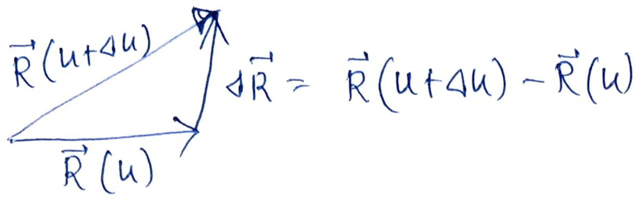
Vector Differentiation

Ordinary derivative of vectors



Let $\vec{R}(u)$ be a vector depending on a single scalar variable u. Then

$$\frac{\Delta \vec{R}}{\Delta u} = \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

where Δu denotes an increment in u.

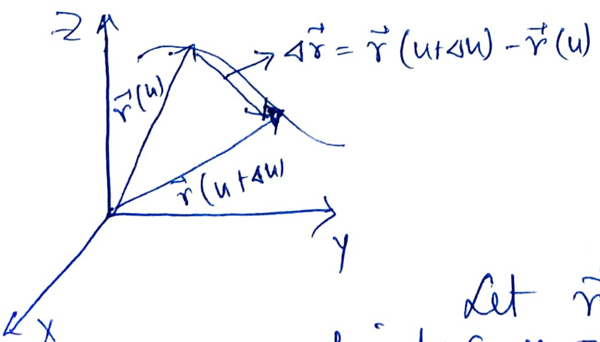
The ordinary derivative of the vector $\vec{R}(u)$ w.r.t scalar u is given by

$$\frac{d\vec{R}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{R}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

if the limit exists.

Since $\frac{d\vec{R}}{du}$ is itself a vector depending upon u, we can define its higher derivatives w.r.t u such as $\frac{d^2\vec{R}}{du^2}$ if this derivative exists and so on.

Space curves



Let $\vec{r}(u)$ be the position vector of a point (x, y, z) then we can write

$$\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$$

As u changes the terminal point of $\vec{r}(u)$ describes a space curve having parametric eqn.

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then $\frac{\Delta\vec{r}}{\Delta u} = \frac{\vec{r}(u+\Delta u) - \vec{r}(u)}{\Delta u}$ is a vector in the direction of $\Delta\vec{r}$.

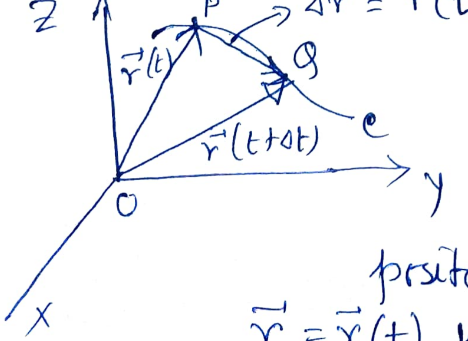
If $\lim\limits_{\Delta u \to 0} \frac{\Delta\vec{r}}{\Delta u} = \frac{d\vec{r}}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x,y,z) and is given by

$$\frac{d\vec{r}}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}$$

If u is the time t, $\frac{d\vec{r}}{dt}$ represents the velocity vector $\vec{v}$.
Similarly $\frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ represents the acceleration vector $\vec{a}$ along the curve.

Velocity

$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$



Suppose that a particle moves along a path or curve C. Let the position vector of point P at time t be $\vec{r} = \vec{r}(t)$ while the position vector of point Q at time $t+\Delta t$ is $\vec{r} + \Delta\vec{r} = \vec{r}(t+\Delta t)$.

Then the instantaneous velocity of the particle at P is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

and $\vec{v}$ is a vector tangent to the curve c at P.
So instantaneous velocity of a particle is everywhere tangent to the trajectory.

If $\vec{r} = \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$
then we can write

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

The magnitude of the velocity is called the speed and is given by

$$v = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}$$

where s is the arc length along c.

Similarly we can define instantaneous acceleration vector $\vec{a}$, like

If $\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity of a particle, we define the instantaneous acceleration at P as

$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

Notation for time derivatives

$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt}$, $\vec{a} = \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}$ and so on...

