Lecture - 16

Z-Transform

1.1 Introduction :

Fourier transform provides a valuable technique for frequency domain analysis and design of continuous time signals and LTI systems. While the Z-transform provides a valuable technique for analysis and design of discrete time signals and discrete time LTI systems.

The Z-transform has real and imaginary parts like fourier transform. A plot of imaginary part versus real part is called as Z-plane or complex Z-plane. The poles and zeros of discrete time system are plotted in the complex Z-plane. The pole-zero plot is main characteristic of discrete time LTI systems. We can also check the stability of system using pole-zero plot.

1.1.1 Advantages of Z-transform :

The importance of Z-transform is as follows :

1. Discrete time signals and LTI systems can be completely characterized using Z-transform.
2. The stability of LTI system can be determined using Z-transform.
3. Mathematical calculations are reduced using Z-transform. For example convolution operation is transformed into simple multiplication operation.
4. By calculating Z-transform of given signal, DFT and FT can be determined.
5. Entire family of digital filters can be obtained from one proto-type design using Z-transform.
6. The solution of differential equations can be simplified using Z-transform.

1.2 Z-Transform :

There are two types of Z-transform :

1. Single sided Z-transform
2. Double sided Z-transform.

1. Single sided Z-transform :

Definition : A single sided Z-transform of discrete time signal x (n) is defined as,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad \ldots(1)$$

Here "Z" is a complex variable. In Equation (1), limits of summation are from 0 to ∞. So while expanding the summation we will put only positive values of n (from n = 0 to n = ∞). So this is single sided or one-sided Z-transform.

2. Double sided Z-transform :

Definition : A double sided Z-transform of discrete time signal x (n) is defined as,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(2)$$

While expanding the summation, we will put both positive and negative values of 'n'. Thus this is double-sided Z- transform.

What is Z-domain ?

Observe Equations (1) and (2). Here we are transforming discrete time sequence x(n) into X(Z). We know that discrete time sequence x(n) is drawn by plotting amplitude versus 'n'. This is called as discrete domain. In case of Z-transform we are plotting real part of Z on X-axis and imaginary part of Z on Y-axis as shown in Fig. U-1. This is called as Z-domain.

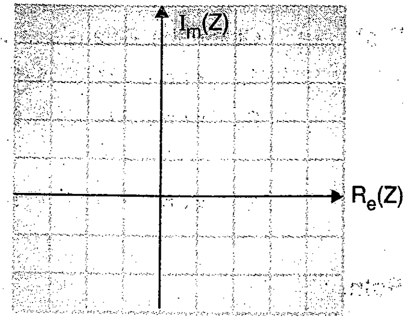


Fig. U-1 : Z-domain

How to denote Z-transform ?

The relationship between x(n) and X(Z) is indicated as follows :

$$x(n) \overset{Z}{\longleftrightarrow} X(Z) \quad ...(3)$$

Note that X(Z) is 'Z' transform of x(n). Always, when Z-transform of sequence is obtained then it is denoted by capital letter, 'X'(Z). Here the arrow is bidirectional. This is because we can also obtain x(n) from X(Z) using inverse Z-transform.

The 'Z' transform of x(n) is also denoted as,

$$X(Z) = \mathbf{Z}\{x(n)\} \quad ...(4)$$

Here x(n) and X(Z) are called as Z-transform pairs.

1.2.1 Region of Convergence (ROC) :

In case of Z-transform; the limits of summation are from $n = -\infty$ to $n = \infty$. So if we will expand this summation then we will get an infinite power series. This infinite power series (that means Z-transform) will exist only for those values of Z for which the series attains a finite value. (That means the series converges).

Definition of ROC :

The region of convergence (ROC) of X(Z) is set for all the values of Z for which X(Z) attains a finite value. Everytime when we find the Z-transform, we must indicate its ROC.

Significance of ROC :

1. ROC will decide whether a system (filter) is stable or unstable.
2. ROC also determines the type of sequence that means.
   (i) Causal or non-causal
   (ii) Finite or infinite.

Solved Problems :

Prob. 1 : Obtain the Z-transform of following finite duration sequences.

1. $x(n) = \{1, 2, 4, 5, 0, 7\}$
2. $x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\}$
3. $x(n) = \{1, 2, \underset{\uparrow}{4}, 5, 0, 7\}$

Soln. :

1. Here arrow is not mentioned. So by default it is at first position.

$$\therefore \quad x(n) = \{\underset{\uparrow}{1}, 2, 4, 5, 0, 7\} \quad ...(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(2)$$

But x (n) is present from n = 0 to n = 5. The different values are as follows :

$x(0) = 1$	$x(3) = 5$
$x(1) = 2$	$x(4) = 0$
$x(2) = 4$	$x(5) = 7$

Thus we will change the limits of summation from n = 0 to n = 5.

$$\therefore \quad X(Z) = \sum_{n=0}^{5} x(n) Z^{-n} \quad ...(3)$$

Expanding the summation we get,

$$\therefore \quad X(Z) = x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + x(4)Z^{-4} + x(5)Z^{-5}$$

$$\therefore \quad X(Z) = 1.Z^{0} + 2Z^{-1} + 4Z^{-2} + 5Z^{-3} + 0.Z^{-4} + 7Z^{-5}$$

But $Z^0 = 1$

$$\therefore X(Z) = 1 + \frac{2}{Z} + \frac{4}{Z^2} + \frac{5}{Z^3} + \frac{7}{Z^5} \quad ...(4)$$

ROC : This is finite duration sequence. For this sequence we will check for which values of 'Z'; the value of X (Z) becomes infinity. The simple method to decide the ROC is, put Z = 0 and Z = ∞ in the equation of X (Z).

