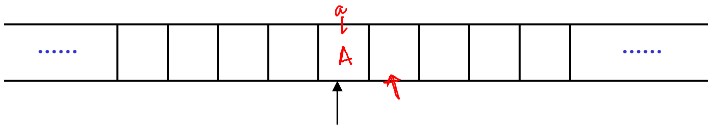
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



Read-Write head

The head at each time step:

a, A, R

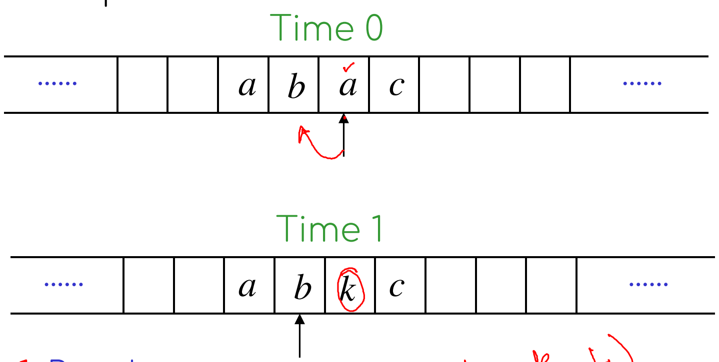
No update

S

Saturation

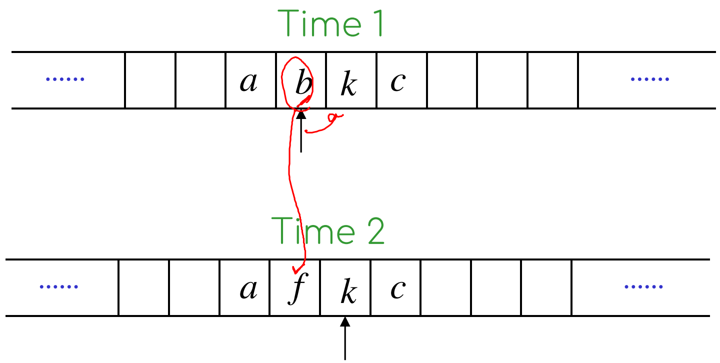
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:



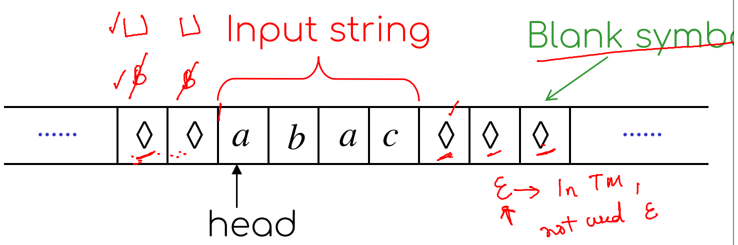
1. Reads a
2. Writes k
3. Moves Left

(a, k, L)



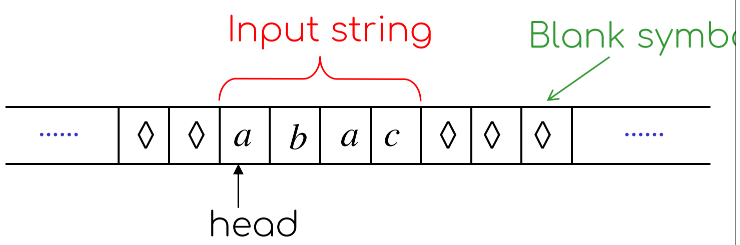
1. Reads b
2. Writes f
3. Moves Right

The Input String



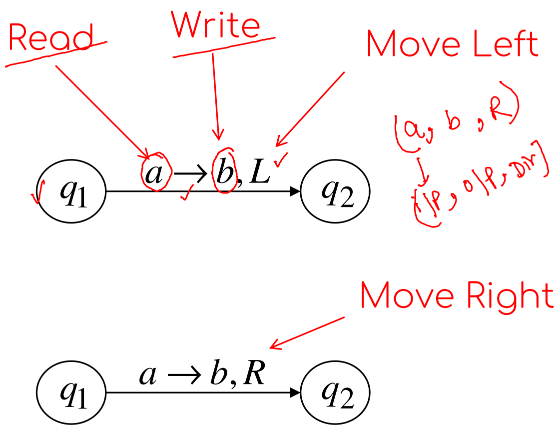
Head starts at the leftmost position of the input string

By default L → R

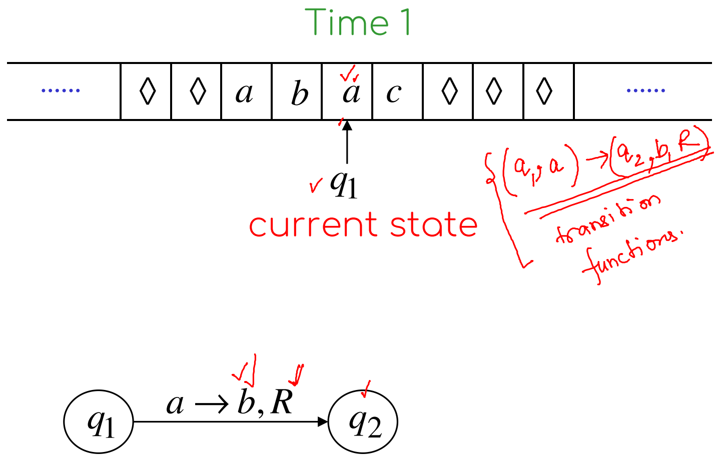


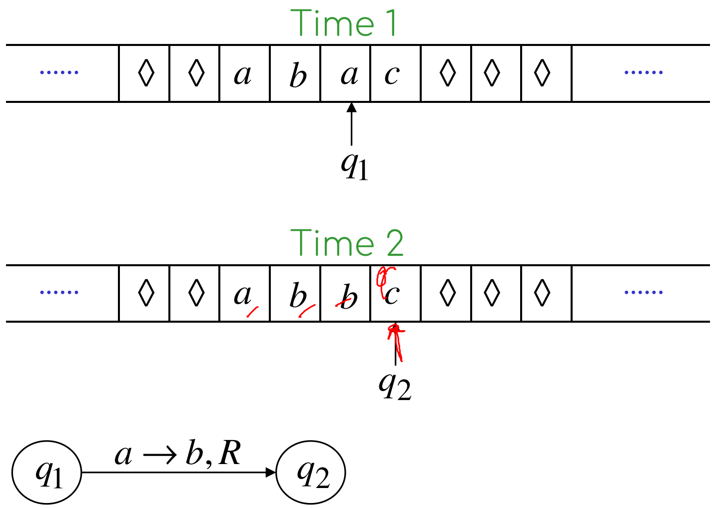
Remark: the input string is never empty

States & Transitions

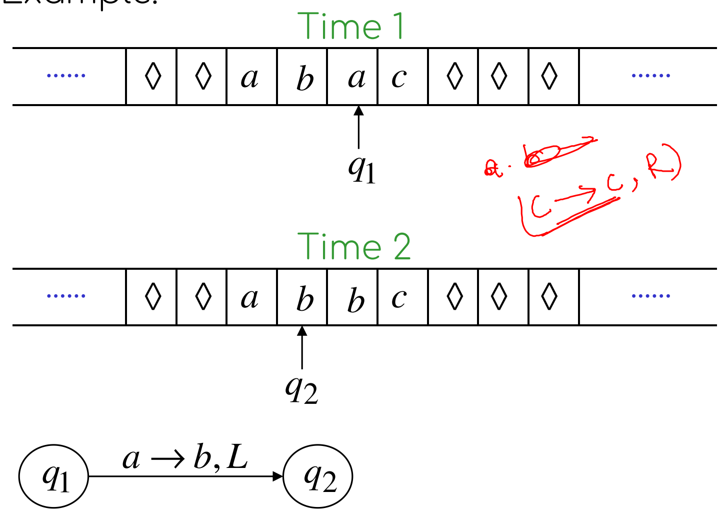


Example:



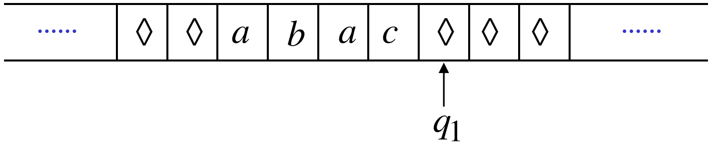

Time 1
...... ◊ ◊ a b a c ◊ ◊ ◊ ......
q1
Time 2
...... ◊ ◊ a b b c ◊ ◊ ◊ ......
q2
q1
a → b, R
q2

Example:

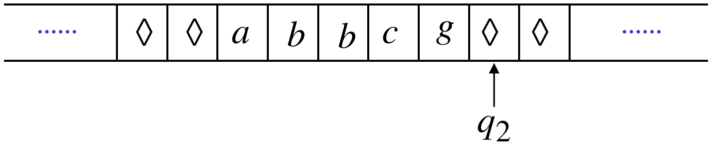

Time 1
...... ◊ ◊ a b a c ◊ ◊ ◊ ......
q1
Time 2
...... ◊ ◊ a b b c ◊ ◊ ◊ ......
q2
q1
a → b, L
q2

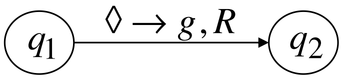
Example:

Time 1



Time 2

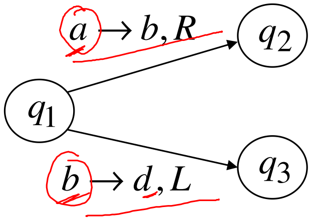




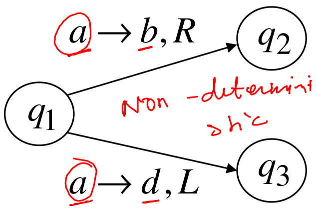
Determinism

Turing Machines are deterministic

Allowed



✗ Not Allowed

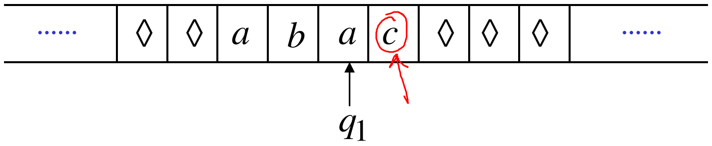


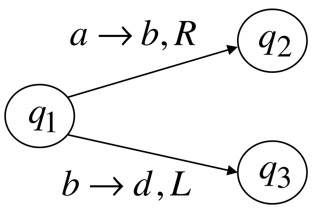
No ~~lambda~~ ε transitions allowed

Partial Transition Function

Example:

i/p Direction





Allowed:

No transition
for input symbol c c

Halting

The machine ***halts*** if there are no possible transitions to follow

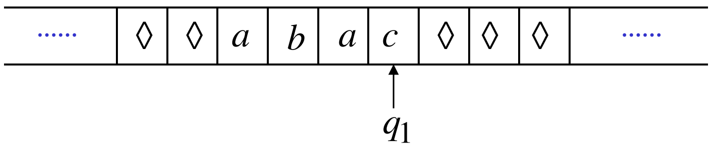
FA
PDA

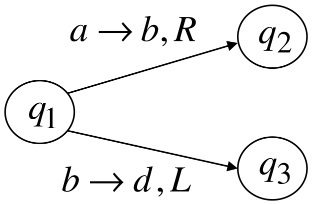
halts at final state

halts at non final state

TM

Example:

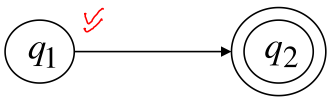




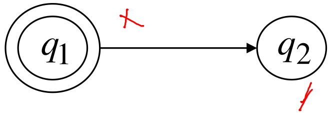
No possible transitio

HALT!!!

Final States



Allowed



Not Allowed

Dead state →X

- Final states have no outgoing transitio
- In a final state the machine halts

Acceptance

Accept Input → If machine halts in a final state

Accept by Empty stack

PDA

Reject Input → If machine halts in a non-final state

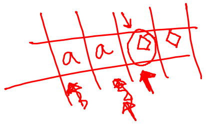
or

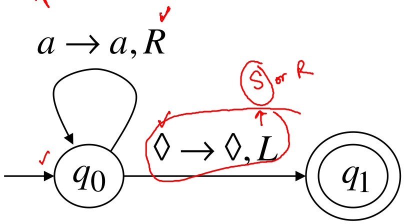
If machine enters an *infinite loop*

Turing Machine Example

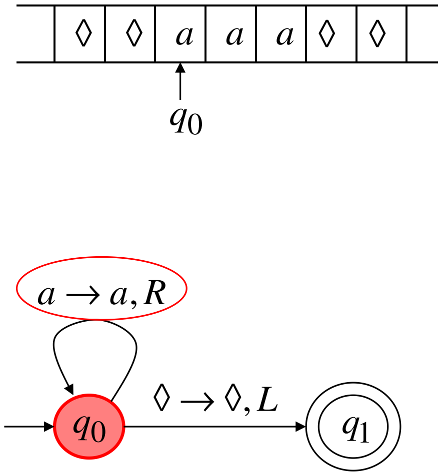
A Turing machine that accepts the langu

aa^*

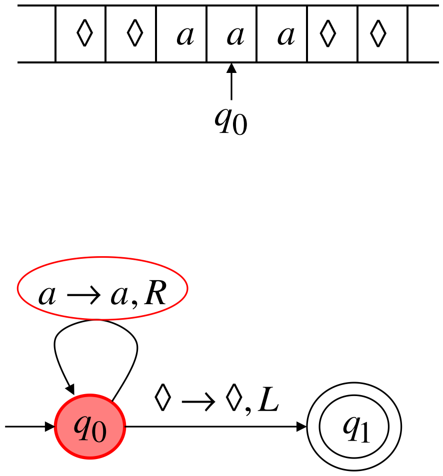




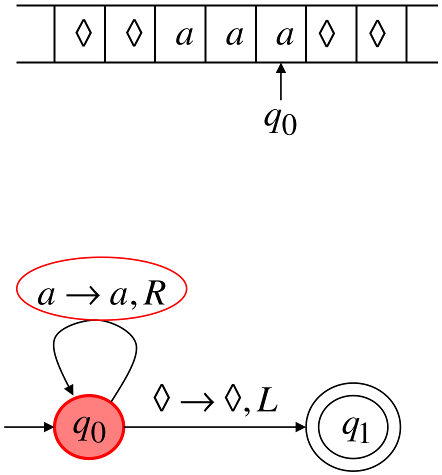
Time 0



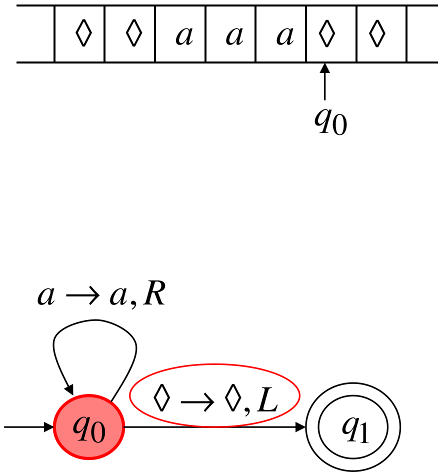
Time 1



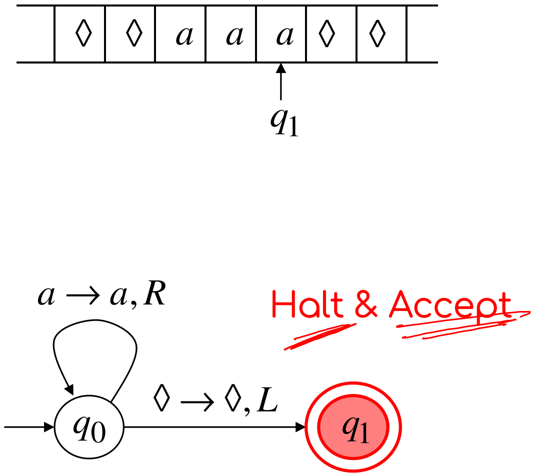
Time 2


◊ ◊ a a a ◊ ◊
q_0
$a \to a, R$
$\Diamond \to \Diamond, L$
q_0
q_1

Time 3

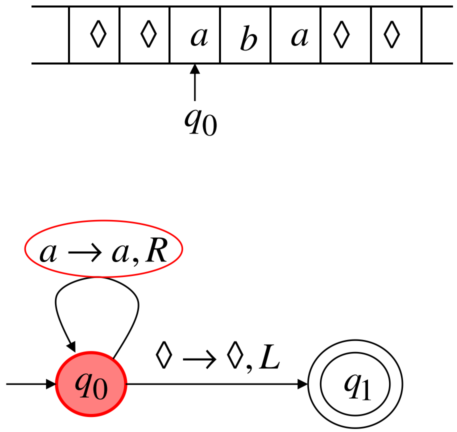

◊ ◊ a a a ◊ ◊
q_0
$a \to a, R$