Partial Derivative of Vectors

If $\vec{A}$ is a vector depending on more than one scalar variable say x, y, z then we write $\vec{A} = \vec{A}(x, y, z)$.
The partial derivative of $\vec{A}$ w.r.t x is defined as

$$\frac{\partial \vec{A}}{\partial x} = \underset{\Delta x \to 0}{\text{Lt.}} \frac{\vec{A}(x+\Delta x, y, z) - \vec{A}(x, y, z)}{\Delta x}$$ if this limit exists

Similarly

$$\frac{\partial \vec{A}}{\partial y} = \underset{\Delta y \to 0}{\text{Lt.}} \frac{\vec{A}(x, y+\Delta y, z) - \vec{A}(x, y, z)}{\Delta y}$$ if this limit exists

and $$\frac{\partial \vec{A}}{\partial z} = \underset{\Delta z \to 0}{\text{Lt.}} \frac{\vec{A}(x, y, z+\Delta z) - \vec{A}(x, y, z)}{\Delta z}$$ if this limit exists.

Ordinary Derivatives

Suppose we have a function of one variable $f(x)$. Now the question is: what does the derivative $\frac{df}{dx}$ do for us? The answer is that it tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount dx:

$$df = \frac{df}{dx}\, dx$$

That is if we increment x by an infinitesimal amount dx, then the function f changes by an amount df; here the derivative $\frac{df}{dx}$ is the proportionality factor.

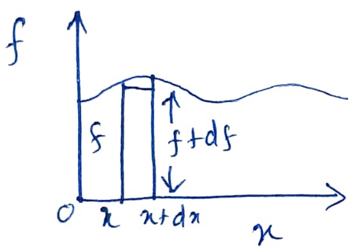


Fig (a)

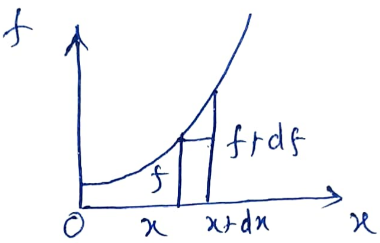


Fig (b)

For example, in Fig (a), the function 'f' varies slowly with x, and the derivative is correspondingly small. In Fig (b), the function f increases rapidly and the derivative is large, as one moves away from $x=0$.

Geometrical interpretation: The derivative df/dx is the slope of the curve of f versus x.

Gradient

Suppose that we have a function of three variables — say, the temperature $T(x, y, z)$ of a room. Now we want to generalize the notion of derivative to functions like T, which depends on three variables instead of one.

A derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what direction we move. If we go straight up, then temperature will probably increase fairly rapidly, but if we move horizontally it may not change much at all. In fact the question "how fast does T vary?" has an infinite number of answers, one for each direction

Theorem on partial derivative which states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \quad \text{———} (1)$$

This tells us how T changes when we change three variables (x, y, z) by the infinitesimal amount dx, dy, dz simultaneously

Equation (1) is reminiscent of a dot product:

$$dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$\boxed{\text{or,} \quad dT = \vec{\nabla} T \cdot d\vec{r}} \quad \text{———} (2)$$

where $\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ is known as gradient of $T(x, y, z)$. $\vec{\nabla} T$ (gradient or grad of T) is a vector quantity with three components. This (Eqn. 2) is the 3-dimensional version of $df = \frac{df}{dx} dx$ equation.

Geometrical interpretation of gradient

Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning, let us rewrite the dot product (Eq^n. ②)

$$dT = \vec{\nabla}T \cdot d\vec{r} = |\vec{\nabla}T||d\vec{r}|\cos\theta \quad \text{---} (3)$$

where θ is the angle between $\vec{\nabla}T$ and $d\vec{r}$. Now if we fix the magnitude $|d\vec{r}|$, then maximum change in T occurs when $\theta = 0$ (ie $\cos\theta = 1$). That is, for a fixed distance $|d\vec{r}|$, dT (change in T) is greatest when $\vec{\nabla}T$ and $d\vec{r}$ both are in the same direction. Thus:

The gradient $\vec{\nabla}T$ points in the direction of maximum increase of the function T.

Moreover:

The magnitude $|\vec{\nabla}T|$ gives the slope (rate of increase) along this maximal direction.

Example

Imagine you are standing on a hillside. Look all around you, and find the ~~steepest~~ direction of steepest ascent. That is the direction of the gradient. Now measure the slope in that direction (that is rise over run). That is the magnitude of the gradient. (Here the function we are talking about is the height of the hill and the coordinates it depends on are position — latitude and longitude, say. This function (height) depends on two variables (latitude and longitude) only instead of three.

