Complex function: A function f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z i.e, $f(z)=w$

$$w=f(z)=u(x,y)+iv(x,y)$$

Here u & v are real functions of x & y. e.g.

let $w=z^2$ $\quad f(z)=z^2=\underbrace{x^2-y^2}_{u(x,y)}+i\underbrace{2xy}_{v(x,y)}$

Limit: A function $f(z)$ is said to have limit l as z approaches a point z_0, if f is defined in neighbourhood of z_0 (except perhaps at z_0 itself) and if the values of f are close to l for all z close to z_0 i.e, for every positive real $\epsilon > 0$, there exists a positive real $\delta > 0$, such that for all $z \neq z_0$ in the disk $|z-z_0|<\delta$,

$$|f(z)-l|<\epsilon$$

In such case we say $\lim_{z\to z_0} f(z)=l$

Continuity: A function $f(z)$ is said to be continuous at point $z=z_0$ if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z)=f(z_0)$

Derivative: Derivative of a complex function f at a point z_0 is given by

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

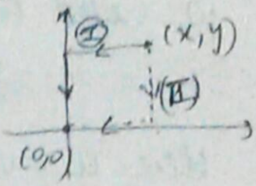
Ex Find out whether $f(z)$ is continuous at $z=0$ if $f(0)=0$ and for $z\neq 0$ f is given by

(1) $\dfrac{\operatorname{Im} z}{|z|}$ $\quad$ (2) $\dfrac{\operatorname{Re}(z^2)}{|z|}$ $\quad$ (3) $\dfrac{(\operatorname{Re} z^2-\operatorname{Im} z^2)}{|z|^2}$

soln: (1) $f(z)=\begin{cases}\dfrac{\operatorname{Im} z}{|z|}, & z\neq 0\\ 0 & z=0.\end{cases}$

First we will check whether the limit exists at $z=0$ or not: Along path I:

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{y}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0}\left(\frac{y}{y}\right)=1$$



Along path II:

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{y}{\sqrt{x^2+y^2}}\right) = 0$$

$\therefore$ limit along path I $\neq$ limit along path II

$\Rightarrow$ limit does not exist at $z=0$

Hence, function is not continuous at $z=0$

(ii) $f(z) = \begin{cases} \mathrm{Re}(z^2)/|z| & z\neq 0 \\ 0 & z=0 \end{cases} = \begin{cases} \dfrac{x^2-y^2}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Along path I:

$$\lim_{z\to 0} f(z) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0}\left(\frac{-y^2}{y}\right)=0$$

Along path II:

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0}\left(\frac{x^2}{x}\right)=0$$

[illegible]: Take the path $y=mx$. Then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0}\frac{x^2-m^2x^2}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0}\frac{x^2(1-m^2)}{x\sqrt{1+m^2}} = 0$$

Since ~~all~~ along all paths the value of limit is same hence, $\lim_{z\to 0} f(z)$ exists and is zero.

(3) $f(z) = \begin{cases} \dfrac{\mathrm{Re}\, z^2 - \mathrm{Im}\, z^2}{|z|^2} & z\neq 0 \\ 0 & z=0 \end{cases} = \begin{cases} \dfrac{x^2-y^2-2xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Along path I: $\lim_{z\to 0} f(z) = \lim_{y\to 0}\left(\lim_{x\to 0}\dfrac{x^2-y^2-2xy}{x^2+y^2}\right) = -1$

Along path II: $\lim_{z\to 0} f(z) = \lim_{x\to 0}\left(\lim_{y\to 0}\dfrac{x^2-y^2-2xy}{x^2+y^2}\right) = 1$

limit along path I $\neq$ limit along path II. Hence limit does not exist at $z=0$

Ex Check the differentiability of following functions:

1. $f(z) = \begin{cases} \frac{Re(z^2)}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$

Solution This function we have seen is continuous at $z=0$. Hence, may be differentiable at $z=0$. Let's check.

Along path I

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{y \to 0} \left(\lim_{x \to 0} \frac{\frac{x^2 - y^2}{\sqrt{x^2 + y^2}} - 0}{x + iy} \right)$$

$$= \lim_{y \to 0} \left(\lim_{x \to 0} \frac{x^2 - y^2}{(x+iy)\sqrt{x^2+y^2}} \right) = \lim_{y \to 0} \left(\frac{x^2}{x \cdot x} \right) = 1$$

Along path II

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{x \to 0} \frac{\left[\lim_{y \to 0} \frac{x^2 - y^2}{\sqrt{x^2+y^2}} - 0 \right]}{x + iy}$$

$$= \lim_{x \to 0} \left(\frac{-y^2}{y (iy)} \right) = \lim_{x \to 0} \left(\frac{-1}{i} \right) = i$$