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 1+\frac{2}{0}+\frac{4}{0}+\frac{5}{0}+\frac{7}{0} = 1+\infty = \infty$$

We know that $\frac{2}{0}, \frac{4}{0}$ etc. are equal to ∞. And $1+\infty$ is again ∞. Thus Z = 0 is not valid as it results $X(Z) = \infty$.

(ii) Putting Z = ∞ in Equation (4) we get,

$$X(Z) = 1+\frac{2}{\infty}+\frac{4}{\infty}+\frac{5}{\infty}+\frac{7}{\infty} = 1+0 = 1$$

Here $\frac{2}{\infty}, \frac{4}{\infty} = 0$. Thus putting Z = ∞ we get finite value. So Z = ∞ is allowed.

Now ROC of given sequence is written as follows :

ROC : Entire Z-plane except | Z | = 0

This ROC is shown in Fig. U-2.

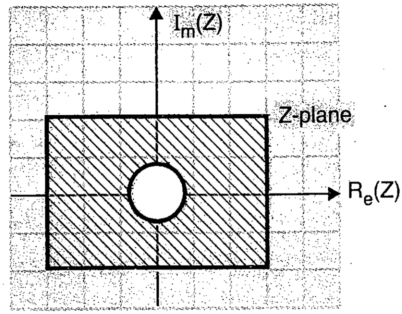


Fig. U-2 : ROC

> **Note :** This sequence is causal, since x(n) is present only for positive values of n. Thus for a causal finite duration sequence ROC is entire Z-plane except | Z | = 0.

2. Given sequence is,

$$x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\} \quad ...(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(2)$$

Note that the range of given sequence is from n = −5 to n = 0. The different values of x(n) are :

$x(-5) = 1$ $\quad$ $x(-2) = 5$

$x(-4) = 2$ $\quad$ $x(-1) = 0$

$x(-3) = 4$ $\quad$ $x(0) = 7$

We will change the limits of summation from n = − 5 to n = 0.

$$\therefore \quad X(Z) = \sum_{n=-5}^{0} x(n) Z^{-n} \quad ...(3)$$

Expanding the summation we get,

$$X(Z) = x(-5)Z^{+5} + x(-4)Z^{+4} + x(-3)Z^{+3} + x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0}$$

$$\therefore \quad X(Z) = 1Z^5 + 2Z^4 + 4Z^3 + 5Z^2 + 0Z^1 + 7Z^0$$

$$X(Z) = Z^5 + 2Z^4 + 4Z^3 + 5Z^2 + 7 \qquad ...(4)$$

ROC : We will determine ROC by putting $Z = 0$ and $Z = \infty$.

(i) Putting $Z = 0$ in Equation (4) we get,

$$X(Z) = 0+0+0+0+7 = 7$$

This is a finite value. So $Z = 0$ is allowed.

(ii) Putting $Z = \infty$ in Equation (4) we get,

$$X(Z) = \infty+\infty+\infty+\infty+7 = \infty+7 = \infty$$

This is because $\infty + 7$ etc. $= \infty$. So $Z = \infty$ is not allowed.

Thus ROC is entire Z-plane except $|Z| = \infty$. This ROC is shown in Fig. U-3.

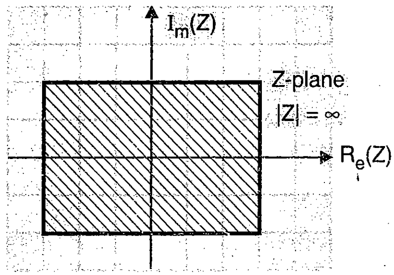


Fig. U-3 : ROC

Note : The given sequence $x(n)$ is anticausal. This is because $x(n)$ is present only for negative values of 'n'. Thus for anticausal finite duration sequence, ROC is entire Z-plane except $|Z| = \infty$

3. Given sequence is,

$$x(n) = \{1, 2, \underset{\uparrow}{4}, 5, 0, 7\} \qquad ...(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

But $x(n)$ is present from $n = -2$ to $n = 3$. The different values of $x(n)$ are as follows :

$x(-2) = 1$ $\quad$ $x(1) = 5$

$x(-1) = 2$ $\quad$ $x(2) = 0$

$x(0) = 4$ $\quad$ $x(3) = 7$

We will change the limits of summation from $n = -2$ to $n = 3$.

$$\therefore \quad X(Z) = \sum_{n=-2}^{n=3} x(n) Z^{-n} \qquad ...(3)$$

Expanding the summation we get,

$$X(Z) = x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$\therefore \quad X(Z) = 1Z^2 + 2Z^1 + 4Z^0 + 5Z^{-1} + 2Z^{-2} + 7Z^{-3}$$

$$\therefore X(Z) = Z^2 + Z + 4 + \frac{5}{Z} + \frac{2}{Z^2} + \frac{7}{Z^3} \quad ...(4)$$

ROC : We will determine ROC by putting Z = 0 and Z = ∞ in Equation (4).

