Basics of Complex Numbers:

$z = x + iy \rightarrow$ complex number $x, y \in \mathbb{R}$.

$\bar{z} = x - iy \rightarrow$ conjugate of z

$z \cdot \bar{z} = |z|^2 = x^2 + y^2 \rightarrow$ modulus of z squared.

$x = \text{Re } z$ (Real part of z)

$y = \text{Im } z$ (Imaginary part of z)

Addition & Substraction: Let $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

Then

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Multiplication:
$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Division:
$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z_2}}{\bar{z_2}} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$
$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Ex $z = 2 + 3i$ can also be written as $z = (2, 3)$ Real part Imaginary part

Hence, any complex number can be plotted on xy-plane (also called Argand diagram)

Ex1 If $z_1 = 4 + 3i$ and $z_2 = 2 - 5i$ find

(1) $\frac{1}{z_2}$ (2) $\text{Re}(z_1^2)$ (3) $(\text{Re } z_1)^2$ (4) $\text{Re}\left(\frac{z_2}{z_1}\right)$

Polar Form representation of complex number:

we know relation between x, y & r, θ is

$x = r\cos\theta \quad y = r\sin\theta$

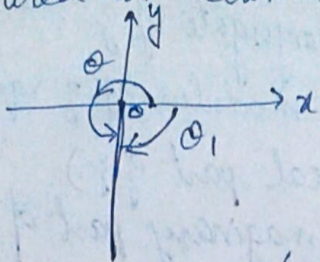
$\therefore z = x + iy) = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

$|z| = r = \sqrt{x^2 + y^2} \rightarrow$ modulus of z

$\arg z = \theta = \tan^{-1}(y/x) \rightarrow$ argument of z.

<u>Note</u>: Angle θ measured ~~is~~ is measured in radians. The value of θ is positive if it is measured in counterclockwise direction.

~~Ex~~ e.g.



$\theta = \frac{3\pi}{2}$ (−ive y-axis)

$\theta_1 = -\pi/2$ [measured in clockwise direction]

Both θ & θ_1 are angles [measured from x-axis] that −ive y-axis makes with x-axis. So +ive

$\theta \to$ measured in counterclockwise direction, " −ive

$\theta_1 \to$ " " clockwise "

<u>Principal value</u>: If the value of θ lies in the interval $\boxed{-\pi < \theta \le \pi}$, then it is called ~~prin~~ principal value. It is written as $\boxed{\text{Arg } z}$ (with A → capital)

<u>How to ~~Princip~~ find θ</u>: Let $z = x+iy$ be given complex number whose argument is to be found.

x, y can be +ive, −ive or zero

Consider $z = |x| + i|y|$ → This will lie in 1st quadrant

$\therefore \theta_1 = \tan^{-1}(|y|/|x|)$

Quadrant	Principal value of θ	θ (general value) ($=\arg z$)
z in first quadrant	$\theta = \theta_1$	$\theta = \theta_1$
z in 2nd quadrant	$\theta = \pi - \theta_1$	$\theta = \pi - \theta_1$
z in 3rd quadrant	$\theta = -\pi + \theta_1$	$\theta = -\pi + \theta_1$ or $\pi + \theta_1$
z in 4th quadrant	$\theta = -\theta_1$	$\theta = 2\pi - \theta_1$ or $-\theta_1$

Hence, given z principal value and general value of θ can be found using above table.

<u>Ex 12</u> Find polar form of following complex number.

(1) $1+i$ (2) $-2+2i$ (3) -10 (4) $3i$

(5) $\frac{1-i}{1+i}$ (6) $\frac{i}{3+3i}$ (7) $-1-i\sqrt{3}$.

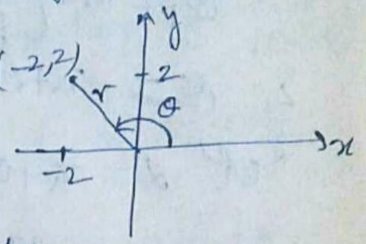
Soln 2 (1) $z = 1+i$ $\quad x=1 \quad y=1 \quad \therefore r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$