Derivative along path I $\neq$ Derivative along path II at $z=0$

$\therefore$ Function is not differentiable at $z=0$

Note! The paths considered above may not but sufficient in all cases. The following example shows this:

Ex Check differentiability of $f(z) = \begin{cases} \frac{x^3 y (y - ix)}{x^6 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$

Solution: If you check differentiability along path I, path II or along $y = mx$, $f'(0) = 0$

Now, consider along $y = x^3$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \boxed{\lim_{x \to 0} \cancel{\frac{x^6 (x^3 - ix)}{2x^6 (\ldots)}}}$$

P.T.O.

$$f'(z) = \lim_{x\to 0} \frac{x^3 . x^3(x^3 - ix)}{(x^6 + yx^6)(x + ix^3)}$$

$$= \lim_{x\to 0} \frac{x^7(x^2 - i)}{2x^7(1+ix^2)} = \frac{-i}{2} \neq 0 .$$

Hence, the function is not differentiable at $z=0$.

Cauchy - Reimann Equations:

Analytic functions: A function is said to be analytic in ~~an~~ a domain D if $f(z)$ is defined and differentiable at all points of D

Analytic at a point: Function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is differentiable at everypoint in some neighbourhood of z_0.

Cauchy - Reiman Equations: $f(z) = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad -(1)$$

These provide a criteria for the analylicity of a complex function, given in theorem below:

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and differentiable at z itself. Then at that point first-order partial derivatives of u and v exist and satisfy C-R eqns (1)

Hence, if $f(z)$ is analytic in a Domain D then partial derivatives u_x, v_y, v_x, u_y exist and are continuous and satisfy (1) at all points of D

Also, $f'(z) = u_x + iv_x$ (Path II), $f'(z) = v_y - iu_y$ (Path I)

'or' $f'(z) = u_x - iu_y$ $\quad f'(z) = v_y + iv_x$

Ex Check for the analyticity of the following functions:

(1) z^2 (2) $i|z|^3$ (3) $e^x(\cos y + i\sin y)$

(4) $z\bar{z}$ (5) $\ln|z| + i\,\mathrm{Arg}\,z$ (6) $\dfrac{\mathrm{Re}\,z}{\mathrm{Im}\,z}$

Solution (1) $f(z) = z^2 = (x^2 - y^2) + i\,2xy$

Here $u = x^2 - y^2$ $v = 2xy$

$u_x = 2x$, $u_y = -2y$. $v_x = 2y$ $v_y = 2x$.

Here C-R eqns $u_x = v_y$ & $u_y = -v_x$

$2x = 2x$ $-2y = -(2y)$

are satisfied for all x & y and partial derivatives are continuous, hence $f(z) = z^2$ is analytic fn.

(2) $f(z) = i|z|^3 = i(x^2+y^2)^{3/2}$

Here $u = 0$ $v = (x^2+y^2)^{3/2}$

$u_x = u_y = 0$ $v_x = \frac{3}{2}(x^2+y^2)^{1/2} \cdot 2x = x(x^2+y^2)^{1/2}$

$v_y = y(x^2+y^2)^{1/2}$

C-R eqns: $u_x = v_y$ & $u_y = -v_x$

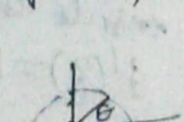
$0 = y(x^2+y^2)^{1/2}$ & $0 = -x(x^2+y^2)^{1/2}$

These two eqns are satisfied only at $(0,0)$ i.e. origin

Hence function may be analytic only at $z=0$

To check analyticity at $z=0$:

Take a disk of radius 'ϵ' around $z=0$



$|z| < \epsilon$. This is a nbd of $z=0$.

Take a point $(\epsilon/2, 0)$ on real axis inside the disk

$f'(z) = u_x + i v_x$ & $f'(z) = v_y - i u_y$

$f'((\epsilon/2, 0)) = 0 - i\frac{\epsilon}{2}\left(\frac{\epsilon^2}{4}+0\right)^{1/2} = -\frac{i\epsilon^2}{4}$ | $f'((\epsilon/2, 0)) = 0 - i0 = 0$

Along two different paths value of derivative at $(\epsilon/2, 0)$ are different. Hence fn is not differentiable at $z=0$. Hence not analytic at $z=0$.

(3) $f(z) = e^x(\cos y + i\sin y)$

$u = e^x\cos y$ $v = e^x\sin y$

$u_x = e^x\cos y$ $u_y = -e^x\sin y$ $v_x = e^x\sin y$

$v_y = e^x\cos y$.

Now, C-R eqns

$u_x = v_y$ & $u_y = -v_x$

$e^x\cos y = e^x\cos y$ $-e^x\sin y = -(e^x\sin y)$

are satisfies for all x, y. Hence function is analytic every where.

(4) $f(z) = z\cdot\bar{z} = (x+iy)(x-iy) = x^2+y^2$.