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 0 + 0 + 4 + \frac{5}{0} + \frac{2}{0} + \frac{7}{0} = 4 + \infty = \infty$$

This is because $\frac{5}{0}, \frac{2}{0}$ and $\frac{7}{0} = \infty$

Thus Z = 0 is not allowed.

(ii) Putting Z = ∞ in Equation (4) we get,

$$X(Z) = \infty + \infty + 4 + \frac{5}{\infty} + \frac{2}{\infty} + \frac{7}{\infty} = \infty$$

Thus Z = ∞ is not allowed.

ROC : ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

This ROC is shown in Fig. U-4.

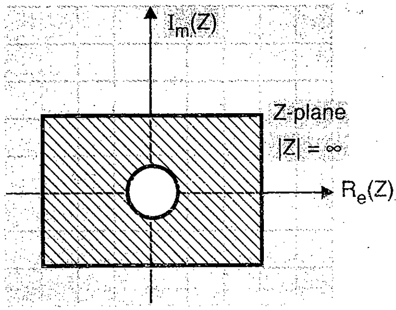


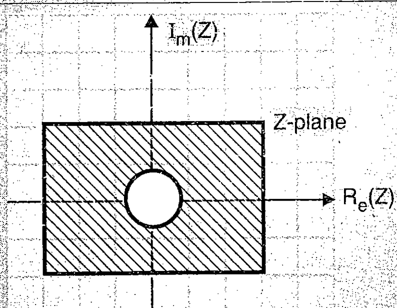
Fig. U-4 : ROC

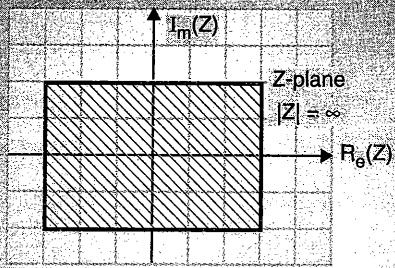
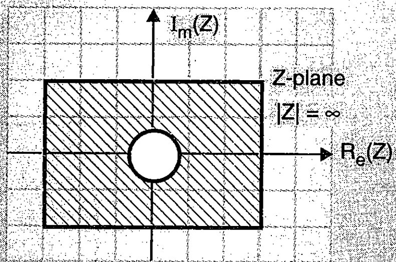
> **Note :** The given sequence is a bothsided sequence. This is because x (n) is present for both positive and negative values of 'n'. Thus for bothsided finite duration sequence the ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

Summary of ROC :

Table U-1 shows the ROC for different finite duration sequences.

Table U-1

Sr. No.	Finite duration sequence	ROC (Diagram)	ROC (Statement)
1)	Causal sequence		Entire Z-plane except | Z | = 0

Sr. No.	Finite duration sequence	ROC (Diagram)	ROC (Statement)
2)	Anticausal sequence		Entire Z-plane except $\lvert Z \rvert = \infty$
3)	Two sided sequence		Entire Z-plane except $\lvert Z \rvert = 0$ and $\lvert Z \rvert = \infty$

1.2.2 Z-transform of Standard Sequences :

In this section we will obtain Z-transform of some standard sequences. Most of the standard sequences are having infinite duration. But there are some exceptions like unit impulse, $\delta(n)$.

1. Z-transform of unit impulse $\delta(n)$:

Unit impulse $\delta(n)$ is shown in Fig. U-5. It is given by,

$$\delta(n) = 1 \quad \text{only at } n = 0$$
$$= 0 \quad \text{otherwise}$$

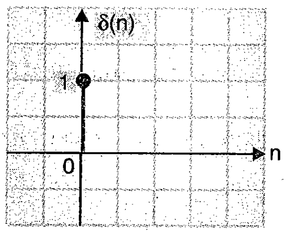


Fig. U-5 : Unit impulse $\delta(n)$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(1)$$

Here $x(n) = \delta(n)$

$$\therefore \; X(Z) = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n} \qquad ...(2)$$

Since $\delta(n)$ is present only at $n = 0$ we can directly write,

$$X(Z) = \delta(0) Z^0$$

But $\delta(0) = 1$ and $Z^0 = 1$

$$\therefore \; X(Z) = 1 \qquad \text{...(3)}$$

ROC : In Equation (3), there is no 'Z' term. So *ROC is entire Z-plane.* That means Z can have any value.

Thus the Z-transform pair is

$$\delta(n) \overset{Z}{\longleftrightarrow} 1$$

2. Z-transform of delayed unit impulse, $\delta(n-k)$:

Here $\delta(n-k)$ is a delayed unit impulse. It indicates that $\delta(n)$ is delayed by 'k' samples. It is shown in Fig. U-6.