$\Diamond \to \Diamond, L$
q_0
q_1

Time 4

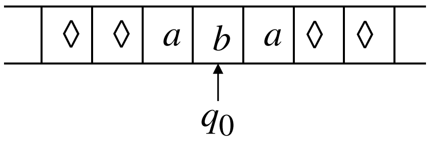


Rejection Example

Time 0

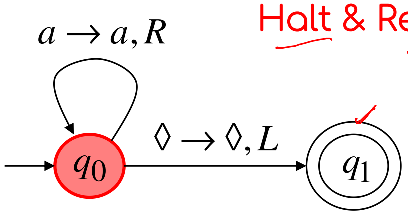


Time 1



No possible Transition

Halt & Reject

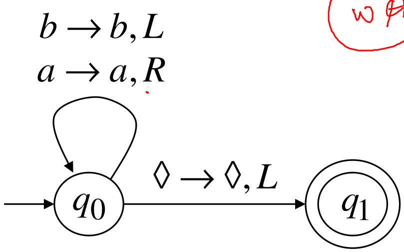


Infinite Loop Example

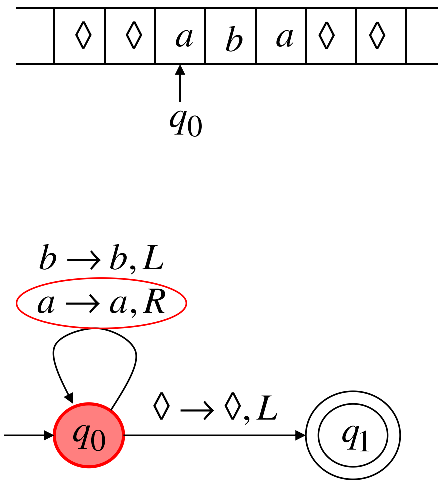
a a a b b b a   ∆ b a

w ∈ L , Accept
Halt

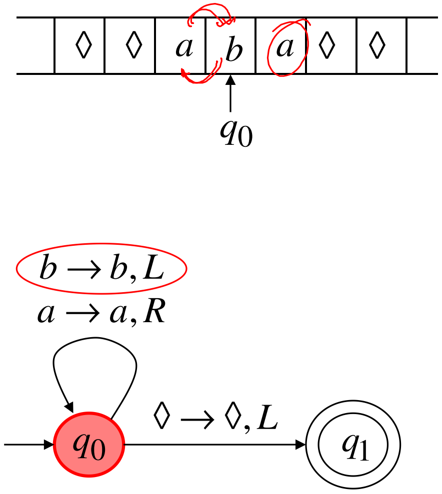
w ∉ L, Halt
Non accept
&
Loop



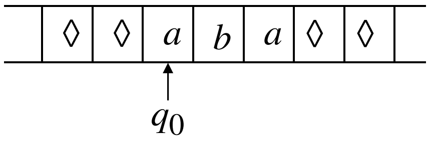
Time 0

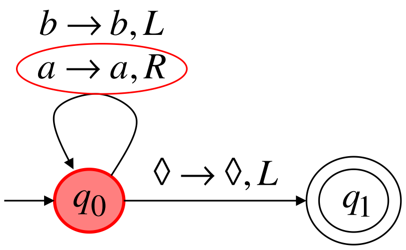

◊ ◊ a b a ◊ ◊
q_0
$b \to b, L$
$a \to a, R$
$\Diamond \to \Diamond, L$
q_0
q_1

Time 1


◊ ◊ a b a ◊ ◊
q_0
$b \to b, L$
$a \to a, R$
$\Diamond \to \Diamond, L$
q_0
q_1

Time 2



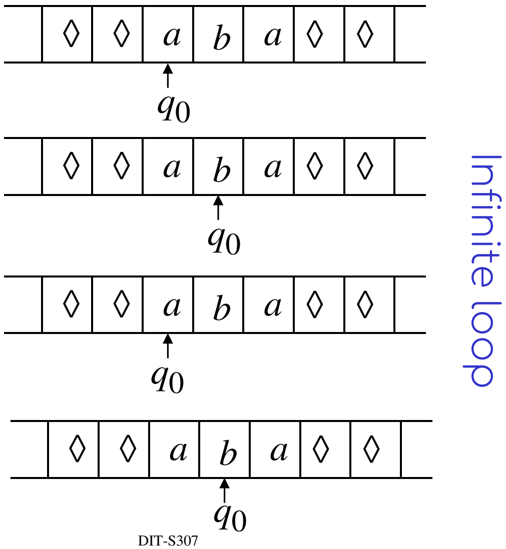


Time 2

Time 3

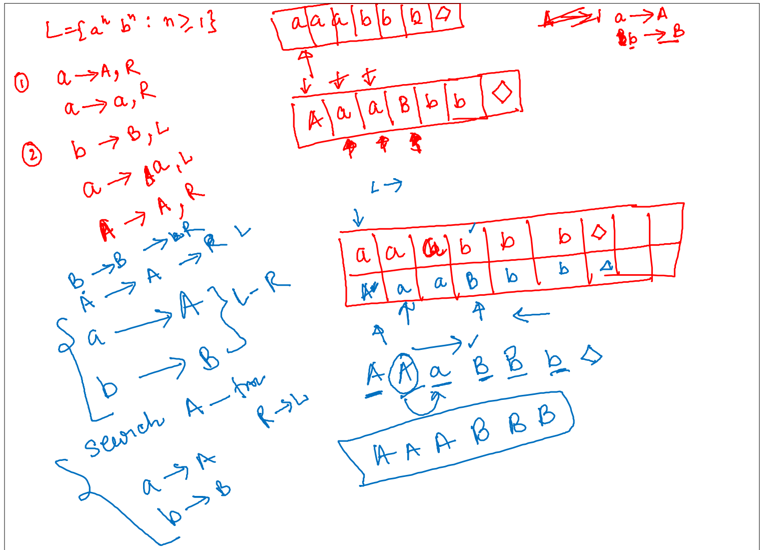
Time 4

Time 5



Because of the infinite loop:

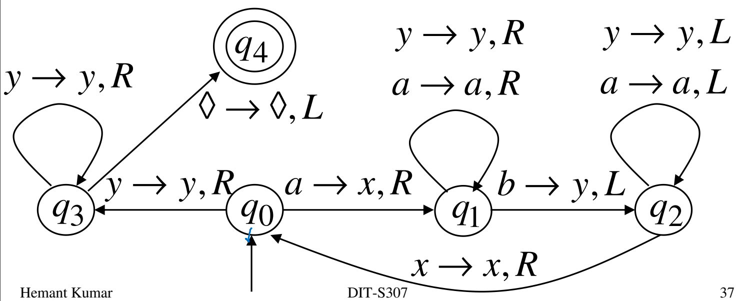
- The final state cannot be reached
- The machine never halts
- The input is not accepted



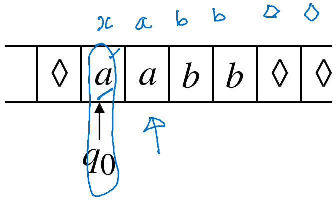
Another Turing Machine Example

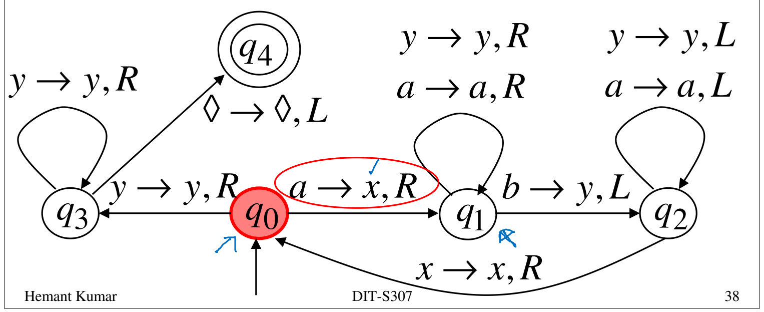
Turing machine for the language $\{a^n b^n : n \geq 1\}$

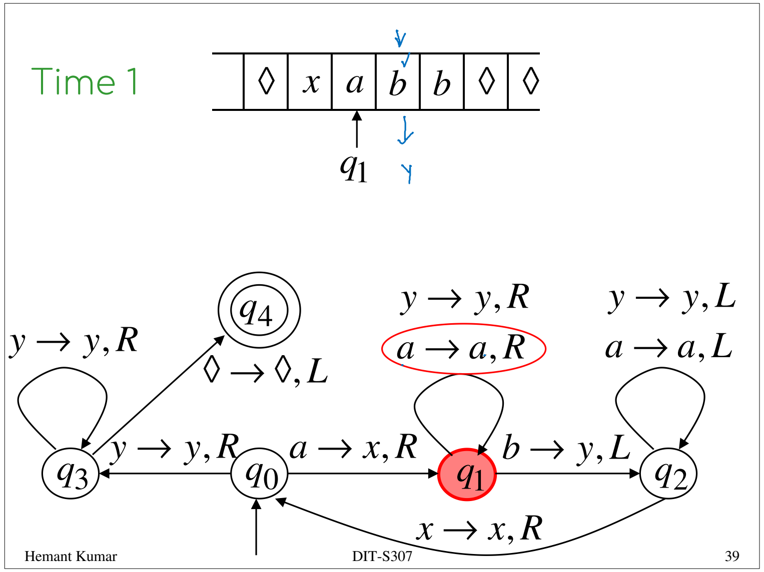
$a^n b^n \rightarrow$ $a \rightarrow x$ $b \rightarrow y$