$\therefore z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ — Ans

(2) $z = -2+2i \quad x=-2 \quad y=2 \quad r = \sqrt{(-2)^2+2^2} = \sqrt{8}$

Now, $\theta_1 = \tan^{-1}\left(\frac{|2|}{|-2|}\right) = \pi/4$



$\therefore \theta = \pi - \theta_1 = \pi - \pi/4 = 3\pi/4$

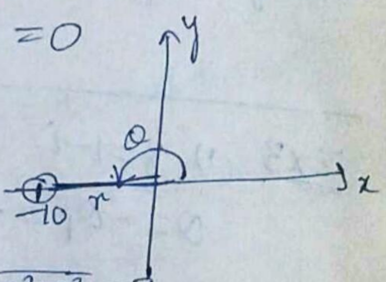
$\therefore z = \sqrt{8}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin(3\pi/4)\right]$ — Ans

(3) $z = -10 \quad x=-10 \quad y=0 \quad r = \sqrt{(-10)^2+0^2} = 10$

$\theta_1 = \tan^{-1}\left(\frac{0}{|-10|}\right) = \tan^{-1}0 = 0$

$\therefore \theta = \pi - \theta_1 = \pi - 0 = \pi$

$\therefore z = 10(\cos\pi + i\sin\pi)$ → Ans

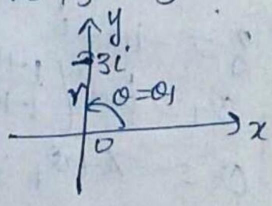


(4) $z = 3i \quad x=0 \quad y=3 \quad r = \sqrt{0^2+3^2} = 3$

$\theta_1 = \tan^{-1}(3/0) = \pi/2$

$\therefore \theta = \theta_1 = \pi/2$

$\therefore z = 3\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)$ — Ans

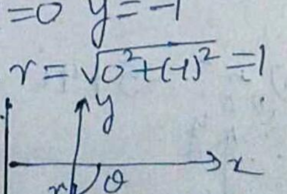


(5) $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{-2i}{2} = -i \quad x=0 \quad y=-1$

$\therefore r = \sqrt{0^2+(-1)^2} = 1$

$\theta_1 = \tan^{-1}\left(\frac{|-1|}{0}\right) = \pi/2$

$\therefore \theta = -\theta_1 = -\pi/2$ or $\theta = 2\pi - \pi/2 = 3\pi/2$



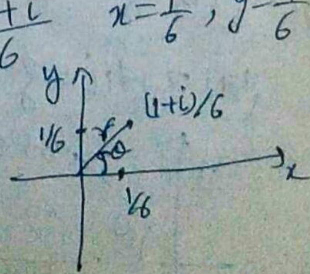
$\therefore z = 1\left[\cos(-\pi/2) + i\sin(-\pi/2)\right]$

or $z = 1\cdot\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$ → Ans

(6) $\frac{i}{3+3i} = \frac{1}{3}\left(\frac{i}{1+i} \times \frac{1-i}{1-i}\right) = \frac{1}{3}\frac{(i+1)}{2} = \frac{1+i}{6} \quad x = \frac{1}{6};\ y = \frac{1}{6}$

$\therefore r = \sqrt{\frac{1}{36}+\frac{1}{36}} = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}}$

$\theta = \theta_1 = \tan^{-1}\left(\frac{1/6}{1/6}\right) = \pi/4$



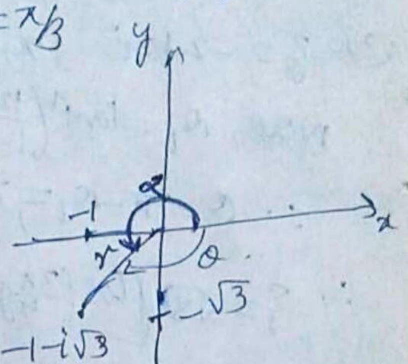
$\therefore\ z = \frac{1}{3\sqrt{2}}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] \rightarrow$ Ans

(7) $z = -1 - i\sqrt{3}$ $\quad x = -1 \quad y = -\sqrt{3} \quad r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

$\theta_1 = \tan^{-1}\left(\frac{|-\sqrt{3}|}{|-1|}\right) = \tan^{-1}\sqrt{3} = \pi/3$

$\angle\ \theta = -\pi + \theta_1 = -2\pi/3$

OR $\alpha = \pi + \theta_1 = 4\pi/3$



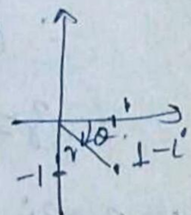
$\therefore\ z = 2\left[\cos(-2\pi/3) + i\sin(-2\pi/3)\right]$