It is given by,

$$\delta(n-k) = 1 \quad \text{only at } n = k \text{ and } k > 0$$
$$= 0 \quad \text{otherwise}$$

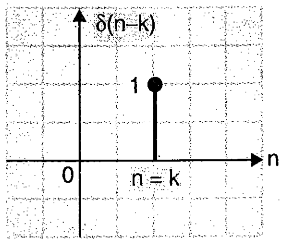


Fig. U-6 : Delayed unit impulse

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \text{...(1)}$$

Here $x(n) = \delta(n-k), \quad k > 0$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} \delta(n-k) Z^{-n}$$

But $\delta(n-k) = 1$ only at $n = k$. Thus we get,

$$X(Z) = 1 \cdot Z^{-k}$$

$$\therefore \; X(Z) = Z^{-k}$$

ROC :

Here $X(Z) = Z^{-k} = \frac{1}{Z^k}$. Since k is positive, $(k > 0)$ for any value of 'Z' (except $Z = 0$) we will get finite value of $X(Z)$. Thus *ROC is entire Z-plane except $Z = 0$.* This is because if we put $Z = 0$ then $X(Z) = \infty$. Thus the Z-transform pair is,

$$\delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}$$

3. Z-transform of advanced unit impulse, $\delta(n+k)$, $k > 0$:

Here $\delta(n+k)$, $k > 0$ is an advanced unit impulse. It indicates that $\delta(n)$ is advanced by '+ k' samples. It is shown in Fig. U-7.

It is given by,

$$\delta(n+k) = 1 \quad \text{only at } n = -k,\ k > 0$$
$$= 0 \quad \text{otherwise}$$

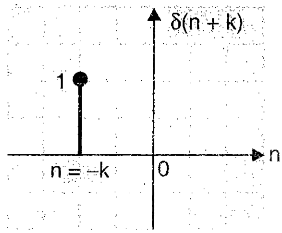


Fig. U-7 : Advanced unit impulse

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Here $x(n) = \delta(n+k)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} \delta(n+k) Z^{-n}$$

But $\delta(n+k) = 1$ only at $n = k$. Thus we get,

$$X(Z) = 1 \cdot Z^k$$

$$\therefore \quad X(Z) = Z^k$$

ROC :

Here $k > 0$. So we will get finite values of $X(Z)$ for all 'Z' except $Z = \infty$. Because for $Z = \infty$ we get, $X(Z) = (\infty)^k = \infty$. Thus *ROC is entire Z-plane except $Z = \infty$.*

Thus Z-transform pair is,

$$\delta(n+k) \overset{Z}{\longleftrightarrow} Z^{+k}$$

4. Z-transform of unit step, u (n) :

We know that u (n) is unit step as shown in Fig. U-8.

It is given by,

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise}$$

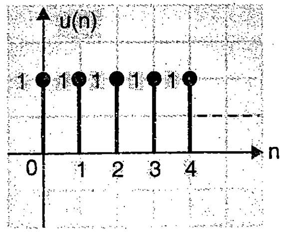


Fig. U-8 : Unit step

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(1)}$$

Here x (n) = u (n). Since u (n) is present from n = 0 to n = ∞ we will change the limits of summation.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} u(n) Z^{-n} \quad \text{...(2)}$$

But the magnitude of u (n) is always 1.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} 1 \cdot Z^{-n}$$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (Z^{-1})^n \quad \text{...(3)}$$

The standard equation of geometric series is,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + ... = 1 + A + A^2 + A^3 + ... = \frac{1}{1-A} \text{ if } |A| < 1 \quad \text{...(4)}$$

Let $A = Z^{-1}$. Thus Equation (3) becomes,

$$X(Z) = \frac{1}{1-A} \quad \text{if} \quad |A| < 1 = \frac{1}{1-Z^{-1}} \text{ if } |Z^{-1}| < 1$$

∴ Multiplying numerator and denominator by Z we get,

$$X(Z) = \frac{Z}{Z - Z^{-1} Z} \quad \text{if} \quad |Z^{-1}| < 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} \quad \textbf{if } |Z^{-1}| < 1$$

In Equation (4), the condition "if $|A| < 1$" indicates that if this condition is satisfied then only the geometric series is convergent otherwise not. In the given example we get this condition as, $|Z^{-1}| < 1$. That means this is the condition of ROC.

Thus ROC : $|Z^{-1}| < 1$

that means $\left|\frac{1}{Z}\right| < 1$

$\therefore \ |1| < |Z| \cdot 1$

$\therefore$ ROC is $|Z| > 1$

This ROC is plotted as shown in Fig. U-9.

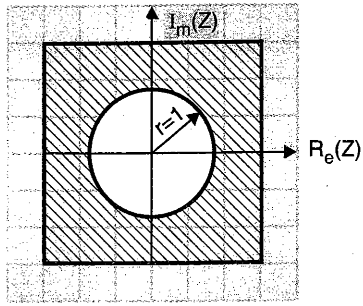


Fig. U-9 : ROC of unit step

Thus ROC is exterior part of circle having radius 1.

Thus the Z-transform pair is,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1} \text{ if } |Z| > 1$$

Why to draw circle only ?

