Bounded function: Let $f: A \to \mathbb{R}$ be a function so that $f(A)$ is image set A under f. Then f is bounded if the set $f(A)$ is bounded - i.e, $\exists\ k \,\&\, K$ s.t $k \leq f(x) \leq K\ \forall x \in A$

Theorem: If f is continuous on closed interval $[a,b]$, then it is bdd in that interval

Proof: Let f be continuous on $[a,b]$

Assume f is not bdd above.

$\Rightarrow \forall n \in \mathbb{N}$, $\exists\ x_n \in [a,b]$ s.t $f(x_n) > n$

(i.e, $f(x_1) > 1$, $f(x_2) > 2$, .., $f(x_n) > n$ - -)

$\Rightarrow \langle x_n \rangle$ is a sequence in $[a,b]$. & hence bounded. ①

$\Rightarrow \exists$ a subseq $\langle x_{m_n} \rangle \to x$ as $n \to \infty$, $x \in [a,b]$ (by Bolzano-Weierstrass thm for seq.)

$\because f(x_n) > n \Rightarrow f(x_{m_n}) > m_n\ \forall n$ [$\because \langle m_n \rangle \uparrow$ seq.)

$\Rightarrow \langle f(x_{m_n}) \rangle$ diverges to ∞ - i.e, $\lim_{n\to\infty} f(x_{m_n}) = \infty$

But from ① $\lim_{n\to\infty} f(x_{m_n}) \to f(x) \neq \infty$ ($\because f$ is continuous)

~~contradiction i.e $f(x_{m_n})$ diverges.~~

$\Rightarrow f$ is not continuous. contradiction

$\Rightarrow$ Our assumption was wrong. Hence, f is bdd above

Similarly prove f is bdd below.

$\Rightarrow f$ is bdd on $[a,b]$ //

Note: Converse of above theorem is not true

Let $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ f is bdd [$\because |f(x)| \leq 1$]

But f is not continuous at $x = 0$.

Theorem (Extreme Value Theorem): If a fn is continuous on a closed interval $[a,b]$, then it attains its supremum and infimum at least once in $[a,b]$

Proof: If f is constant fn, clearly it attains its bounds at every point in the interval

Let $f \neq$ constant

$\because f$ is continuous on $[a,b]$ $\therefore$ Bdd

Let $m = \operatorname{Inf}_{x\in[a,b]} f(x)$ and $M = \operatorname{Sup}_{x\in[a,b]} f(x)$

To prove: $\exists\ \alpha, \beta \in [a,b]$ s.t. $f(\alpha) = m$ and $f(\beta) = M$

Consider Supremum: Suppose f does not attain supremum, M in $[a,b]$

$\Rightarrow f(x) \neq M$ for any $x \in [a,b]$

Consider $g(x) = \frac{1}{M - f(x)}$ $\forall x \in [a,b]$. Then $g(x) > 0$ $\forall x \in [a,b]$

and g is continuous & bdd on $[a,b]$. Let $k = \operatorname{Sup}_{x\in[a,b]} g(x)$

$\Rightarrow \frac{1}{M - f(x)} \leq k$ $\forall x \in [a,b]$

$\Rightarrow f(x) \leq M - \frac{1}{k}$ $\forall x \in [a,b]$ Contradiction as $M = \operatorname{Sup} f(x)$

Hence, f attains supremum for atleast one value i.e, $\exists \beta$ in $[a,b]$ s.t $f(\beta) = \operatorname{Sup} f(x) = M$.

Similarly prove $f(\alpha) = \operatorname{Inf} f(x)$, $x \in [a,b]$

Hence, $f(x)$ attains Sup & Inf atleast once in $[a,b]$

Note: In above thm closed interval is must.

~~Let~~ Consider $f(x) = x$ in $(0,1)$. Then f is continuous & bdd but $\sup f\ (=1)$ & $\inf f\ (=0)$ ~~are not~~ do not belong to $(0,1)$.

Theorem: If f is continuous at $x=c$, where $f(c)\neq 0$, then $\exists$ a $\delta>0$ s.t $f(x)$ has same sign as $f(c)$ for all $x\in(c-\delta, c+\delta)$.

Proof: $\because f$ is continuous at $x=c$, for any given $\epsilon>0$, $\exists\,\delta>0$ s.t $x\in(c-\delta, c+\delta) \Rightarrow f(c)-\epsilon < f(x) < f(c)+\epsilon$ --- (*)

$\because f(c)\neq 0$, either $f(c)>0$ or $f(c)<0$.