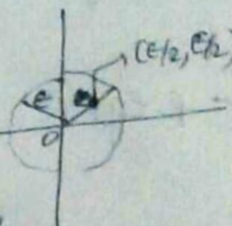
Here $u = x^2+y^2$ $v = 0$

$u_x = 2x$ $u_y = 2y$ $v_x = 0$ $v_y = 0$

C-R eqns $u_x = v_y$ & $u_y = -v_x$

$2x = 0$ $2y = 0$

are satisfied only at $(0,0)$. Hence, fn may be analytic at $(0,0)$ only.



To check analyticity at $(0,0)$

Consider neighbourhood of $z=0$ $|z| < \epsilon$. Take a point $(\epsilon/2, \epsilon/2)$ inside the neighbourhood

$f'(z) = u_x + iv_x$ & $f'(z) = v_y - iu_y$

$f'(z) = 0 - i\,2\cdot\epsilon/2$

$\therefore f'(\epsilon/2, \epsilon/2) = 2\cdot\frac{\epsilon}{2} + 0$

$= \epsilon$ $\neq$ $= -i\epsilon$

Since derivative at $(\epsilon/2, \epsilon/2)$ is different along different paths. Hence, fn is not differentiable at $(\epsilon/2, \epsilon/2)$.

$\Rightarrow f(z)$ is not analytic at $z=0$.

Hence, given fn is not analytic.

(5) $f(z) = \ln|z| + i\,\text{Arg}\,z$

$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)$

$u = \frac{1}{2}\ln(x^2+y^2)$ $\quad v = \tan^{-1}(y/x)$

$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}2x = \frac{x}{x^2+y^2}$. $\quad v_x = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

$u_y = \frac{y}{x^2+y^2}$ $\quad v_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$

C-R eqns $u_x = v_y$ & $u_y = -v_x$ are satisfied and are continuous at all x, y except at $(0,0)$ at which they ~~exact~~ ~~sat~~ do not exist. Hence, $f(z)$ is ~~Hence, f(z) is analytic every~~ analytic everywhere except at $(0,0)$.

(6) $f(z) = \frac{\text{Re}\,z}{\text{Im}\,z} = \frac{x}{y}$. Here $u = \frac{x}{y}$ $\quad v = 0$

$u_x = \frac{1}{y}$. $u_y = \frac{-x}{y^2}$ $\quad v_x = 0,\ v_y = 0$

C-R eqns $u_x = v_y$ & $u_y = -v_x$ are not satisfied ~~expect at~~ and partial derivatives do not exist at $y=0$. Hence, $f(z)$ is not analytic anywhere.

<u>Cauchy-Reimann Equations in Polar form</u>: $z = r(\cos\theta + i\sin\theta)$

$f(z) = u(r,\theta) + i v(r,\theta)$

$$\boxed{r u_r = v_\theta \text{ and } r v_r = -u_\theta}$$

<u>Ex</u> Check analyticity of following functions:

(1) $f(z) = z^6$. $\quad$ (2) $1/z^5$ $\quad$ (3) $z + \frac{1}{z}$.

<u>Soln</u> $f(z) = z^6 = r^6 e^{i6\theta} = r^6(\cos 6\theta + i\sin 6\theta)$.

$u = r^6\cos 6\theta \quad v = r^6\sin 6\theta$

$r u_r = r(6r^5\cos 6\theta) = 6r^6\cos 6\theta$

$v_\theta = 6r^6\cos 6\theta$.

$\Rightarrow r u_r = v_\theta$

Similary you can show $rV_r = -u_\theta = -6r^6 \sin 6\theta$

Hence $f(z)$ is analytic everywhere.

(2) $f(z) = \frac{i}{z^5} = iz^{-5} = ir^{-5}(\cos 5\theta - i\sin 5\theta)$

$= r^{-5}(i\cos 5\theta + \sin 5\theta)$

$u = r^{-5}\sin 5\theta \quad V = r^{-5}\cos 5\theta$

$rU_r = r(-5r^{-6}\sin 5\theta) = -5r^{-5}\sin 5\theta$

$V_\theta = -5r^{-5}\sin 5\theta$ $\Big] \Rightarrow rU_r = V_\theta$

$rV_r = r(-5r^{-6}\cos 5\theta) = -5r^{-5}\cos 5\theta$

$u_\theta = 5r^{-5}\cos 5\theta$ $\Big] \Rightarrow rV_r = -u_\theta$

C-R eqns are satisfied every where except at $z=0$, (as partial derivatives do not exist at $z=0$)

Ex Let $f(z)$ be analytic. Prove that if $\text{Re}\, f(z) = \text{Const}$ then $f(z)$ is constant.

soln $f(z) = u + iV = k + iV$ $\quad u = k$ (given)

$u_x = u_y = 0$ $\quad \therefore \boxed{u_x = V_y} \Rightarrow V_y = 0$

[This is possible as $f(z)$ is given to be analytic] $\Rightarrow V = \varphi(x) + c_1$ — (i)

$\boxed{u_y = -V_x} \Rightarrow V_x = 0$

$\Rightarrow V = \psi(y) + c_2$ — (ii)

From (1) & (2) $\varphi(x) + c_1 = \psi(y) + c_2$ $\quad \varphi(x) = \psi(y) = C$

This is possible only when $V = \text{const}$.