OR $z = 2\left[\cos(4\pi/3) + i\sin(4\pi/5)\right]$

Ans

Ex3 (1) $z = 1 - i$ $\quad x = 1 \quad y = -1 \quad \theta_1 = \pi/4$

$\theta = -\theta_1 = -\pi/4$ - Ans

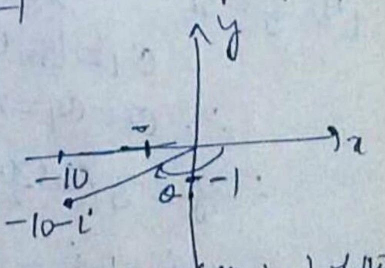


(2) $z = -10 - i$ $\quad x = -10 \quad y = -1$

$\theta_1 = \tan^{-1}\left(\frac{|-1|}{|-10|}\right) = \tan^{-1}\left(\frac{1}{10}\right)$

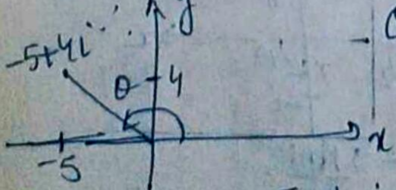
$\theta = -\pi + \theta_1 = -\pi + \tan^{-1}\left(\frac{1}{10}\right)$

$\hookrightarrow$ Ans



(3) (i) $z = -5 + 4i$ $\quad x = -5,\ y = 4 \quad \theta_1 = \tan^{-1}\left(\frac{4}{|-5|}\right) = \tan^{-1}\left(\frac{4}{5}\right)$

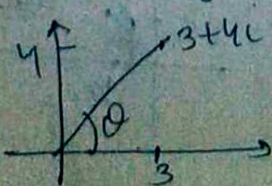
$\theta = \pi - \theta_1 = \pi - \tan^{-1}\left(\frac{4}{5}\right)$ - Ans



(ii) $z = -5 - 4i$ $\quad \theta = -\pi + \tan^{-1}\left(\frac{4}{5}\right)$ (see (2))

(4) $3 + 4i = z$ $\quad x = 3 \quad y = 4 \quad \theta_1 = \tan^{-1}(4/3)$

$\theta = \theta_1 = \tan^{-1}(4/3)$ - Ans



Ex.3 Determine the principal value of the argument
(1) $1-i$ (2) $-10-i$ (3) $-5 \pm 4i$, (4) $3+4i$

If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$

[using De Moivre's Formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$]

Roots of complex numbers: Let w be any complex number. Then, if $z = w^n$ $(n = 1, 2, \ldots)$ then to each value of n, there corresponds one value of z (obviously $z \neq 0$). Conversely, for each z there correspond precisely n-distinct values of w

i.e, $\sqrt[n]{z} = w$.

If $z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\varphi + i\sin\varphi)$

then $z = w^n$ becomes

$R^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $R^n = r$ and $n\varphi = \theta + 2k\pi$.

or, $\boxed{R = \sqrt[n]{r} \text{ and } \varphi = \frac{\theta + 2k\pi}{n}}$ $k = 0, 1, \ldots, (n-1)$.

Hence, $\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$

$k = 0, 1, 2, \ldots, (n-1)$.

What will happen if $k = n, n+1, \ldots$??

Q What will happen if $k = n, n+1, \ldots$?? you have a formula For finding square root of z, you have a formula

$$\boxed{\sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + i(\operatorname{sign} y)\sqrt{\tfrac{1}{2}(|z| - x)}\right]}$$

where $\operatorname{sign} y = \begin{cases} 1 & y > 0 \\ -1 & y < 0 \end{cases}$.

Ex Find $\sqrt{2+3i}$. $|z| = \sqrt{4+9} = \sqrt{13}$. $x = 2$ $y = 3 > 0$

$\therefore \sqrt{z} = \pm\left[\sqrt{\frac{1}{2}(\sqrt{13}+2)} + i\sqrt{\frac{1}{2}(\sqrt{13}-2)}\right]$ $\rightarrow$ Using this formula does not require computation of 'θ'.