Case I: $f(c)>0$. Choose $\epsilon>0$ s.t $f(c)>\epsilon$. Then $f(c)\pm\epsilon>0$. Hence, $f(x)>0$ $\forall x\in(c-\delta, c+\delta)$ (from (*))

$\Rightarrow f(x)$ has same sign as $f(c)$ $\forall x\in(c-\delta, c+\delta)$

Case II: $f(c)<0$. Choose $\epsilon>0$ s.t $-f(c)>0$

$\Rightarrow f(c)\pm\epsilon<0 \Rightarrow f(x)<0$ $\forall x\in(c-\delta, c+\delta)$ (from (*))

$\Rightarrow f(x)$ has same sign as $f(c)$ $\forall x\in(c-\delta, c+\delta)$

Theorem: If f is continuous on $[a,b]$ and $f(a)$ & $f(b)$ have opposite signs, then $\exists$ atleast one value of x in $[a,b]$ for which $f(x)$ vanishes (i.e zero)

Proof: Assume $f(a)<0$ & $f(b)>0$ (You can take otherwise also)

Let $S=\{x : x\in[a,b], f(x)<0\}$. $S\neq\phi$ ($\because a\in S$)

& b is upper bound for S. $\Rightarrow$ Sup S exists (Bolzano Weierstrass thm for sets).

Let $c=\text{Sup}\,S$. To prove: $f(c)=0$

~~no point of S can lie to the RHS of c−δ~~

If $f(c)>0$, from above theorem $f(x)>0$ $\forall x\in(c-\delta, c+\delta)$

$\Rightarrow$ No point of S can be to right of $c-\delta$.

$\Rightarrow c-\delta$ is upper bound of S contradiction to $c=\text{Sup}\,S$.

$\therefore f(c)\not> 0$ (1)

If $f(c) < 0$ then from above theorem $f(x) < 0 \ \forall \ x \in (c-\delta, c+\delta)$

$\Rightarrow f(x) < 0$ for some $x > c$ ~~contradiction~~

But $c = \text{Sup } S \Rightarrow f(x) < 0 \ \forall x \leq c$ contradiction

$\Rightarrow f(c) \nless 0$ — (ii)

From (i) & (ii) $f(c) = 0$. $\because f(a) < 0$ & $f(b) > 0 \Rightarrow c \neq a$ & $c \neq b$

$\therefore c \in (a, b)$ //

<u>Intermediate value Theorem</u>: If f is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, then $f(x)$ takes atleast once ~~the~~ all the values b/w $f(a)$ & $f(b)$

<u>Proof</u>: $\because f(a) \neq f(b)$ either $f(a) > f(b)$ or $f(b) > f(a)$

Let $f(a) < f(b)$. Let 'k' be any number b/w $f(a)$ & $f(b)$

Define $g(x) = f(x) - k \ \forall \ x \in [a, b]$

$\Rightarrow g(x)$ is continuous & $g(a) < 0$ & $g(b) > 0$ (from above theorem)

$\Rightarrow \exists \ c \in (a, b)$ s.t $g(c) = 0$

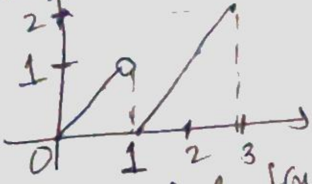
$\Rightarrow f(c) = k$

Hence, $f(x)$ takes value k atleast once, Since k is arbitrary we have the result

<u>H.W.</u> Prove above theorem when $f(a) > f(b)$

<u>Note</u> Converse of above thm is not true.

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x-1 & 1 \leq x \leq 3 \end{cases}$$



Here $f(0) = 0$
$f(3) = 2$

$f(x)$ takes all values b/w 0 & 2 but $f(x)$ is not continuous at $x = 1$.

<u>Corollary</u>: If f is continuous on $[a, b]$ then $f([a,b]) = [m, M]$ where $m = \text{Inf } f$ and $M = \text{Sup } f$ on $[a, b]$ & thus $f([a,b])$ is a closed set.

Uniform Continuity – A fn defined on $[a,b]$ is said to be uniformly continuous on $[a,b]$ if for every $\epsilon>0$, $\exists\ \delta>0$ (depending only on ϵ) s.t. $|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\epsilon$

Note :- (1) Continuity $\not\Rightarrow$ uniform continuity

(2) In order to prove uniform continuity make $|f(x_1)-f(x_2)| \le K|x_1-x_2|$

So, by taking $\delta=\frac{\epsilon}{K}$, $|f(x_1)-f(x_2)|<\epsilon$ whenever $|x_1-x_2|<\delta$.

Ex Show that $f(x)=x^2$ is uniformly continuous on $[0,1]$

Soln Let $x_1, x_2 \in [0,1] \Rightarrow |x_i|<1 \quad i=1,2$

Let $\epsilon>0$ be given. Then

$|f(x_1)-f(x_2)|=|x_1^2-x_2^2| \le |x_1-x_2|(|x_1|+|x_2|)$

$< 2|x_1-x_2|<\epsilon$ provided $|x_1-x_2|<\frac{\epsilon}{2}$.