$\Rightarrow f(z) = k + i\,\text{Const} = \text{Const}$ //.

H.W Ex Do the above problem with $\text{Im}\, f(z) = \text{Const}$ in place of $\text{Re}(f(z)) = \text{Const}$.

Ex Let $f(z)$ be analytic function and $f'(z)=0$. Then prove that $f(z) = \text{constant}$.

Soln: $f(z)$ is analytic fn. So it satisfies C-R eqns

$u_x = v_y$ & $u_y = -v_x$.

$\therefore f'(z) = u_x + iv_x$

$= u_x - iu_y$ (using C-R eqns)

Now, $f'(z) = 0 = u_x - iu_y \Rightarrow u_x = 0$ & $u_y = 0$

$u_x = 0 \Rightarrow u = \phi(y) + c_1$

$u_y = 0 \Rightarrow u = \psi(x) + c_2$

$\Big] \Rightarrow \phi(y) + c_1 = \psi(x) + c_2$

$\Rightarrow u = \text{constant}$.

Similarly if you take $f'(z) = v_y + iv_x$ then $v = \text{constant}$ (prove it)

$\therefore f(z) = u + iv = \text{const} + i\,\text{const} = \text{const}$ //

Ex Let $f(z)$ be analytic function & $|f(z)| = \text{constant}$. Prove that $f(z) = \text{const}$.

Soln $f(z)$ is analytic $\Rightarrow$ C-R eqns are satisfied ie, $u_x = v_y$ & $u_y = -v_x$

$|f(z)| = \text{const} \Rightarrow \sqrt{u^2+v^2} = \text{const}$

$\Rightarrow u^2 + v^2 = \text{const}$ — (1)

Differentiate (1) partially w.r.t 'x':

$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0$ — (2)

Differentiate (1) partially w.r.t 'y':

$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0$ — (3)

Using C-R eqns convert (2) & (3) in terms of partial derivatives of u [You can convert in terms of p.d's of v also]

$uu_x - vu_y = 0 \,]\times u$ and substract

& $uu_y + vu_x = 0 \,]\times v$

$\Rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = 0$ ($\because$ from (1))

$\Rightarrow u = \phi(y) + c_1$ — (4)

Similarly we can show $(u^2+v^2)u_y = 0 \Rightarrow u_y = 0$

$\Rightarrow u = \psi(x) + c_2$ — (5)

from (4) & (5) we have $u = $ Const.
Similarly we can prove $v =$ Const.
$\therefore f(z) = u + iv =$ Const //

<u>Harmonic Functions</u>: If $f(z) = u + iv$ is analytic in a domain D, then u and v satisfy Laplace's equation i.e.,

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \& \quad \nabla^2 v = v_{xx} + v_{yy} = 0.$$

If two harmonic functions u & v satisfy the C-R eqns in a domain D, they are real & imaginary parts of an analytic function f in D. Then v is said to be <u>harmonic conjugate</u> of u in D

<u>Ex</u> Determine 'a' & 'b' such that the given functions are harmonic.
(1) $u = ax^3 + by^3$ (2) $v = e^{ax}\sin 5y$ (3) $u = ax^3 + bxy$

<u>Soln</u> (1) $u = ax^3 + by^3$

$u_{xx} = 6ax$ $\quad \nabla^2 u = 0 \Rightarrow 6ax + 6by = 0$
$u_{yy} = 6by$ $\quad \Rightarrow \boxed{a = 0 \ \& \ b = 0}$

(2) $v = e^{ax}\sin 5y$

$v_{xx} = a^2 e^{ax}\sin 5y$. $v_{yy} = -25e^{ax}\sin 5y$

$\therefore \nabla^2 v = 0 \Rightarrow a^2 e^{ax}\sin 5y - 25e^{ax}\sin 5y = 0$
$\Rightarrow (a^2 - 25)e^{ax}\sin 5y = 0$
$\Rightarrow a^2 - 25 = 0 \Rightarrow \boxed{a = 5}$

(3) $u = ax^3 + bxy$

$u_{xx} = 6ax$ $\quad u_{yy} = 0$

$\therefore \nabla^2 u = 0 \Rightarrow 6ax = 0 \Rightarrow \boxed{a = 0}$

b → can have any value.