Ex Find all roots of

(1) $\sqrt[3]{1+i}$ (2) $\sqrt[3]{8i}$ (3) $\cancel{\sqrt[4]{1296}}$ (4) $\sqrt[4]{-4}$

(3) $\sqrt[4]{1296}$

5) $\sqrt{-7+24i}$

Solution (1) $z = 1+i$ $r = \sqrt{1^2+1^2} = \sqrt{2}$ $\theta = \tan^{-1}(1/1) = \pi/4$

$$\therefore z = \sqrt{2}\left[\cos(\pi/4 + 2k\pi) + i\sin(\pi/4 + 2k\pi)\right]$$

($\because$ As cos and sin are periodic functions with period 2π)

$$\therefore \sqrt[3]{1+i} = z^{1/3} = \left(\sqrt{2}\left[\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right]\right)^{1/3}$$

$$= 2^{1/6}\left[\cos\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4}+2k\pi}{3}\right)\right] \quad k=0,1,2$$

For $k=0$

$$z_0 = 2^{1/6}\left[\cos\left(\frac{\pi}{12}\right) + i\sin(\pi/12)\right]$$

For $k=1$

$$z_1 = 2^{1/6}\left[\cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right)\right] = 2^{1/6}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= 2^{1/6}\left[\cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right] = 2^{1/6}\left[-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= 2^{1/6}\left[\frac{-1+i}{\sqrt{2}}\right]$$

For $k=2$ $z_2 = 2^{1/6}\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$

$$= 2^{1/6}\left[\cos\left(\pi + \frac{5\pi}{12}\right) + i\sin\left(\pi + \frac{5\pi}{12}\right)\right]$$

$$= -2^{1/6}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

$\therefore$ $\sqrt[3]{1+i}$ are z_0, z_1, z_2.

(2) $\sqrt[3]{8i}$ $z = 8i$ $x=0$ $y=8$ $r = \sqrt{0^2+8^2} = 8$

$\theta = \tan^{-1}(8/0) = \pi/2$

$$\therefore \sqrt[3]{8i} = z^{1/3} = \left\{8\left[\cos(\pi/2 + 2k\pi) + i\sin(\pi/2 + 2k\pi)\right]\right\}^{1/3}$$

$$= 2\left[\cos\left(\frac{(4k+1)\pi}{6}\right) + i\sin\left(\frac{(4k+1)\pi}{6}\right)\right], \quad k=0,1,2$$

For $k=0$

$$z_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{3} + i$$

For $k=1$

$$z_1 = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = 2\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$$
$$= 2\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = -\sqrt{3} + i$$

For $k=2$

$$z_2 = 2\left[\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right] = 2\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = -2i$$

$\therefore \sqrt[3]{8i}$ are z_0, z_1, z_2.

(3) $z = 1296$ $\quad x = 1296 \quad y = 0 \quad r = 1296$.

$\theta = \tan^{-1}\left(\frac{0}{1296}\right) = 0$

$$\therefore \sqrt[4]{1296} = z^{1/4} = \left(1296\left[\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)\right]\right)^{1/4}$$
$$= 6\left[\cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right)\right] \quad k = 0, 1, 2, 3$$

For $k=0$

$z_0 = 6(\cos 0 + i\sin 0) = 6$

For $k=1$

$z_1 = 6\left[\cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right)\right]$
$= 6i$

For $k=2$

$z_2 = 6\left[\cos\left(\frac{4\pi}{4}\right) + i\sin\left(\frac{4\pi}{4}\right)\right]$
$= -6$

For $k=3$

$z_3 = 6\left[\cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right)\right]$
$= -6i$

$\therefore \sqrt[4]{1296}$ are z_0, z_1, z_2, z_3.

(4) $\sqrt[4]{-4}$ $\quad z = -4 \quad x = -4 \quad y = 0 \quad r = 4$

$\theta = \pi$ (z lies on $-$ive x-axis)

$$\therefore \sqrt[4]{-4} = z^{1/4} = \left(4\left[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right]\right)^{1/4}$$
$$= 4^{1/4}\left[\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right] \quad k = 0, 1, 2, 3.$$

$z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right) = 1 + i$

$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1 + i$

$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1 - i$

$z_3 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
$= 1 - i$

(5) $\sqrt{-7+24i}$, $z = -7+24i$

$\therefore |z| = \sqrt{7^2+24^2} = 25$

$x = -7$, $y = 24 > 0$ $\therefore$ sign $y = 1$

$\therefore \sqrt{-7+24i} = \pm\left[\sqrt{\frac{1}{2}(25+(-7))} + i\sqrt{\frac{1}{2}(25-(-7))}\right]$

(See formula for $\sqrt{z}$)

$= \pm[3+i4] \rightarrow$ Ans.