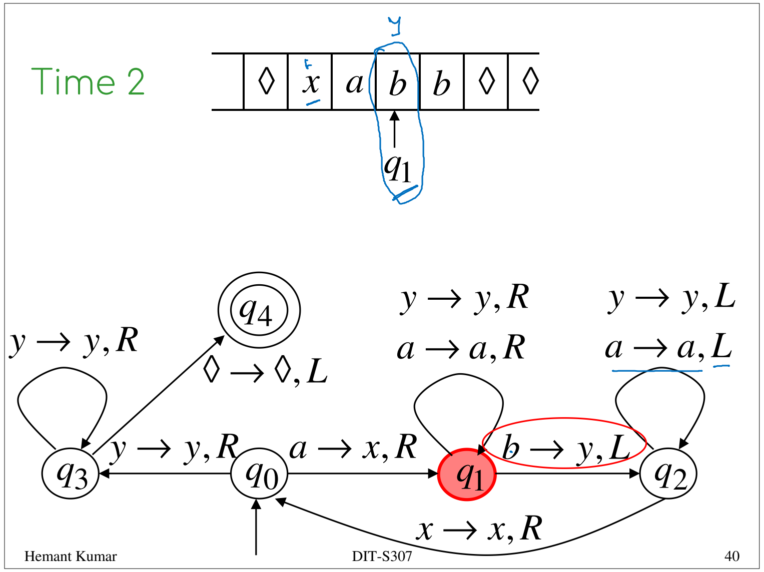


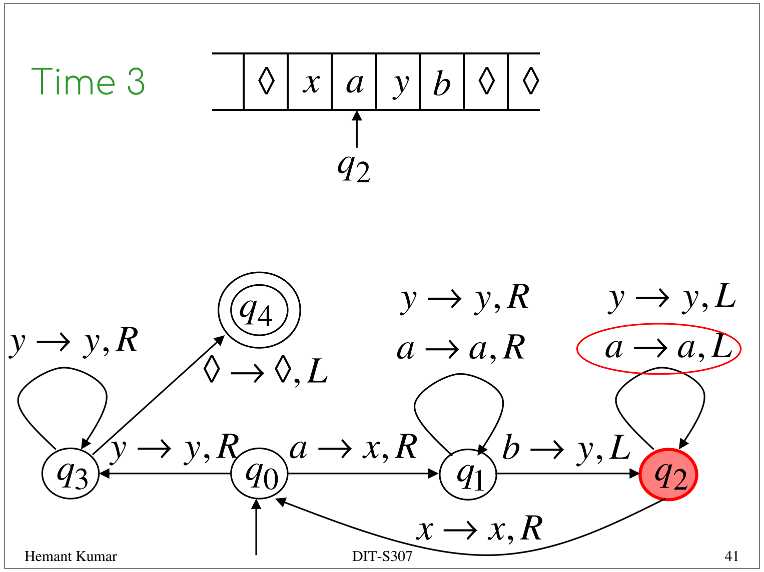
Time 0

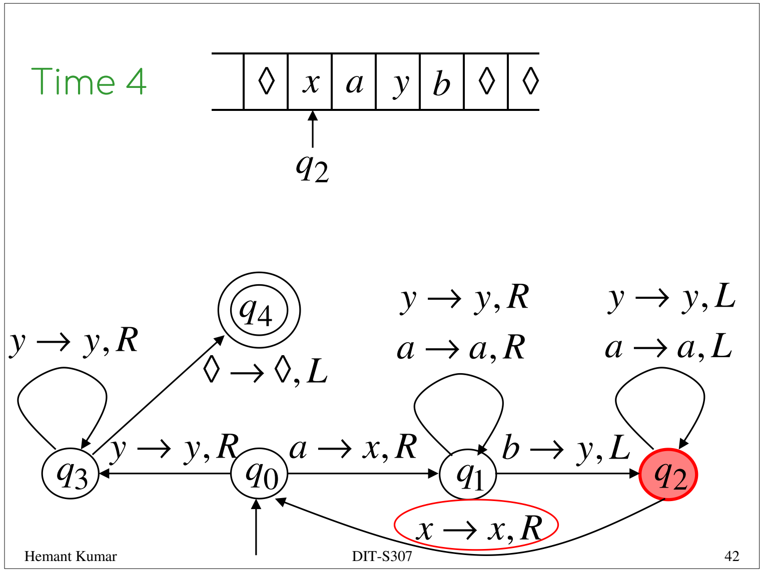




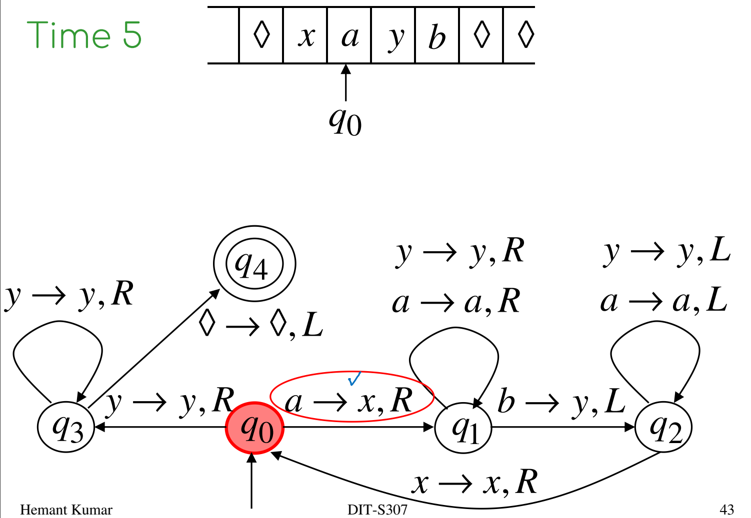

Time 1
◊ x a b b ◊ ◊
q1
y
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
x → x, R
q0 q1 q2 q3 q4
Hemant Kumar
DIT-S307
39


Time 2
◊ x a b b ◊ ◊
y
q1
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
x → x, R
q0 q1 q2 q3 q4
Hemant Kumar
DIT-S307
40

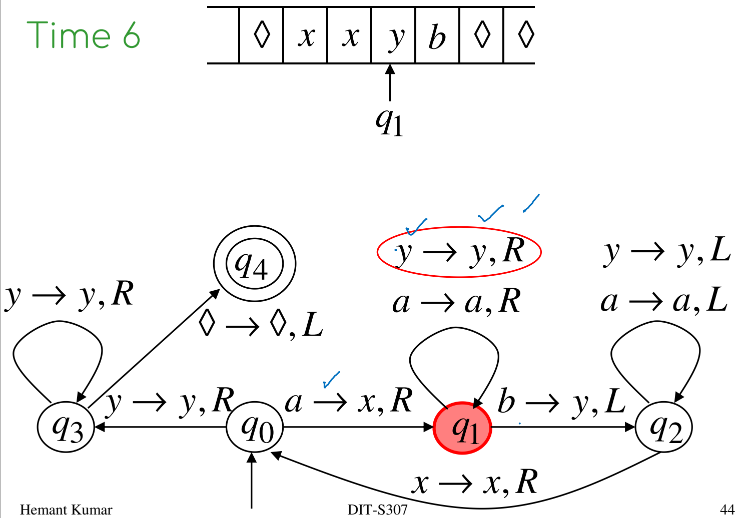

Time 3
◊ x a y b ◊ ◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
Hemant Kumar
DIT-S307
41

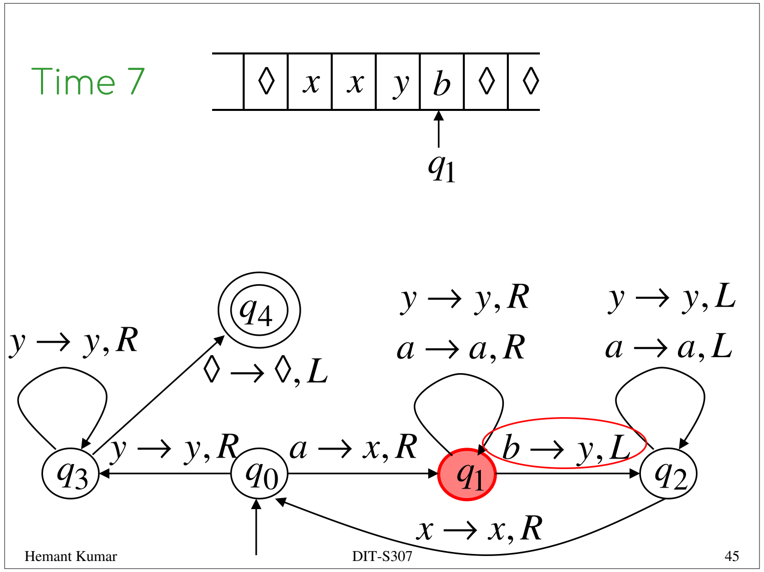

Time 4
◊ x a y b ◊ ◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3 q_0 q_1 q_2
$x \rightarrow x, R$
Hemant Kumar
DIT-S307
42

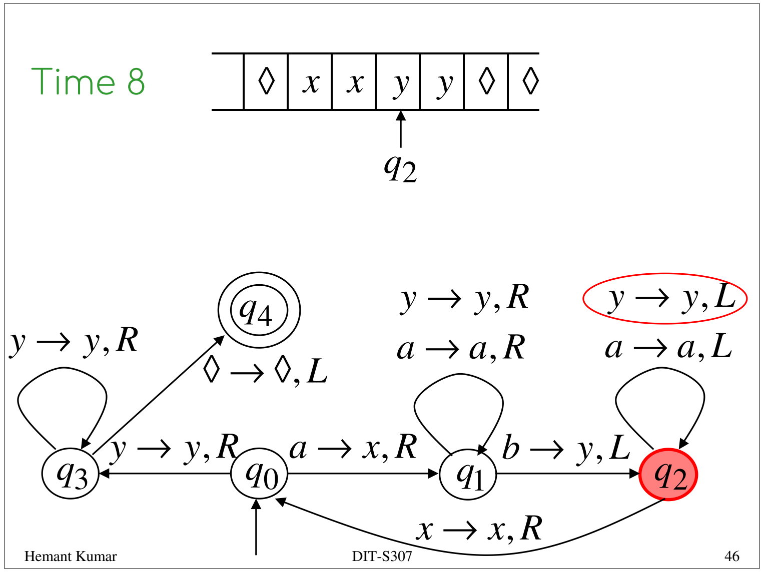
Time 5


◊ x a y b ◊ ◊
q_0
q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

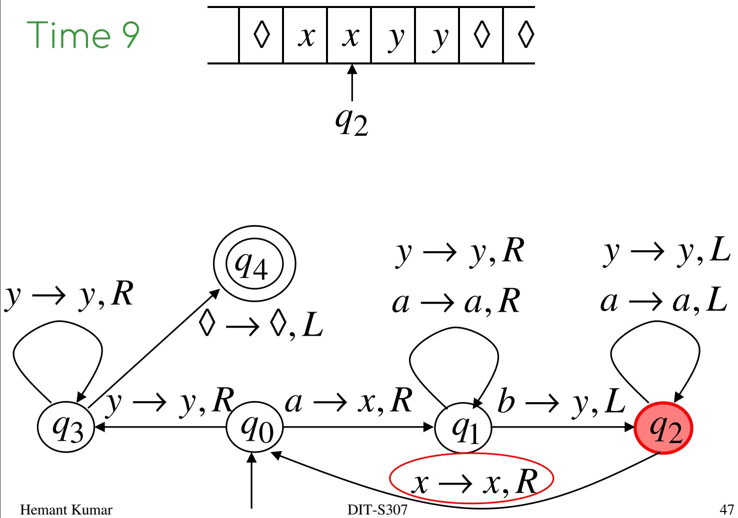
Time 6


◊ x x y b ◊ ◊
q_1
q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

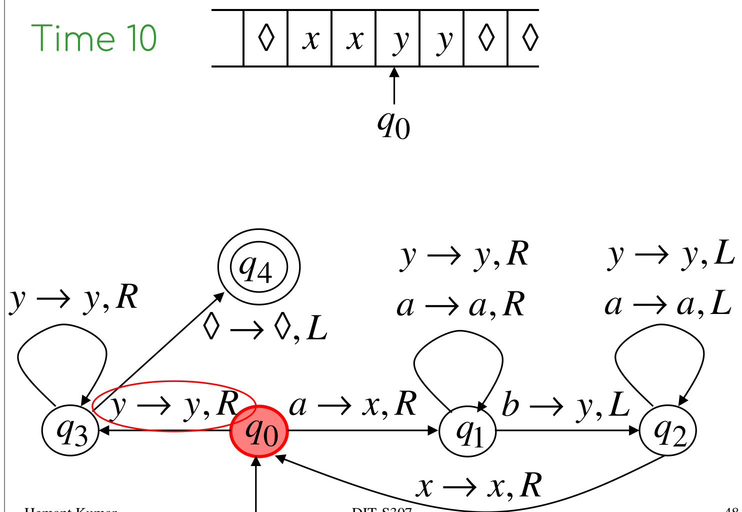

Time 7
◊ x x y b ◊ ◊
q1
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
x → x, R
q4
q3
q0
q1
q2
Hemant Kumar
DIT-S307
45