The given ROC is $|Z| > 1$. In case of circle having radius 1, every point outside the periphery of circle satisfies this condition ($|Z| > 1$). For other diagrams like square or triangle this condition cannot be satisfied.

Note : The unit step is an infinite sequence and as shown in Fig. U-9, the range of $u(n)$ is from $n = 0$ to $n = \infty$. That means this sequence is present only for positive values of n. So it is a causal sequence.

5. Z-transform of unit ramp :

We know that unit ramp sequence is denoted by $r(n)$. It is as shown in Fig. U-10.

Its magnitude is as follows :

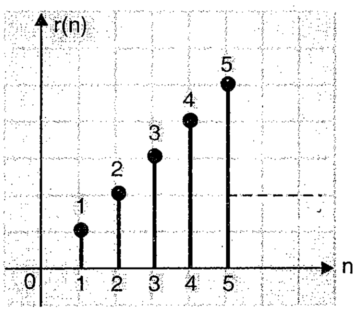


Fig. U-10 : Unit ramp

at $n = 0$, $r(n) = 0$

at $n = 1$, $r(n) = 1$

at $n = 2$, $r(n) = 2$

at $n = 3$, $r(n) = 3$ ...

Thus it is expressed as,

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise} \qquad ...(1)$$

Since this sequence is again causal sequence we can write

$$r(n) = n\,u(n) \qquad ...(2)$$

Note : Whenever any sequence is multiplied by u (n) then the magnitude of that sequence is not changed. But since u (n) is present, only for positive values of n ($n \geq 0$); the result of multiplication is a causal sequence. Thus multiplication of any sequence with u (n) always results in causal sequence.

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(3)$$

But here $x(n) = r(n) = n\,u(n)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} n\,u(n) Z^{-n} \quad ...(4)$$

Just now we discussed that the multiplication by u (n) results in a causal sequence. So the limits of summation will be from $n = 0$ to $n = \infty$. Remember that since the magnitude of u (n) is 1; it will not affect the calculation. It is just used to change the limits. So now onwards we will not write u (n) in the calculations.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} n Z^{-n} = \sum_{n=0}^{\infty} n (Z^{-1})^n \quad ...(5)$$

Now use standard summation formula,

$$\sum_{n=0}^{\infty} n A^n = \frac{A}{(1-A)^2} \quad \text{if } |A| < 1$$

Let $A = Z^{-1}$. Thus Equation (5) becomes,

$$X(Z) = \frac{Z^{-1}}{(1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

To convert this equation into positive powers of 'Z' we will multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2 \cdot Z^{-1}}{Z^2(1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2(1-2Z^{-1}+Z^{-2})} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2-2Z+1} \quad \text{if } |Z^{-1}| < 1$$

$$\therefore X(Z) = \frac{Z}{(Z-1)^2} \quad \text{if } |Z^{-1}| < 1$$

ROC : Here the condition "$|Z^{-1}| < 1$" indicates the ROC.

$$\therefore \quad \text{ROC is } \left|\frac{1}{Z}\right| < 1 \Rightarrow 1 < |Z|$$

$$\therefore \text{ROC is } |Z| > 1$$

Thus ROC is exterior part of circle having radius 1. This ROC is same as shown in Fig. U-9.

Thus the Z-transform pair is

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \quad \text{if } |Z| > 1$$

6. Z-transform of right hand exponential sequence :

As the name indicates, it is an exponential sequence present at the right hand that means only for positive values of 'n'. So it is a causal exponential sequence. Thus it is expressed as,

$$x(n) = \alpha^n u(n) = \alpha^n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0 \qquad ...(1)$$

This exponential sequence is shown in Fig. U-11.

According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

Here $x(n) = \alpha^n u(n)$. Since the sequence is multiplied by $u(n)$, the limits of summation will change. It will be $n = 0$ to $n = \infty$.

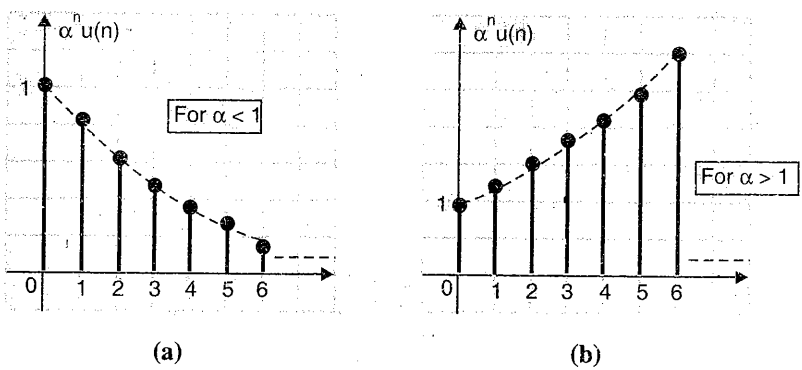


Fig. U-11 : Exponential sequence $\alpha^n u(n)$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} \alpha^n Z^{-n} \qquad ...(3)$$

Let us bring it in the form of A^n.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (\alpha Z^{-1})^n \qquad ...(4)$$

