Unit II

Neighbourhood of a point ($x=c$): An open ~~open~~ interval around $x=c$ is called nbd of point 'c' i.e, $(c-r, c+r)$ where $r>0$

Let $f(x)$ be fn defined at all points in nbd N of 'c' except possibly at 'c' itself.

Limit: The fn $f(x)$ is said to tend to limit as x tends to 'c', if for each $\epsilon>0$, $\exists \delta>0$ such that

$$0<|x-c|<\delta \Rightarrow |f(x)-l|<\epsilon$$

$$\cong f(x) \in (l-\epsilon, l+\epsilon) \ \forall \ x \in (c-\delta, c+\delta) \sim \{c\}$$

Infinite limits: The fn $f(x)$ is said to tend to ∞ ($-\infty$) as x tends to 'c', if for each $G>0$ (however large), $\exists \delta>0$ such that $f(x)>G$, when $|x-c|<\delta$

$(f(x)<-G)$.

i.e, $\lim_{x\to c} f(x)=\infty \ (-\infty)$

Limit approaching ∞: The fn $f(x)$ is said to tend to limit l as x tends to ∞ if for each $\epsilon>0$, $\exists K>0$ such that

$|f(x)-l|<\epsilon$ when $x>K$

Note: 1) In general δ depends on ϵ

(2) δ is not unique

(3) ϵ can be replaced by any of its multiple (i.e, $k\epsilon$, $k\in R$), $\sqrt{\epsilon}$ etc.

Left hand limit: A fn f tends to limit l as x tends to c from left if for each $\epsilon>0$, $\exists \delta>0$ st

$|f(x)-l|<\epsilon$ when $c-\delta<x<c$ $\cong \lim_{x\to c^-} f(x)=l$.

* replace, left by right & $c-\delta<x<c$ by $c<x<c+\delta$ for defn of Right hand limit

Ex Find L.H.L & R.H.L of fn defined as $f(x) = \begin{cases} \frac{x-2}{|x-2|} & x \neq 2 \\ 0 & x = 2 \end{cases}$
Does the limit exist? If yes, find the limit

Soln $|x-2| = \begin{cases} x-2 & x \geq 2 \\ 2-x & x < 2 \end{cases}$

$\therefore$ L.H.L $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x-2}{2-x} = -1$

R.H.L $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$

L.H.L $\neq$ R.H.L
$\therefore$ limit does not exist.

Ex Find $\lim_{x \to 0} \frac{x e^{1/x}}{1+e^{1/x}}$

Soln If $x \to 0^+ \Rightarrow \frac{1}{x} \to \infty$ & hence $e^{1/x} \to \infty$. So, divide

Consider $\frac{x}{e^{-1/x}+1} \to 0$ as $x \to 0^+$

If $x \to 0^- \Rightarrow \frac{1}{x} \to -\infty \quad \therefore \frac{x e^{1/x}}{1+e^{1/x}} \to 0$

Hence zero may be the limit. Now,

$\left| \frac{x e^{1/x}}{1+e^{1/x}} - 0 \right| = |x| \left| \frac{e^{1/x}}{1+e^{1/x}} \right| = |x| \left| 1 - \frac{1}{1+e^{1/x}} \right| < |x| < \epsilon$

Choose $\delta = \epsilon$. Then

$\left| \frac{x e^{1/x}}{1+e^{1/x}} - 0 \right| < \epsilon$ when $|x-0| < \delta$. $\Rightarrow \lim_{x \to 0} \frac{x e^{1/x}}{1+e^{1/x}} = 0$ //

Ex Show that $\lim_{x \to 1} \frac{4}{(x-1)^2} = \infty$

Soln Let $G > 0$ be a large number. Then $\left| \frac{4}{(x-1)^2} \right| > G$

$\Rightarrow (x-1)^2 > \frac{G}{4} \Rightarrow |x-1| < \frac{\sqrt{G}}{2} = \delta$.

$\Rightarrow \frac{4}{(x-1)^2} > G$ when $0 < |x-1| < \delta$ (where $\delta = \frac{\sqrt{G}}{2}$). Hence,

$\lim_{x \to 1} \frac{4}{(x-1)^2} = \infty$ //

Ex Find $\lim_{x\to 0} e^x \operatorname{sgn}(x+[x])$, where signum fn is defined as $\operatorname{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ where $[x]$ means greatest integer $\le x$.

Soln L.H.L: $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} e^{-h}\operatorname{sgn}(-h+[-h])$

$$= \lim_{h\to 0} e^{-h}\operatorname{sgn}(-h-1) = -1$$

R.H.L: $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h) = \lim_{h\to 0} e^{h}\operatorname{sgn}(h+[h])$

$$= \lim_{h\to 0} e^{h}\operatorname{sgn}(h+0] = 1$$

$\because$ LHL $\neq$ R.HL, the limit does not exist.