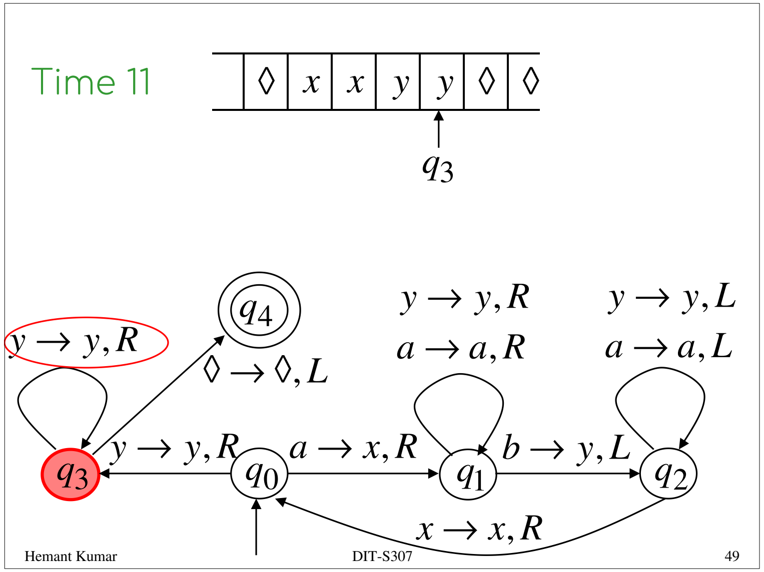

Time 8
◊ x x y y ◊ ◊
q2
y → y, R
◊ → ◊, L
y → y, R
a → a, R
y → y, L
a → a, L
y → y, R
a → x, R
b → y, L
x → x, R
q4
q3
q0
q1
q2
Hemant Kumar
DIT-S307
46

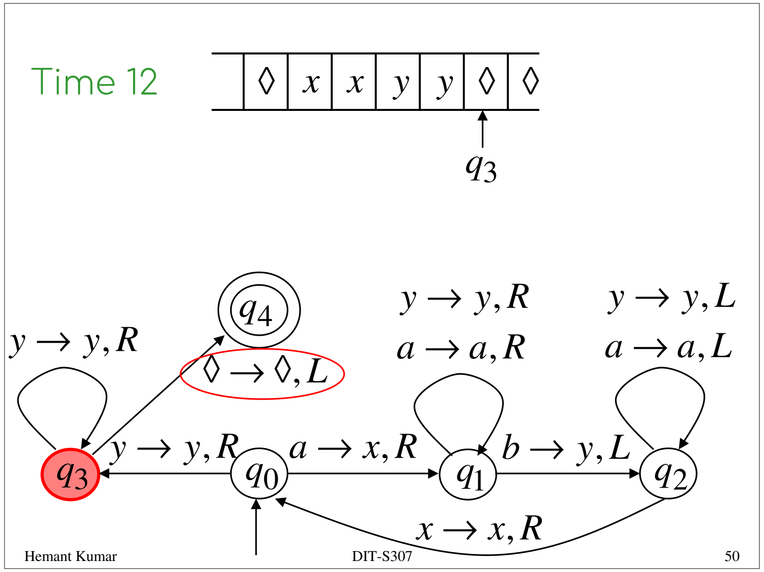
Time 9



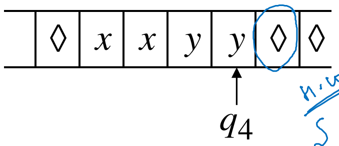
Time 10




Time 11
◊ x x y y ◊ ◊
q_3
q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$
Hemant Kumar
DIT-S307
49


Time 12
◊ x x y y ◊ ◊
q_3
q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$
Hemant Kumar
DIT-S307
50

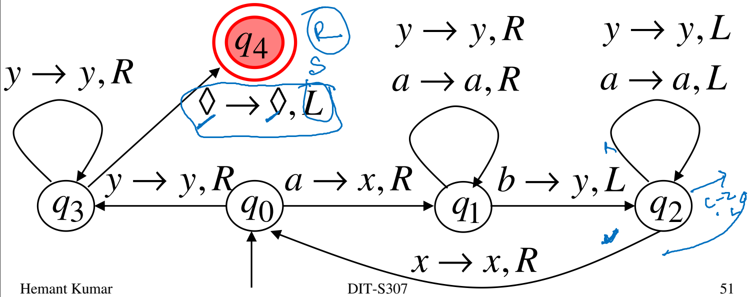
Time 13



$a \to x$
$b \to y$
$c \to z$

H.W
$\{a^n b^n c^n : n \geq 1\}$
No CFL

Halt & Accept



Observation:

We easily modify the machine for the language $\{a^n b^n : n \geq 1\}$

to construct a TM for the language $\{a^n b^n c^n : n \geq 1\}$

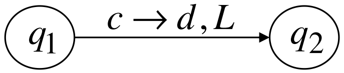
Formal Definitions for Turing Machines

Transition Function

$$q_1 \xrightarrow{a \to b, R} q_2$$

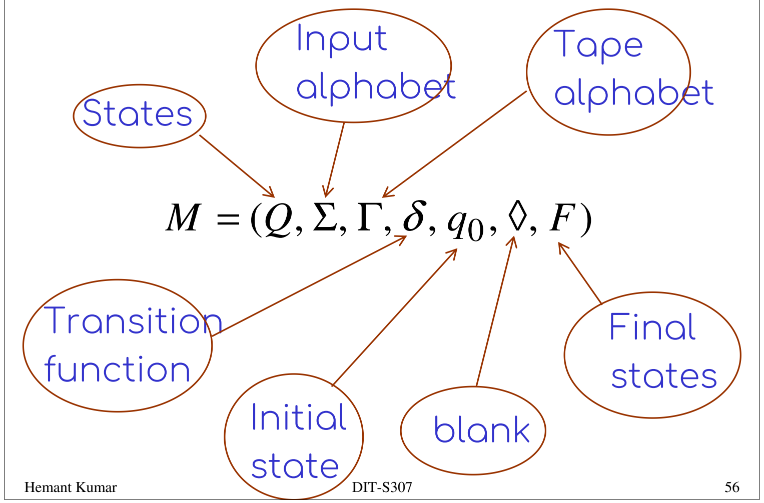
$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function



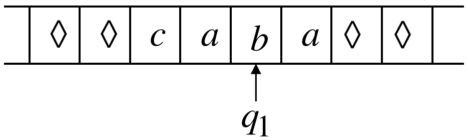
$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:



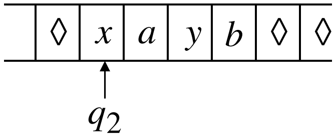
$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

Configuration

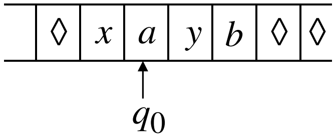


Instantaneous description: $ca\ q_1\ ba$

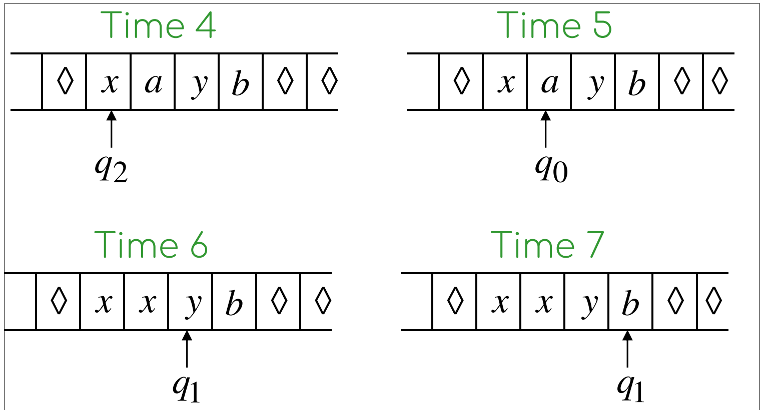
Time 4



Time 5



A Move: $q_2\ xayb \succ x\ q_0\ ayb$



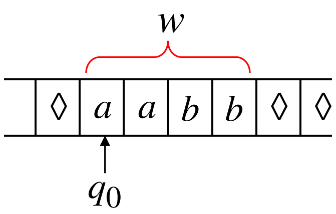
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

Equivalent notation: $q_2\ xayb \overset{*}{\succ} xxy\ q_1\ b$

Initial configuration: $q_0 \; w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \; q_0 \, w \overset{*}{\succ} x_1 \, q_f \, x_2\}$$

Initial state Final state

Standard Turing Machine

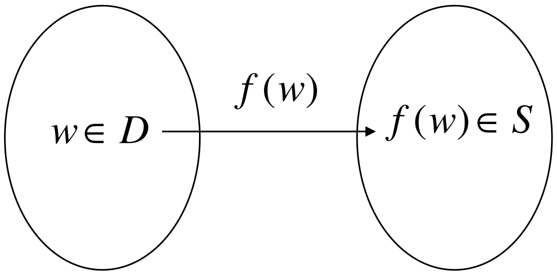
The machine we described is the standar

- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain D Result Region S



A function may have several parameters / arguments:

Example: The *Addition* function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

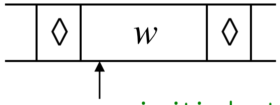
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

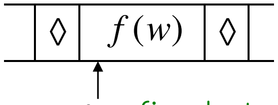
A function f is computable if there is a Turing Machine M such that:

Initial configuration



q_0 initial state

Final configuration



q_f final state

For all $w \in D$ Domain

In other words:

A function f is computable if
there is a Turing Machine M such tha

$$q_0 \, w \overset{*}{\succ} q_f \, f(w)$$

Initial Configurati
on

Final Configuration

$\forall w \in D$ (the domain)

Example

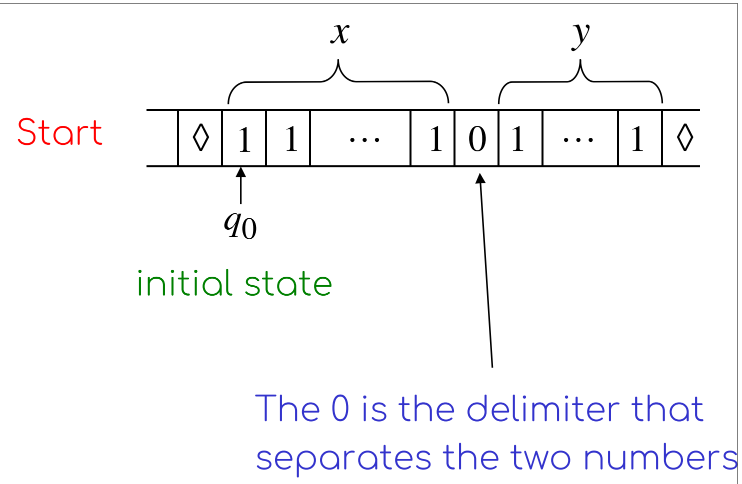
The function $f(x, y) = x + y$ is computab