Now we have the equation of geometric series,

$$\sum_{n=0}^{\infty} A^n = \frac{1}{1-A} \quad \text{if } |A| < 1$$

Let $A = \alpha Z^{-1}$. Thus Equation (4) becomes,

$$X(Z) = \frac{1}{1-\alpha Z^{-1}} \quad \textbf{if } |\alpha Z^{-1}| < 1 \qquad ...(5)$$

To convert it into positive powers of Z; multiply numerator and denominator by Z.

$$\therefore \quad X(Z) = \frac{Z}{Z(1-\alpha Z^{-1})} \quad \text{if } |\alpha Z^{-1}| < 1$$

$$\therefore \quad X(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |\alpha Z^{-1}| < 1$$

ROC :

This condition $|\alpha Z^{-1}| < 1$ indicates the ROC.

$$\therefore \quad \text{ROC is } |\alpha Z^{-1}| < 1 \Rightarrow \left|\frac{\alpha}{Z}\right| < 1 \Rightarrow |\alpha| < |Z|$$

Thus ROC is $|Z| > |\alpha|$.

It indicates that ROC is exterior part of circle having radius α.

This ROC is shown in Fig. U-12.

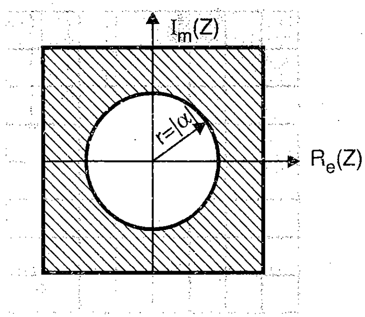


Fig. U-12 : ROC of $\alpha^n u(n)$

Thus we can write Z-transform pair as,

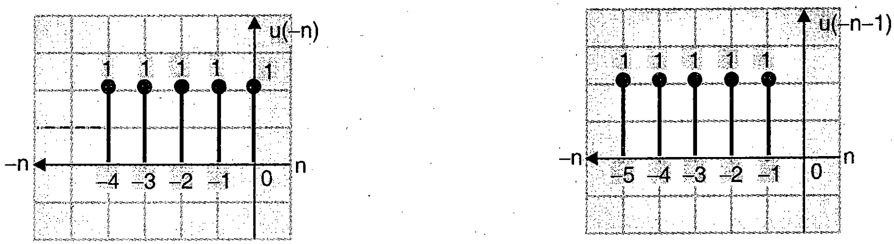
$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha|$$

7. Z-transform of left handed exponential sequence :

As the name indicates; it is an exponential sequence present on left hand side; that means only at negative values of 'n'. So it is anticausal or non-causal exponential sequence. It is expressed as,

$$x(n) = -\alpha^n u(-n-1) \qquad ...(1)$$

Here $u(-n)$ is folded unit step as shown in Fig. U-13(a).



(a) Folded unit step, u (– n) **(b) Sequence u (– n – 1)**

Fig. U-13

While $u(-n-1)$ indicates the advancing of $u(-n)$ by '1' sample. It is obtained by shifting Fig. U-13(a) towards left by 1 sample. This sequence is shown in Fig. U-13(b).

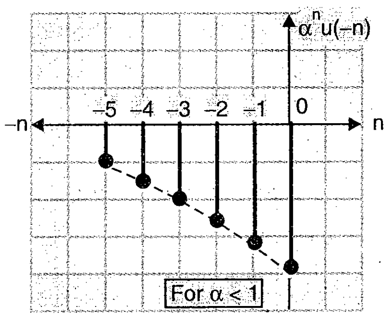


Fig. U-14 : Anticausal exponential sequence

Note that this sequence is having magnitude equals to 1. From $n = -1$ to $n = -\infty$. So we can express it as,

$$u(-n-1) = 1 \quad \text{for } n \leq -1$$
$$= 0 \quad \text{for } n \geq 0 \qquad ...(2)$$

Clearly this is an anticausal sequence. So *multiplication of any sequence with $u(-n-1)$ gives anticausal sequence.* And since its magnitude is unity; it will not change the magnitude of sequence after multiplication. Now anticausal exponential sequence is shown in Fig. U-14.

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(3)$$

But $\quad x(n) = -\alpha^n u(-n-1)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} -\alpha^n u(-n-1) Z^{-n} \qquad ...(4)$$

Here multiplication by $u(-n-1)$ gives anticausal sequence present from $n = -1$ to $n = -\infty$. So we will change the limits of summation. And since the magnitude of $(-\alpha^n)$ is not changed with this multiplication; we will not consider the term $u(-n-1)$ in the calculations.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} (-\alpha^n) Z^{-n}$$

Here also we will have to bring it in the form A^n.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} -(\alpha^{+1} Z^{-1})^n \qquad ...(5)$$

The limits of summation are negative, so to make it positive we will substitute

$$l = -n$$

$$\therefore \quad n = -l$$

$\therefore$ when $n = -\infty$, we get $-\infty = -l$; That means $l = \infty$

and when $n = -1$, we get $-1 = -l$; That means $l = 1$.

$$\therefore \quad X(Z) = \sum_{l=1}^{\infty} -(\alpha^1 Z^{-1})^{-l} \qquad ...(6)$$

Take negative sign outside and change the term inside the bracket.

$$\therefore \quad X(Z) = -\sum_{l=1}^{\infty} (\alpha^{-1} Z)^l \qquad ...(7)$$

Now put $A = \alpha^{-1} Z$

$$\text{Thus} \quad \sum_{l=1}^{\infty} A^l = A + A^2 + A^3 + A^4 + ... = A(1 + A + A^2 + A^3 + ...)$$

The term inside the bracket is standard geometric series which converges to $\frac{1}{1-A}$ if $|A| < 1$.