Theorem: If $\lim_{x\to a} f(x)$ exists, then limit must be unique

Proof: Assume that $f(x) \to l_1 \,\&\, l_2$ where $l_1 \neq l_2$

$\therefore$ For given $\epsilon>0$, $\exists\, \delta_1$ s.t. $|f(x)-l_1| < \epsilon/2$ when $|x-a|<\delta_1$

& for given $\epsilon>0$, $\exists\, \delta_2$ s.t $|f(x)-l_2| < \epsilon/2$ when $|x-a|<\delta_2$

Let $\delta = \min(\delta_1, \delta_2)$. So,

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \le |f(x)-l_1| + |f(x)-l_2| < \epsilon \quad \text{when } |x-a|<\delta$$

$\Rightarrow l_1 = l_2$ // (In unit I we had proved that "if for any $\epsilon>0$ $|b-a|<\epsilon$ then $a=b$)

Theorem on limits: Let f and g be 2 fns defined on some nbd of point 'c', s.t $\lim_{x\to c} f(x) = l$ & $\lim_{x\to c} g(x) = m$, then (1) $\lim_{x\to c} (f\pm g)(x) = l \pm m$ (2) $\lim_{x\to c} (fg)(x) = lm$

(3) $\lim_{x\to c} \left(\frac{f}{g}\right)(x) = \cancel{\frac{f(x)}{g(x)}}\ \frac{l}{m}$, $m\neq 0$ (4) $\lim_{x\to c} (kf)(x) = k f(x)$, $k\in\mathbb{R}$.

Proof! (1) Do it yourself (Hint: Theorem proved above)

(2) Let $\epsilon > 0$ be given. Then

$|(fg)(x) - lm| = |(fg)(x) - mf(x) + mf(x) - lm|$

$\leq |f(x)|\,|g(x) - m| + |m|\,|f(x) - l|$ — (1)

$\because f(x) \to l$ as $x \to c \Rightarrow$ for each $\epsilon > 0$, $\exists\, \delta_1 > 0$ s.t.

$|f(x) - l| < 1$ (Take $\epsilon = 1$) when $0 < |x - c| < \delta_1$,

$\Rightarrow |f(x)| < 1 + |l|$ — (2)

$\because g(x) \to m$ as $x \to c \Rightarrow \exists\, \delta_2 > 0$ s.t. $|g(x) - m| < \dfrac{\epsilon/2}{1+|l|}$ — (3)

when $0 < |x - c| < \delta_2$

$\because f(x) \to l$ as $x \to c$, $\exists\, \delta_3 > 0$ s.t.

$|f(x) - l| < \dfrac{\epsilon/2}{|m|}$ when $0 < |x - c| < \delta_3$ — (4)

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. From (1), (2), (3) & (4) we have

$|(fg)(x) - lm| < \cancel{(1+|l|)}\dfrac{\epsilon/2}{\cancel{(1+|l|)}} + \cancel{|m|}\dfrac{\epsilon/2}{\cancel{|m|}} = \epsilon.$

$\Rightarrow \lim\limits_{x \to c} (fg)(x) = lm$ //

(3) & (4) Do yourself

Ex Evaluate

(1) $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5} = \lim\limits_{x \to 5} \dfrac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}} = 10$ //

(2) $\lim\limits_{x \to 1} \dfrac{1}{(x-1)}\left[\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right] = \lim\limits_{x \to 1} \dfrac{1}{\cancel{(x-1)}}\left[\dfrac{\cancel{(x-1)}}{(x+3)(3x+5)}\right] = \dfrac{1}{32}$ //

(3) $\lim\limits_{x \to 0} \dfrac{1 - 2\cos x + \cos 2x}{x^2} = \lim\limits_{x \to 0} \dfrac{\cancel{1} - 2\cos x + 2\cos^2 x - \cancel{1}}{x^2}$

$= \lim\limits_{x \to 0} \dfrac{2\cos x(\cos x - 1)}{x^2} \times \dfrac{(\cos x + 1)}{(\cos x + 1)}$

$$= \lim_{x\to\infty} \left(\frac{2\cos x}{1+\cos x}\right)\left(\frac{-\sin^2 x}{x^2}\right) = -2\left(\lim_{x\to 0} \frac{\cos x}{1+\cos x}\right)\left(\lim_{x\to 0} \frac{\sin^2 x}{x^2}\right)$$

$$= -2\left(\frac{1}{2}\right)(1) = -1 \qquad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

<u>Ex</u> Show that $\displaystyle\lim_{x\to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}}$ does not exist

<u>Soln</u> <u>L.H.L</u> $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \therefore e^{1/x} \to 0$

$$\therefore \lim_{x\to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}} = \lim_{x\to 0^-} \frac{e^{2/x}-1}{e^{2/x}+1} = -1$$

<u>RHL</u> $\displaystyle\lim_{x\to 0^+} \frac{1}{x} \to \infty \quad \therefore e^{-1/x} \to 0$

$$\therefore \lim_{x\to 0^+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}+e^{-1/x}} = \lim_{x\to 0^+} \frac{1-e^{-2/x}}{1+e^{-2/x}} = 1$$

L.H.L $\neq$ R.H.L

$\therefore$ limit does not exist.