Take $\delta=\epsilon/2$. Therefore,

$|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\epsilon$

Ex Show that $f(x)=\frac{x}{x+1}$ is uniformly continuous on $[0,2]$

Soln Let $\epsilon>0$ be given and $x_1,x_2\in[0,2]$ be arbitrary

$\therefore 0\le x_i\le 2 \Rightarrow 1\le 1+x_i\le 3 \Rightarrow \frac{1}{3}\le\frac{1}{1+x_i}<1 \quad i=1,2$ — (1)

$\therefore |f(x_1)-f(x_2)|=\left|\frac{x_1}{x_1+1}-\frac{x_2}{x_2+1}\right|=\frac{|x_1-x_2|}{|(x_1+1)(x_2+1)|}<|x_1-x_2|<\epsilon$ (from (1))

select $\delta=\epsilon$. Then $|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\epsilon$ //.

Ex Prove that $f(x)=\begin{cases}\sin(\frac{1}{x}) & x\ne 0\\ 0 & x=0\end{cases}$ is not uniformly continuous on $[0,\infty)$

Soln Choose $\epsilon = \frac{1}{2}$ and suppose $\delta > 0$ is arbitrary.

Let $x_1 = \frac{1}{n\pi}$ & $x_2 = \frac{1}{(n\pi + \frac{\pi}{2})}$, $n \in \mathbb{N}$ Clearly $x_1, x_2 \in (0, a)$

Then, $|x_1 - x_2| = \frac{1}{2n(n\pi + \frac{\pi}{2})}$ which can be made less than δ (by choosing n large)

$\therefore |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| = |\sin(\frac{1}{x_1}) - \sin(\frac{1}{x_2})|$

$= |\sin n\pi - \sin(n\pi + \frac{\pi}{2})|$

$= |0 - (-1)^n| = 1 > \frac{1}{2} (= \epsilon)$

Hence, f is not uniformly continuous

Note) If in $f(x)$ it is $\sin(\frac{1}{x^2})$ take $x_1 = \frac{1}{\sqrt{n\pi}}$ & $x_2 = \frac{1}{\sqrt{n\pi + \frac{\pi}{2}}}$

Ex Show that $f(x) = \frac{1}{1-x}$ $\forall x \in (0,1)$ is not uniformly continu...

soln Let $x_1, x_2 \in (0,1) \Rightarrow 0 \le x_i \le 1$

$\Rightarrow 0 \le 1 - x_i \le 1$ or, $1 \le \frac{1}{1-x_i} < \infty$

Take $x_1 = \frac{n-1}{n}$ & $x_2 = \frac{n}{n+1}$ clearly $x_1, x_2 \in (0,1)$, $n \in \mathbb{N}$.

Choose $\epsilon = \frac{1}{2}$. $|x_1 - x_2| = |\frac{n-1}{n} - \frac{n}{n+1}| = \frac{1}{n(n+1)} < \delta$ (choose n large)

$\therefore |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| = |\frac{1}{1 - (\frac{n-1}{n})} - \frac{1}{1 - \frac{n}{(n+1)}}|$

$= |n - (n+1)| = 1 > \frac{1}{2} (= \epsilon)$

$\Rightarrow f$ is not uniformly continuous

Theorem If f is uniformly continuous on interval I, then it is continuous on I.

Proof: f is uniformly continuous $\Rightarrow$ Given for any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$.

Choose $x_2 = c$ & $x_1 = c$

$\Rightarrow |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$\Rightarrow f$ is continuous at $x = c$ and hence in I.

Note: Converse is not true. e.g. $f(x) = \frac{1}{x}$ $\forall x \in (0,1)$ is continuous but not uniformly continuous.

Theorem: If a fn f is continuous on $[a,b]$ then it is uniformly continuous on $[a,b]$

Proof: Let fn f be continuous on $[a,b]$. Assume f is not unif continuous.

$\Rightarrow \exists\, \epsilon > 0$ s.t. for any $\delta > 0$, $x, y \in [a,b]$ for which $|f(x) - f(y)| \not< \epsilon$ whenever $|x - y| < \delta$ can be found s.t.

$\Rightarrow$ For each $n \in \mathbb{N}$ real no's $x_n, y_n \in [a,b]$

$|f(x_n) - f(y_n)| \not< \epsilon$ whenever $|x_n - y_n| < \frac{1}{n}$ — (1)

$\langle x_n \rangle$ & $\langle y_n \rangle$ are sequences in $[a,b]$ and are bdd

$\Rightarrow$ Each has a limit point and as $[a,b]$ is closed it belong to $[a,b]$

Let $\langle x_n \rangle \to x$

& $\langle y_n \rangle \to y$, $x, y \in [a,b]$

$\Rightarrow \exists$ a convergent subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ and $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ s.t. $x_{n_k} \to x$ & $y_{n_k} \to y$ as $k \to \infty$.

From (1) $|f(x_{n_k}) - f(y_{n_k})| \not< \epsilon$ whenever $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$

$\Rightarrow f(x_{n_k}) \neq f(y_{n_k})$

$\therefore \lim_{n \to \infty} f(x_{n_k}) \neq \lim_{n \to \infty} f(y_{n_k})$

$\Rightarrow x \neq y$ — (3)

$\Rightarrow \lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} y_{n_k}$

i.e, $x = y$ — (2)

From (2) & (3) we have contradiction

$\therefore$ Assumption was wrong.

Hence, f is uniformly continuous //