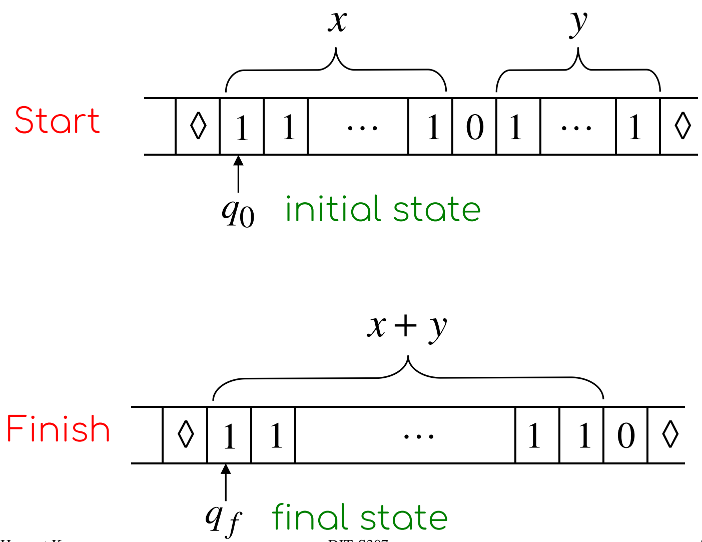
x, y are integers

A Turing Machine M for f:

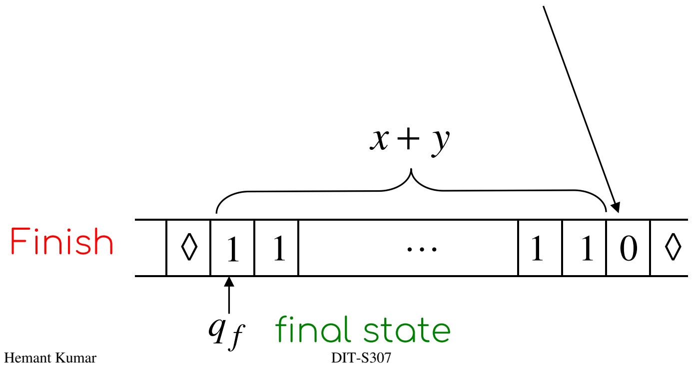
Input string: $x0y$ unary

Output string: $xy0$ unary


Start
x
y
◊ 1 1 ⋯ 1 0 1 ⋯ 1 ◊
q_0
initial state
The 0 is the delimiter that
separates the two numbers

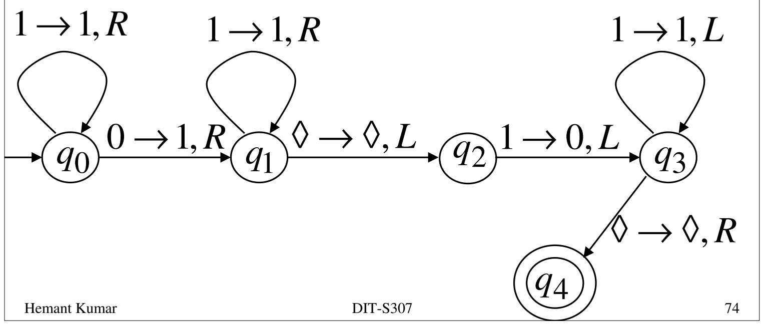

Start
x
y
◊ 1 1 ⋯ 1 0 1 ⋯ 1 ◊
q_0
initial state
x + y
Finish
◊ 1 1 ⋯ 1 1 0 ◊
q_f
final state

The 0 helps when we use
the result for other operations



Turing machine M for the function:

$$f(x, y) = x + y$$



Execution Example:

$x = 11$ (2)

$y = 11$ (2)

Input (time 0)

	x			y		
◊	1	1	0	1	1	◊

$\uparrow$ q_0

Output (final result)