Thus we get,

$$\sum_{l=1}^{\infty} A^l = A \times \frac{1}{1-A} \qquad \text{if} \quad |A| < 1 \qquad ...(8)$$

Thus Equation (7) becomes,

$$X(Z) = -(\alpha^{-1} Z) \times \frac{1}{1 - \alpha^{-1} Z} \qquad \text{if } |\alpha^{-1} Z| < 1$$

$$= -\frac{(Z/\alpha)}{1 - (Z/\alpha)} \qquad \text{if } \left|\frac{Z}{\alpha}\right| < 1$$

$$\therefore \quad X(Z) = \frac{Z/\alpha}{Z/\alpha - 1} \qquad \text{if } |Z| < \alpha$$

Multiplying numerator and denominator by α we get,

$$X(Z) = \frac{Z}{Z-\alpha} \qquad \text{if} \quad |Z| < |\alpha|$$

ROC :

Here ROC is $|Z| < |\alpha|$. That means ROC is interior part of circle having radius α. This ROC is shown in Fig. U-15.

Thus Z-transform pair is

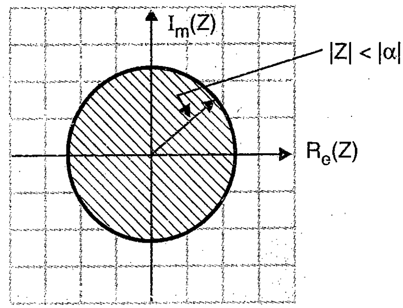


Fig. U-15 : ROC of $-\alpha^n u(-n-1)$

$$-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{if } |Z| < |\alpha|$$

Note : This sequence is an infinite exponential sequence and it is anticausal. Thus ROC of infinite anticausal sequence is interior part of circle having radius $|\alpha|$.

Comment : Z-transforms of $\alpha^n u(n)$ and $-\alpha^n u(-n-1)$ are same but their ROC's are different.

(8) Z-transform of two sided exponential :

Consider two sided exponential sequence, which is addition of causal and non-causal exponential sequences. Thus,

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1) \qquad \text{...(1)}$$

This two sided exponential sequence is shown in Fig. U-16(a).

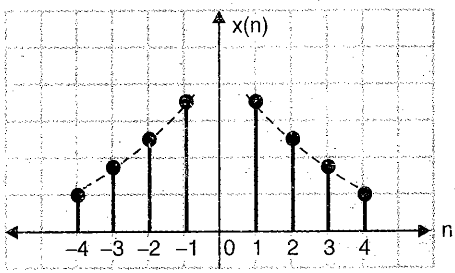


Fig. U-16(a) : Two sided exponential

Let $\quad x_1(n) = \alpha^n u(n)$ and $x_2(n) = \beta^n u(-n-1)$.

Now $\quad X(Z) = X_1(Z) + X_2(Z) \qquad \text{...(2)}$

We have obtained the Z-transform for such sequences separately.

Z-transform of $\alpha^n u(n)$:

We have $\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}$ if $|Z| > |\alpha|$

$$\therefore \; X_1(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha| \qquad ...(3)$$

Z-transform of $\beta^n u(-n-1)$:

We have $-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}$ if $|Z| < |\alpha|$.

From this we can write,

$$\beta^n u(-n-1) = -\frac{Z}{Z-\beta} \quad \text{if } |Z| < |\beta|$$

$$\therefore \; X_2(Z) = \frac{-Z}{Z-\beta} \quad \text{if } |Z| < |\beta| \qquad ...(4)$$

Putting Equations (3) and (4) in Equation (2) we get,

$$X(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

ROC : $|Z| > |\alpha|$ and $|Z| < |\beta|$

Deciding combined ROC :

We have obtained ROC separately. Now for the total sequence we will find ROC for which $X(Z) = X_1(Z) + X_2(Z)$ will convergent. In other words there should be some overlap of two ROC's. Now this will depend on the values of α and β.

We will consider two cases as follows :

Case I : If $|\beta| < |\alpha|$

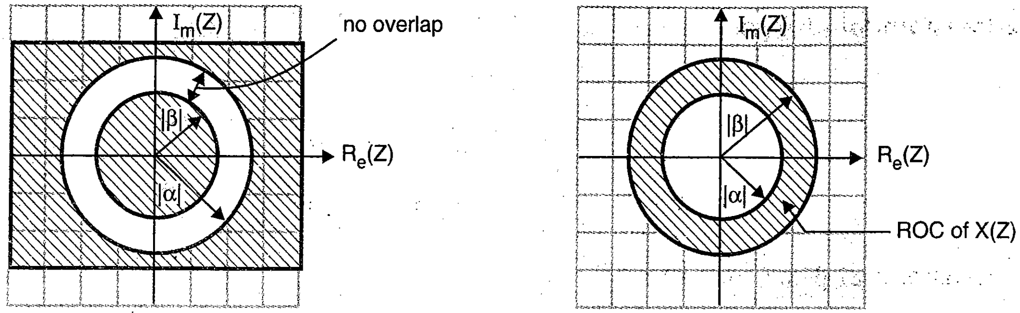
This ROC is plotted as shown in Fig. U-16(b).