The Del operator ($\vec{\nabla}$)

The del operator or vector differential operator is denoted by $\vec{\nabla}$ which is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Of course, $\vec{\nabla}$ is not a vector in the usual sense until we provide it with a scalar function to act upon. This del operator possesses properties analogous to those of ordinary vectors.

The Gradient

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of the scalar field ϕ is defined as

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Note $\vec{\nabla}\phi$ is a vector field.

The component of $\vec{\nabla}\phi$ in the direction of a unit vector $\hat{a}$ is given by $\vec{\nabla}\phi \cdot \hat{a}$. This is called the directional derivative of ϕ in the direction of $\hat{a}$. Physically this is the rate of change of ϕ at (x, y, z) in the direction of $\hat{a}$.

The Divergence

From the definition of $\vec{\nabla}$ we can construct the divergence of a vector field as follows

Let $\vec{v}(x,y,z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be a differentiable vector field in a certain region of space.

We define the divergence of a vector field $\vec{v}(x,y,z)$ as (written as $\vec{\nabla}\cdot\vec{v}$)

$$\vec{\nabla}\cdot\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \Rightarrow \text{scalar quantity.}$$

Geometrical interpretation of divergence

$\vec{\nabla}\cdot\vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question or in the neighbourhood of a point in question

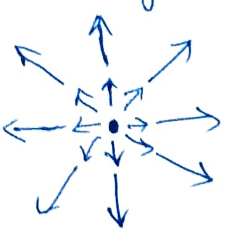


Fig ①

For example the vector function in fig ① has a large +ve divergence (if the arrows pointed in it would be a large −ve divergence).

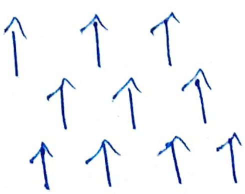


Fig ②

In fig ② the vector function has zero divergence.

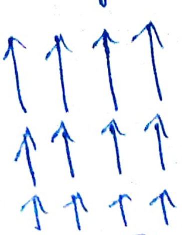


Fig ③

In fig ③ the vector function has again a +ve divergence.

Note that a vector whose divergence is zero ie, $\vec{\nabla}\cdot\vec{V}=0$ then it is known as solenoidal.

The Curl

From the definition of $\vec{\nabla}$ we construct curl.
If $\vec{V}(x,y,z)$ is a differentiable vector field then the curl or rotation of $\vec{V}$ (written as $\vec{\nabla}\times\vec{V}$) is defined as

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times\left(V_1\hat{i}+V_2\hat{j}+V_3\hat{k}\right)$$

It can be written in the determinant form as

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)-\hat{j}\left(\frac{\partial V_3}{\partial x}-\frac{\partial V_1}{\partial z}\right)+\hat{k}\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)$$

Geometrical Interpretation of curl

The curl of a vector field $\vec{V}$ is a measure of how much the vector field $\vec{V}$ curls around the point in question

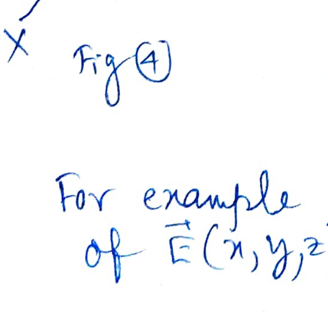


Fig (4)

In fig (4) the vector function has substantial curl pointing in the z-direction as the right-hand rule suggests.

A vector whose curl is zero ie $\vec{\nabla}\times\vec{V}=0$ is known as irrotational vector.

For example electrostatic field $\vec{E}(x,y,z) \Rightarrow$ The curl of $\vec{E}(x,y,z)$ is zero ie $\vec{\nabla}\times\vec{E}=0$, so $\vec{E}$ is known as irrotational vector or conservative

Another example of curl

Imagine you are standing at the edge of a pond. Float a small paddle wheel (a cork with toothpick pointing radially out would do), if it starts to rotate then you placed it at a point of nonzero curl. A whirlpool would be a region of large curl.

Again if we take the curl of static magnetic field $\vec{B}(x,y,z)$ due to a steady current, we find that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ where $\vec{J}$ is the current density.

Thus we see that static magnetic field $\vec{B}$ has nonzero curl.