$x + y$

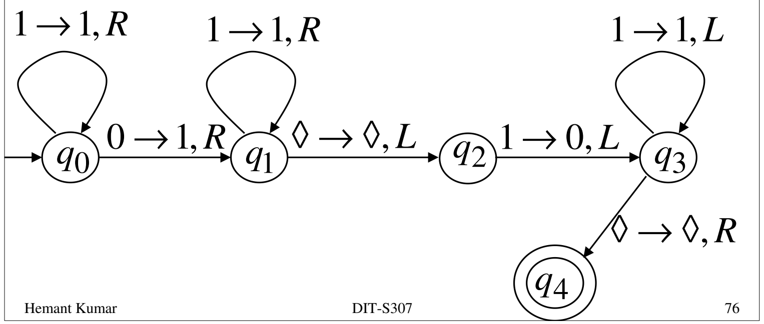
◊	1	1	1	1	0	◊

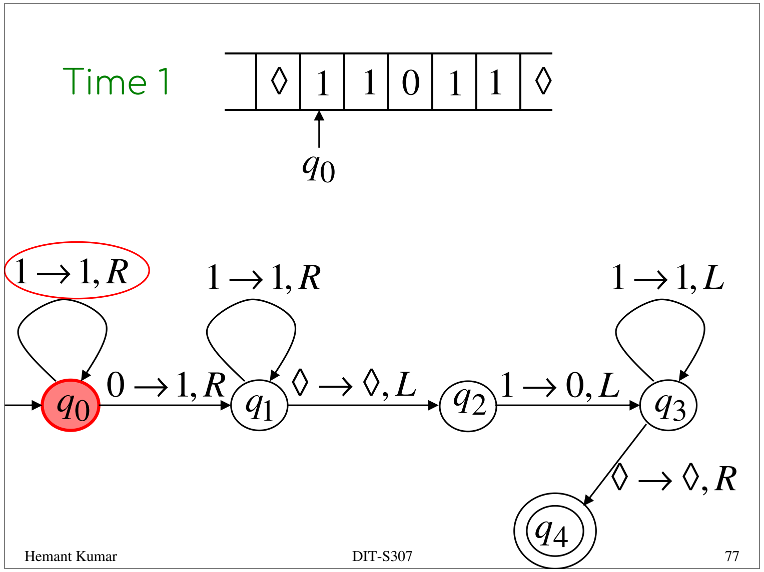
$\uparrow$ q_4

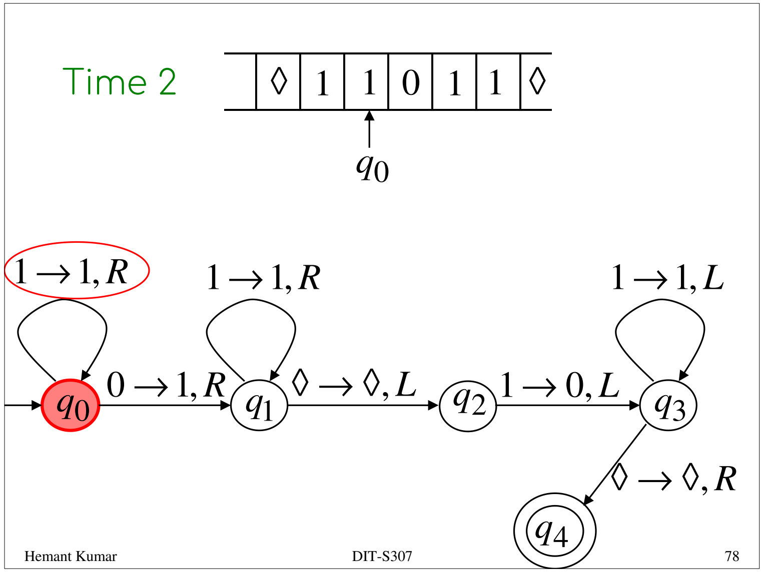
Time 0

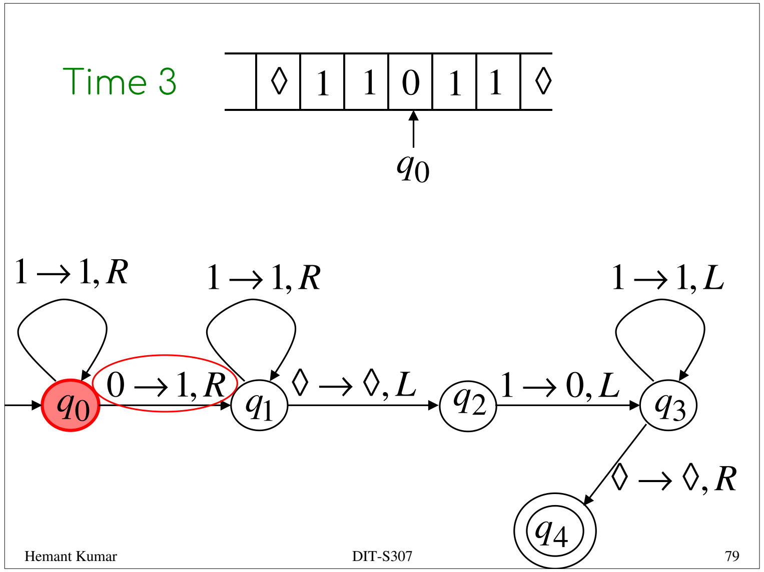
◊	1	1	0	1	1	◊

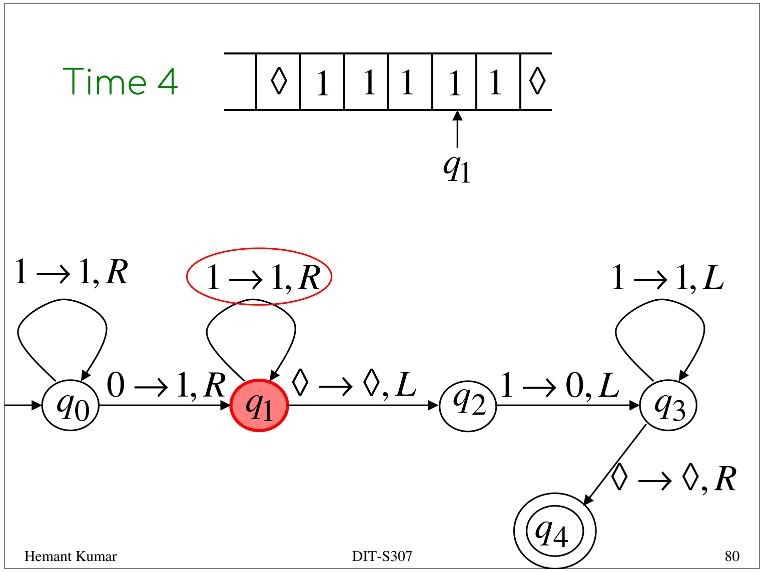
$\uparrow$ q_0

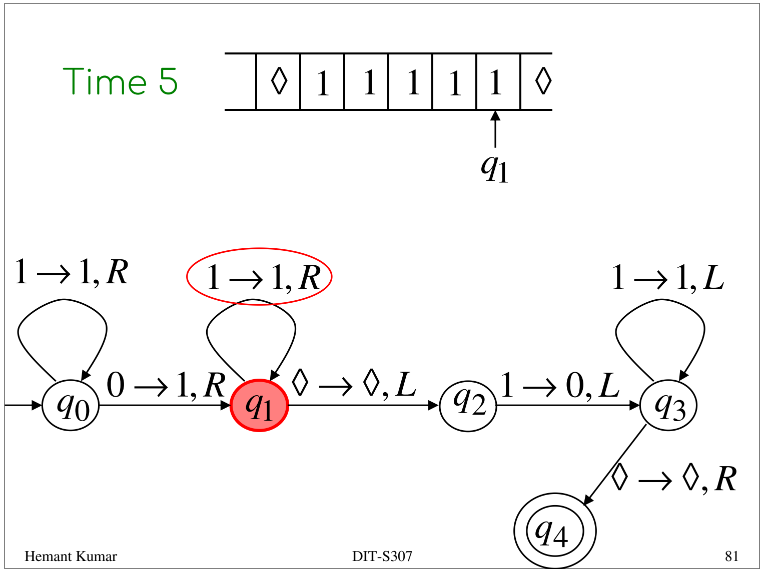


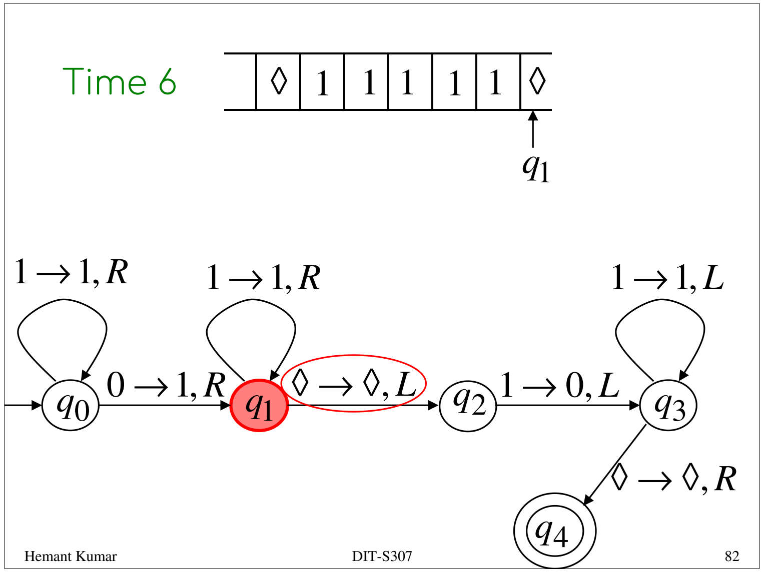

Time 1
◊ 1 1 0 1 1 ◊
q0
1 → 1, R
1 → 1, R
1 → 1, L
0 → 1, R
◊ → ◊, L
1 → 0, L
◊ → ◊, R
q0
q1
q2
q3
q4
Hemant Kumar
DIT-S307
77

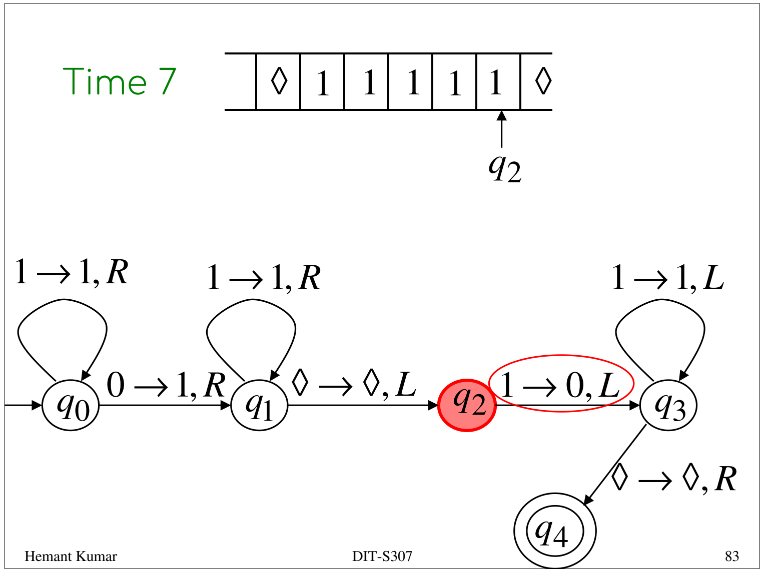

Time 2
◊ 1 1 0 1 1 ◊
q0
1 → 1, R
1 → 1, R
1 → 1, L
0 → 1, R
◊ → ◊, L
1 → 0, L
◊ → ◊, R
q0
q1
q2
q3
q4
Hemant Kumar
DIT-S307
78

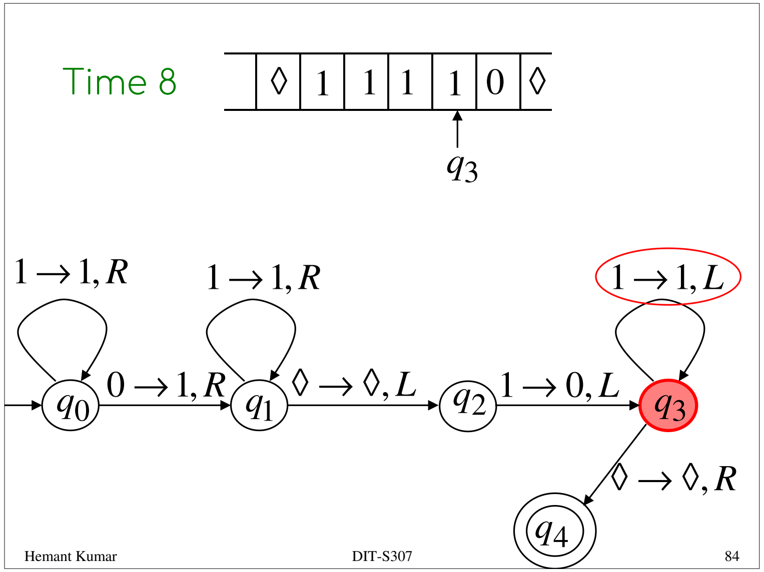

Time 3
◊ 1 1 0 1 1 ◊
q_0
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\lozenge \to \lozenge, L$
$1 \to 0, L$
$\lozenge \to \lozenge, R$
q_0
q_1
q_2
q_3
q_4
Hemant Kumar
DIT-S307
79

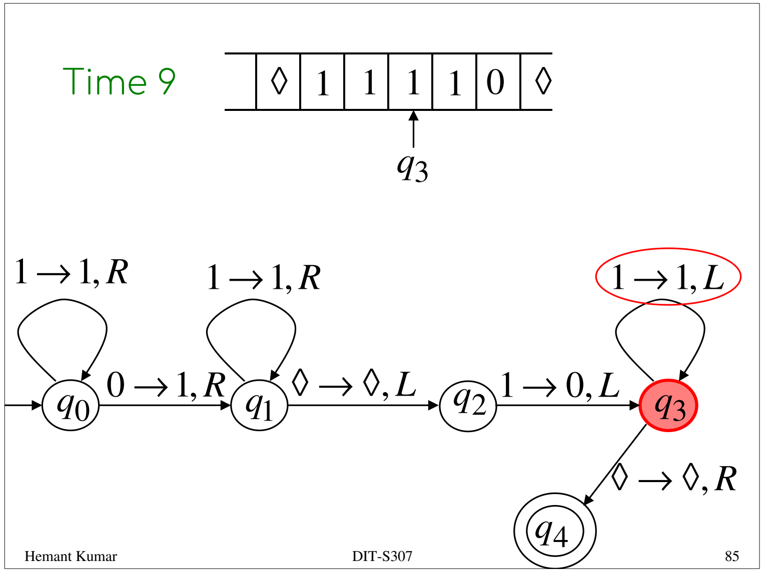

Time 4
◊ 1 1 1 1 1 ◊
q_1
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\lozenge \to \lozenge, L$
$1 \to 0, L$
$\lozenge \to \lozenge, R$
q_0
q_1
q_2
q_3
q_4
Hemant Kumar
DIT-S307
80