We know that for $X_1(Z)$, ROC is $|Z| > |\alpha|$; which is exterior part of circle having radius α, that means exterior part of outer circle. And for $X_2(Z)$, ROC is $|Z| < |\beta|$; which is interior part of circle having radius β. That means interior part of inner circle.

Now as shown in Fig. U-16(a), the two ROC's will not overlap. There is a gap between two ROC's. For this portion, $X(Z)$ will not convergent. That $X(Z)$ will not exist.

Case II : If $|\beta| > |\alpha|$

This ROC is plotted as shown in Fig. U-16(c).



(b) ROC for |β| < |α| **(c) ROC for |β| > |α|**

Fig. U-16

In this case there is a ring in the Z-plane. In this case both power series converge simultaneously. That means X (Z) exist.

$$\therefore \; X(Z) = X_1(Z) + X_2(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

$$\therefore \; X(Z) = \frac{Z}{Z-\alpha} + \frac{Z}{\beta - Z} \quad \text{for } |\alpha| < |Z| < |\beta|$$

ROC :

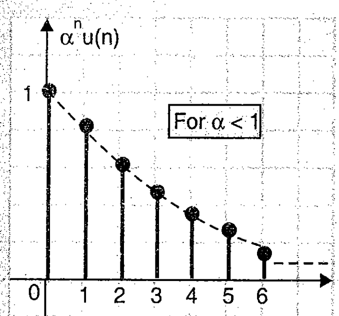
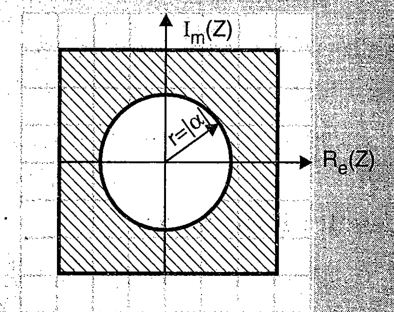
Thus combined ROC is $|\alpha| < |Z| < |\beta|$

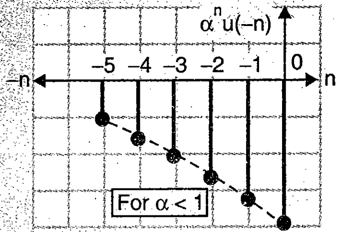
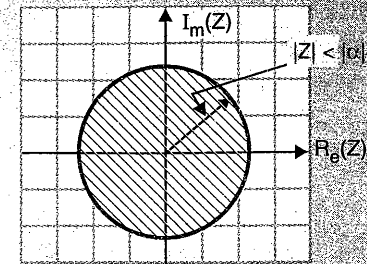
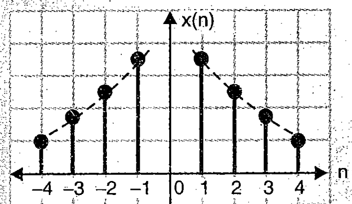
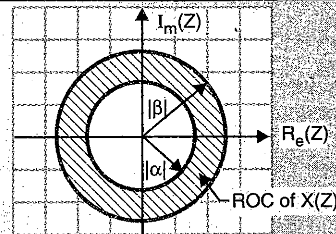
Note : When we combine many Z-transforms the ROC of the overall Z-transform is at least the intersection of the individual Z-transforms.

Summary of ROC for infinite duration sequences :

Table U-2 shows the ROC for infinite duration sequence.

Table U-2

Sr. No.	Infinite duration sequence	ROC
(1)	Causal e.g. $x(n) = \alpha^n u(n)$, 	

Sr. No.	Infinite duration sequence	ROC
(2)	Anticausal e.g. $x(n) = \alpha^n u(-n-1)$, 	
(3)	Two sided e.g. $x(n) = \alpha^n u(n) + \beta^n u(-n-1)$ 	

1.2.3 Properties of ROC :

We have solved some examples using Z transform and we have discussed ROC in each case. Based on this; the properties of ROC are summarized as follows :

(1) The ROC is a ring, whose center is at origin.

(2) ROC cannot contain any pole.

(3) If ROC of X(Z) includes unit circle then and then only the fourier transform of D.T. sequence x(n) converges.

(4) The ROC must be a connected region.

(5) For a finite duration sequence, x(n); the ROC is entire Z plane except $Z = 0$ and $Z = \infty$.

(6) If x(n) is causal then ROC is exterior part of circle of radius say 'α'.

(7) If x(n) is anticausal then ROC is interior part of circle of radius say 'α'.

(8) If x(n) is two sided sequence then ROC is intersection of two circles of radii 'α' and 'β'.