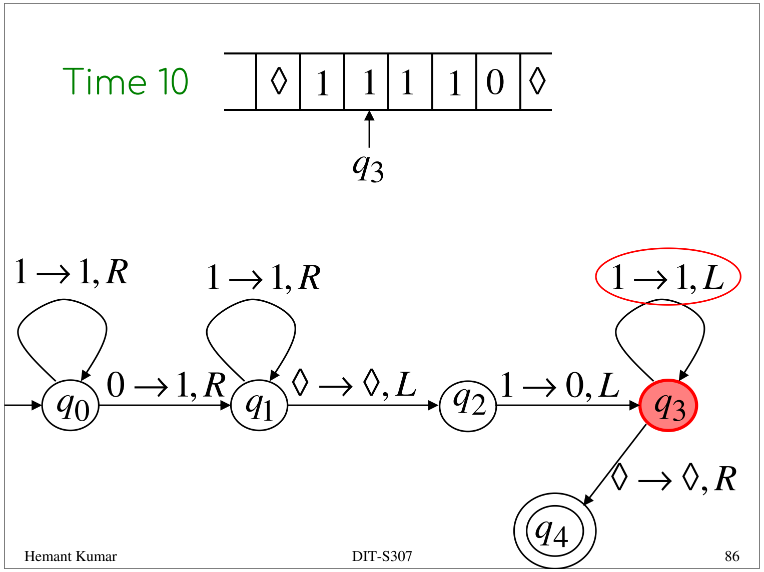

Time 5
◊ 1 1 1 1 1 ◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4
Hemant Kumar
DIT-S307
81

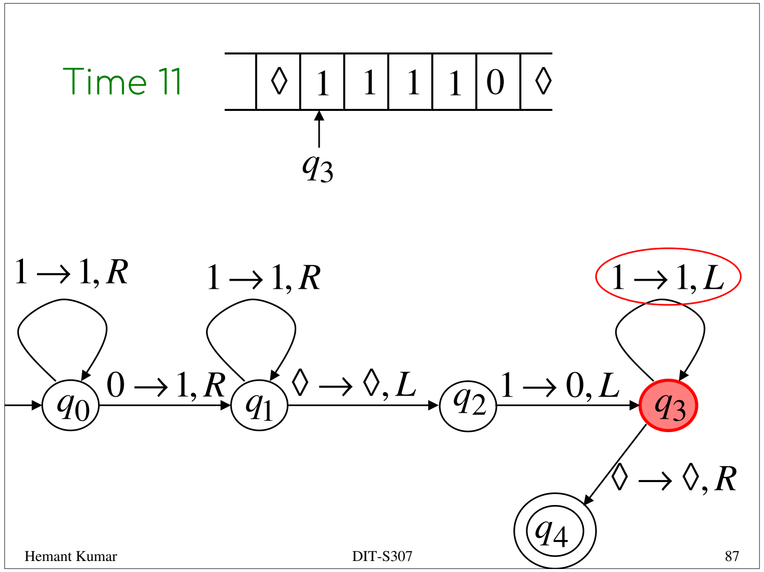

Time 6
◊ 1 1 1 1 1 ◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4
Hemant Kumar
DIT-S307
82

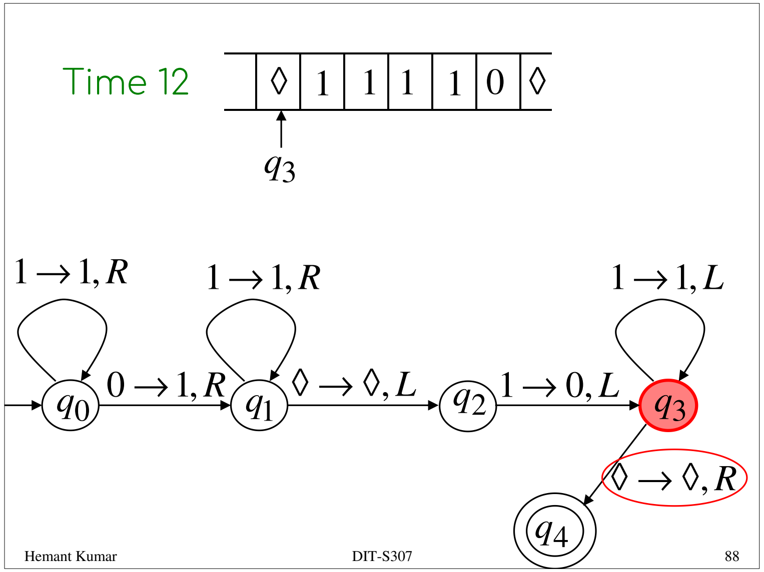

Time 7
◊ 1 1 1 1 1 ◊
q_2
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
q_0
$0 \to 1, R$
q_1
$\lozenge \to \lozenge, L$
q_2
$1 \to 0, L$
q_3
$\lozenge \to \lozenge, R$
q_4
Hemant Kumar
DIT-S307
83

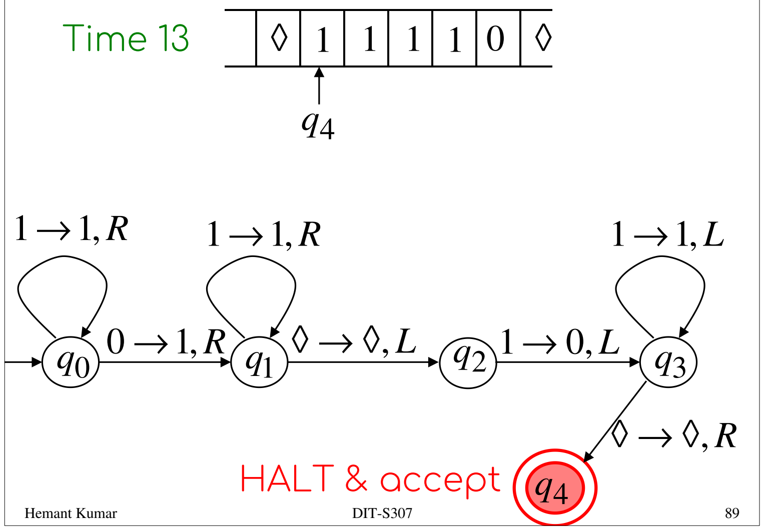

Time 8
◊ 1 1 1 1 0 ◊
q_3
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
q_0
$0 \to 1, R$
q_1
$\lozenge \to \lozenge, L$
q_2
$1 \to 0, L$
q_3
$\lozenge \to \lozenge, R$
q_4
Hemant Kumar
DIT-S307
84


Time 9
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4
Hemant Kumar
DIT-S307
85


Time 10
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4
Hemant Kumar
DIT-S307
86


Time 11
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4
Hemant Kumar
DIT-S307
87


Time 12
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4
Hemant Kumar
DIT-S307
88



Another Example

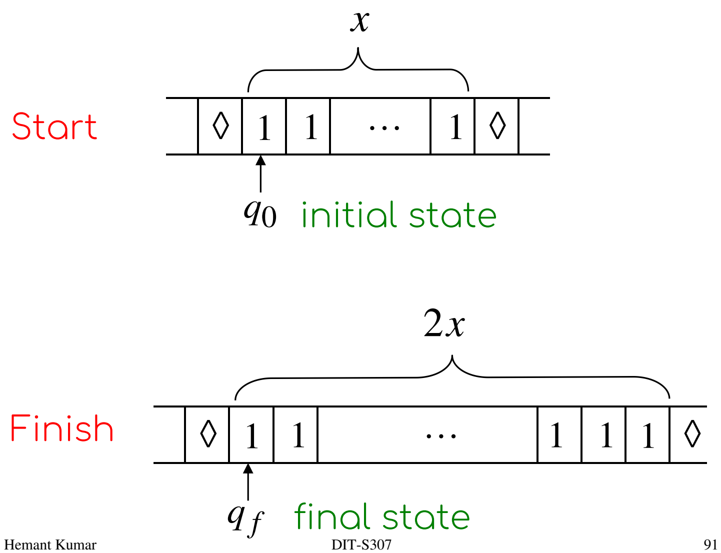
The function $f(x) = 2x$ is computable

x is integer

Turing Machine:

Input string: x unary

Output string: xx unary

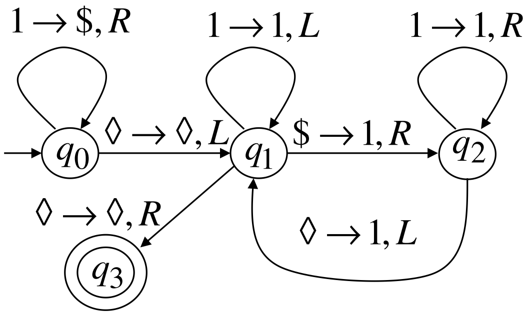


Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with \$
- Repeat:
  - Find rightmost \$, replace it with 1
  - Go to right end, then insert 1

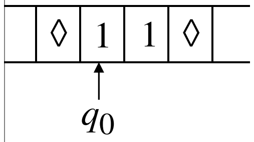
Until no more \$ remains

A Turing Machine for $f(x) = 2x$

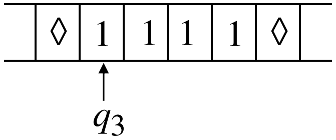


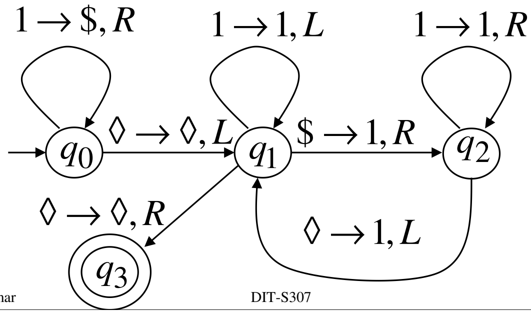
Example

Start



Finish





Another Example

The function
is computable

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

A Turing Machine for

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Pseudocode for our Turing Machine:

- Repeat

  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched

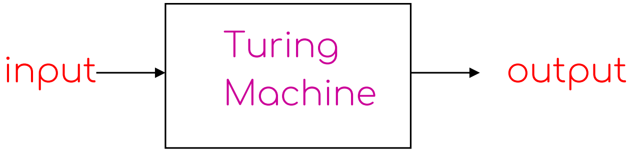
  erase tape, then write 1 $(x > y)$

  else

  erase tape, then write 0 $(x \le y)$

Combining Turing Machines

Block Diagram



Example:

$$f(x,y) = \begin{cases} x+y & \text{if} \quad x > y \\ 0 & \text{if} \quad x \leq y \end{cases}$$